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# A NEW GAUGE FUNCTIONAL CHARACTERIZING A GIVEN ORLICZ CLASS

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Abstract. We define a new gauge functional characterizing a given Orlicz class. This functional is shown to make more computable a formula for the dual of a $\mathcal{K}$-method interpolation space.

## 1. Introduction

Suppose $A$ is a Young function defined by the formula

$$A(t) := \int_0^t a(s)ds, \;\; t \in \mathbb{R}_+ := (0, \infty),$$

in which $a(s)$ is an increasing function on $\mathbb{R}_+$, with $a(0+) = 0$ and $\lim_{s\to\infty} a(s) = \infty$.

Let $(X, \mu)$ be a $\sigma$-finite measure space and denote by $\mathfrak{M}(X)$ the set of $\mu$-measurable functions on $X$. A function $f \in \mathfrak{M}(X)$ is said to belong to the Orlicz class $L_A(X)$ if

$$\int_X A\left(\frac{|f(x)|}{\lambda_f}\right) d\mu(x) < \infty,$$

for some $\lambda_f > 0$. The gauge norm, $\rho_A(f)$, of an $f \in L_A(X)$ is

$$\rho_A(f) := \inf\left\{\lambda > 0 : \quad \int_X A\left(\frac{|f(x)|}{\lambda}\right) d\mu(x) \le 1\right\}. \tag{1.1}$$

See [5, p.97] for the interesting history of the functional (1.1) that justifies the introduction of the term "gauge norm".

With the norms $\rho_A$ in mind, we speak of the Orlicz *spaces* $L_A(X)$. These are examples of rearrangement invariant (r.i) spaces, which are defined by norms $\rho$ whose characteristic property is that $\rho(f) = \rho(g)$ whenever $f, g \in \mathfrak{M}(X)$ are equimeasurable in the sense that $f^* = g^*$; here,

$$f^*(t) := \inf\{\lambda > 0 : \mu(\{x \in X : |f(x)| > \lambda\}) \le t\},$$

$t \in I_\mu := (0, \mu(X))$.

We are now ready to state our principal result, namely,

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**Theorem 1.1.** *Let $A(t) = \int_0^t a(s)ds$, $t \in \mathbb{R}_+$, be a Young function with $a(s)$ absolutely continuous. Define*

$$c(t) := t\frac{d}{dt}\left(\frac{A(t)}{t}\right) = a(t) - \frac{A(t)}{t} = \frac{1}{t}\int_0^t sa'(s)ds$$

*and set*

$$C(t) := \int_0^t c(s)ds, \quad t \in \mathbb{R}_+.$$

*Let $(X, \mu)$ be a $\sigma$-finite measure space and suppose the (increasing) function $C$ satisfies*

$$\int_{\mathbb{R}_+} C\left(\frac{k}{1+t}\right) dt < \infty, \tag{1.2}$$

*for some $k > 0$. Then,*

$$\frac{1}{2}\rho_{\Gamma_C}(f) \le \rho_A(f) \le \rho_{\Gamma_C}(f), \;\; f \in \mathfrak{M}(X), \tag{1.3}$$

*in which*

$$\rho_{\Gamma_C}(f) := \inf\left\{\lambda > 0 : \int_{\mathbb{R}_+} C\left(t^{-1}\int_0^t f^*(s)ds\right) dt \le 1\right\}.$$

**Remarks 1.2**

1. We observe that $A(t) = \int_0^t a(s)ds$ and $\mathbf{A}(t) = \int_0^t \frac{A(s)}{s}ds$ give rise to the same Orlicz class, so there is no essential loss of generality in the assumption of Theorem 1.1 that $a(s)$ is absolutely continuous.
2. When $\mu(X) < \infty$, we may take $a(s)$, and hence $c(s)$, equal to 0 on $(0, 1)$. In this case (1.2) is automatically true, so (1.3) holds with no essential restrictions.

The result of applying (1.3) to the representation of norms dual to the $\mathcal{K}$-method interpolation norms requires some background to even state, so we postpone it to (the last) section 4.

In Section 2 we consider r.i. spaces with special emphasis on the Orlicz case.

Section 3 contains the proof of Theorem 1.1 along with a remark and an example.

## 2. Rearrangement invariant spaces

Let $(X, \mu)$ be a $\sigma$-finite measure space. Denote by $\mathfrak{M}(X)$ the set of $\mu$-measurable real-valued functions on $X$ and by $\mathfrak{M}_+(x)$ the nonnegative functions in $\mathfrak{M}(X)$. A Banach function norm is a functional $\rho : \mathfrak{M}_+(X) \to \mathbb{R}_+$ satisfying

(A1) $\rho(f) = 0$ if and only if $f = 0$ $\mu$- a.e.,
(A2) $\rho(cf) = c\rho(f)$, $c \ge 0$,
(A3) $\rho(f + g) \le= \rho(f) + \rho(g)$,
(A4) $0 \le f_n \uparrow f$ implies $\rho(f_n) \uparrow \rho(f)$,
(A5) $|E| < \infty$ implies $\rho(\chi_E) < \infty$,
(A6) $|E| < \infty$ implies $\int_E f d\mu \le c_E(\rho)\rho(f)$, for some constant $c_E(\rho)$ depending on $E$ and $\rho$ but not on $f \in \mathfrak{M}_+(X)$.

Furthermore, as mentioned in the introduction, a Banach function norm is said to be rearrangement invariant if $\rho(f) = \rho(g)$ whenever $f, g \in \mathfrak{M}_+(X)$ are equimeasurable in the sense that $f^* = g^*$; the nonincreasing rearrangement, $f^*$, of $f \in \mathfrak{M}(X)$ on $\mathbb{R}_+$ is defined as

$$f^*(t) := \inf\{\lambda > 0 : \mu(\{x \in X : |f(x)| > \lambda\}) \leq t\},$$

$t \in I_\mu := (0, \mu(X))$.

It satisfies the property

$$|\{t \in \mathbb{R}_+ : f^*(t) > \tau\})| = \mu(\{x \in X : |f(x)| > \tau\}), \; f \in \mathfrak{M}(X), \; \tau \in I_\mu.$$

Now, although the mapping $f \mapsto f^*$ is not subadditive, the mapping $f \mapsto t^{-1} \int_0^t f^*(s)ds$ is, namely

$$t^{-1} \int_0^t (f+g)^*(s)ds \leq t^{-1} \int_0^t f^*(s)ds + t^{-1} \int_0^t g^*(s)ds, \tag{2.1}$$

for all $f, g \in \mathfrak{M}(X)$, $t \in \mathbb{R}_+$. The Kothe dual of a Banach function norm $\rho$ is another such norm, $\rho'$, with

$$\rho'(g) := \sup_{\rho(f) \leq 1} \int_X fg\mu, \quad f, g \in \mathfrak{M}_+(X).$$

It is obeys the Principle of Duality; that is,

$$\rho'' := (\rho')' = \rho.$$

The space $L_\rho(X)$ is the vector space

$$\{f \in \mathfrak{M}(X) : \rho(|f|) < \infty\},$$

together with the norm

$$\|f\|_{L_\rho} := \rho(|f|).$$

This Banach space is said to be an r.i. space provided $\rho$ is an r.i. function norm.

The gauge norm, $\rho_A$, defined in (1.2) in terms of the Young function $A(t) = \int_0^t a(s)ds$, $t \in \mathbb{R}_+$, is an r.i. norm; indeed,

$$\rho_A(f) = \inf\{\lambda > 0 : \int_{I_\mu} A\left(\frac{f^*(t)}{\lambda}\right) dt \leq 1\}, \quad f \in \mathfrak{M}(X).$$

Its Kothe dual, $\rho'_A$, satisfies

$$\rho_{\widetilde{A}}(g) \leq \rho'_A(g) \leq 2\rho_{\widetilde{A}}(g), \quad g \in \mathfrak{M}(X),$$

with

$$\widetilde{A}(t) := \int_0^t a^{-1}(s)ds, \quad t \in \mathbb{R}_+,$$

being called the Young function complementary to $A$.

## 3. Proof of Theorem 1.1

We will require the following inequalities, which are analogues of ones for the Hardy-Littlewood maximal function, $Mf$, in [6, pp.6-7 and p.27]. Namely, for all $\tau > 0$

$$\frac{1}{\tau}\int_{\{t\in I_\mu : f^*(t)>\tau\}} f^*(t)dt \le |\{t\in I_\mu : (Pf^*)(t) > \tau\}| \le \frac{2}{\tau}\int_{\{t\in I_\mu : f^*(t)>\frac{\tau}{2}\}} f^*(t)dt. \tag{3.1}$$

Their proofs are even simpler than the ones for $Mf$. Thus, let $t_0$ be the least $t$ for which $(Pf^*)(t) = \tau$. (The inequalities are trivial if there is no such $t_0$). Then,

$$|\{t\in I_\mu : (Pf^*)(t) > \tau\}| = t_0 = \frac{1}{\tau}\int_0^{t_0} f^*(t)dt \ge \frac{1}{\tau}\int_{\{t\in I_\mu : f^*(t)>\tau\}} f^*(t)dt.$$

Again, defining

$$f_\tau(t) := \min\,[f^*(t), \frac{\tau}{2}]$$

and

$$f^\tau(t) := f^*(t) - f_\tau(t),$$

one has

$$\begin{aligned}|\{t\in I_\mu : (Pf^*)(t) > \tau\}| &\le |\{t\in I_\mu : (Pf^\tau)(t) > \frac{\tau}{2}\}|\\ &\le \frac{2}{\tau}\int_{I\mu} f^\tau(t)dt\\ &\le \frac{2}{\tau}\int_{\{t\in I_\mu : f^*(t)>\frac{\tau}{2}\}} f^\tau(t)dt\\ &\le \frac{2}{\tau}\int_{\{t\in I_\mu : f^*(t)>\frac{\tau}{2}\}} f^*(t)dt.\end{aligned}$$

Next, we observe that

$$A(t) = t\int_0^t c(s)\frac{ds}{s}.$$

Now, the first inequality in (3.1) ensures that for all $\lambda > 0$,

$$\int_{\mathbb{R}_+}\int_{\{t\in I_\mu : f^*(t)>\tau\}} f^*(t)dtc(\tau)\frac{d\tau}{\tau} \le \int_{\mathbb{R}_+} |\{t\in I_\mu : (Pf^*)(t) > \tau\}|c(\tau)d\tau;$$

that is,

$$\begin{aligned}\int_{I_\mu} A\left(\frac{f^*(t)}{\lambda}\right) dt &= \int_{I_\mu} \frac{f^*(t)}{\lambda} \int_0^{\frac{f^*(t)}{\lambda}} c(\tau)\frac{d\tau}{\tau} dt \\ &= \int_{\mathbb{R}_+} \int_{\{t\in I_\mu : f^*(t)>\tau\}} \frac{f^*(t)}{\lambda} dt c(\tau)\frac{d\tau}{\tau} \\ &\le \int_{\mathbb{R}_+} |\{t \in I_\mu : \frac{(Pf^*)(t)}{\lambda} > \tau\}| c(\tau)d\tau \\ &= \int_{I_\mu} C\left(\frac{(Pf^*)(t)}{\lambda}\right) dt.\end{aligned}$$

Again, the second inequality in (3.1) yields

$$\begin{aligned}\int_{I_\mu} C\left(\frac{(Pf^*)(t)}{\lambda}\right) dt &= \int_{\mathbb{R}_+} |\{t \in I_\mu : \frac{(Pf^*)(t)}{\lambda} > \tau\}| c(\tau)d\tau \\ &\le 2\int_{\mathbb{R}_+} \int_{\{t\in I_\mu : \frac{f^*(t)}{\lambda}>\frac{\tau}{2}\}} \frac{f^*(t)}{\lambda} dt c(\tau)\frac{d\tau}{\tau} \\ &= \int_{I_\mu} \frac{2f^*(t)}{\lambda} \int_0^{\frac{2f^*(t)}{\lambda}} c(\tau)\frac{d\tau}{\tau} dt \\ &= \int_{I_\mu} A\left(\frac{2f^*(t)}{\lambda}\right) dt.\end{aligned}$$

We conclude

$$\frac{1}{2}\rho_{\Gamma_C}(f) \le \rho_A(f^*) \le \rho_{\Gamma_C}(f),$$

which completes the proof of Theorem 1.1, since $\rho_A(f) = \rho_A(f^*)$. □

**Corollary 3.1.** *Let $A(t) = \int_0^t a(s)ds$, $t \in \mathbb{R}_+$, be a Young function, for which*

$$\int_{\mathbb{R}_+} A\left(\frac{k}{1+t}\right) dt < \infty,$$

*for some constant $k > 0$, so, in particular,*

$$\int_0^t a(s)\frac{ds}{s} < \infty, \quad t \in \mathbb{R}_+.$$

*Set*

$$\mathcal{A}(t) := t\int_0^t a(s)\frac{ds}{s}, \quad t \in \mathbb{R}_+. \tag{3.2}$$

*Then, $\mathcal{A}(t)$ is a Young function such that*

$$a(t) = \mathcal{A}'(t) - \frac{\mathcal{A}(t)}{t}, \quad t \in \mathbb{R}_+,$$

*whence, for any $\sigma$-finite measure space $(X,\mu)$,*

$$\frac{1}{2}\rho_{\Gamma_{A}}(f) \le \rho_{\mathcal{A}}(f) \le \rho_{\Gamma_{A}}(f), \quad f \in \mathfrak{M}(X).$$

**Remark 3.2.** The complementary Young function, $\widetilde{\mathcal{A}}$, of $\mathcal{A}$ satisfies

$$\frac{3}{2}\int_0^{\frac{t}{3}} b^{-1}(s)ds \le \widetilde{\mathcal{A}}(t) \le \int_0^t b^{-1}(s)ds,$$

where

$$b(t) := \int_0^t a(s)\frac{ds}{s}, \quad t \in \mathbb{R}_+.$$

For, observe that

$$\begin{aligned} b(t) \le \mathcal{A}'(t) &= \int_0^t a(s)\frac{ds}{s} + a(t) \\ &\le \frac{\ln 2 + 1}{\ln 2}\int_0^{2t} a(s)\frac{ds}{s} \\ &\le 3b(2t), \end{aligned}$$

and so

$$\frac{1}{2}b^{-1}\left(\frac{t}{3}\right) \le (\mathcal{A}')^{-1}(t) \le b^{-1}(t), \quad t \in \mathbb{R}_+.$$

**Example 3.3.** Consider the Young function

$$A(t) = \int_0^t \ln^{\beta}(1+s)ds, \quad 0<\beta<1, \quad t \in \mathbb{R}_+.$$

Then,

$$c(t) = \frac{\beta}{t}\int_0^t \frac{s}{1+s}\ln^{\beta-1}(1+s)ds \sim \beta\ln^{\beta-1}(1+t), \quad \text{as} \quad t \to \infty,$$

from which we see that $c(t)$ essentially decreases rather than increases. That is, $C$ is *not* convex.

## 4. An application to interpolation theory

Given Banach spaces $X_1$ and $X_2$ imbedded in a common Hausdorff topological vector space, $\mathcal{H}$, the $\mathcal{K}$-method of interpolation provides a concrete way to construct new Banach spaces $X$ which lie between them, in the sense that, for any linear operator $T$ satisfying $T : X_i \to X_i$, $i = 1, 2$, one has $T : X \to X$.

The key element in the method is the Peetre $K$-functional defined at $x \in X_1 + X_2$ and $t \in \mathbb{R}_+$ by

$$K(t, x; X_1, X_2) := \inf_{x=x_1+x_2} \left[\|x_1\|_{X_1} + t\|x_2\|_{X_2}\right].$$

For our purposes, each of the so-called interpolation spaces, $X$, will correspond to an r.i. norm $\rho$ on $\mathfrak{M}_+(\mathbb{R}_+)$, with $\rho\left(\frac{1}{1+t}\right) < \infty$; more specifically, the norm of $X$ is defined as

$$\|x\|_X := \rho\left(\frac{K(t,x;X_1,X_2)}{t}\right), \quad x \in X_1 + X_2.$$

The following is a special case of a result proved in [3, Theorem 7.2] for $X_1$ and $X_2$ r.i. spaces and $\rho = \rho_A$ an Orlicz norm. It elaborates, in a particular instance, the deep duality theorem of Brudnyi and Krugljak [2].

**Theorem 4.1.** *Let $(X,\mu)$ be a $\sigma$-finite measure space and suppose $\rho_1$ and $\rho_2$ are r.i. norms on $\mathfrak{M}_+(X)$. Assume, further, that*

$$L_{\rho_1'}(X) \cap L_{\rho_2'}(X) \quad \text{is dense in} \quad L_{\rho_2'}(X)$$

*and*

$$\rho_2'(\chi_{E_k}) \downarrow 0 \quad \text{as} \quad E_k \downarrow \emptyset, \quad E_k \subset X.$$

*Consider a Young function $A(t) = \int_0^t a(s)ds$, $t \in \mathbb{R}_+$, satisfying*

$$\int_{\mathbb{R}_+} A\left(\frac{k}{1+t}\right) dt < \infty,$$

*for some constant $k > 0$. Then, the functional*

$$\rho(f) := \rho_A\left(\frac{K(t,f;L_{\rho_1}(X),L_{\rho_2}(X))}{t}\right), \quad f \in L_{\rho_1}(X) + L_{\rho_2}(X),$$

*is an r.i. norm on $\mathfrak{M}_+(X)$ and the r.i. space, $L_\rho(X)$, to which it gives rise is an interpolation space between $L_{\rho_1}(X)$ and $L_{\rho_2}(X)$.*

*Moreover, if, in addition,*

$$\int_{\mathbb{R}_+} \widetilde{A}\left(\frac{k}{1+t}\right) dt < \infty,$$

*for some constant $k > 0$, and if $\mathcal{A}$ is the Young function defined in (3.2) and $\widetilde{\mathcal{A}}$ is its complementary function, one has*

$$\rho'(g) \approx \rho_{\widetilde{\mathcal{A}}}\left(\frac{d}{dt}K(t,g;L_{\rho_2'}(X),L_{\rho_1'}(X))\right), g \in L_{\rho_2'}(X) + L_{\rho_1'}(X).$$

Now $\frac{d}{dt}K(t,x,X_1,X_2)$ can be computed only in the case when the $K$-functional is known exactly. More often, the latter is only known to within constant multiples. The motivation behind Theorem 1.1 is the following consequence of Theorem 4.1. A version of this result involving further assumptions on the Young function $A$ is given in [3, Theorem 8.2 ].

**Theorem 4.2.** *Let $X$, $\rho_1$, $\rho_2$, $A$, $\rho$ and $\mathcal{A}$ be as in Theorem 4.1, with $a(t)$ absolutely continuous. Define the the increasing function $C$ by*

$$C(t) := \int_0^t c(s)ds,$$

*in which*

$$c(t) := \widetilde{\mathcal{A}}'(t) - \frac{\widetilde{\mathcal{A}}(t)}{t} = \frac{1}{t}\int_0^t s\widetilde{\mathcal{A}}''(s)ds, \quad t \in \mathbb{R}_+.$$

*Then, provided*

$$\int_{\mathbb{R}_+} C\left(\frac{k}{1+t}\right) dt < \infty,$$

*for some constant $k > 0$, one has*

$$\rho'(g) \approx \inf\left\{\lambda > 0 : \int_{\mathbb{R}_+} C\left(\frac{K(t, g; L_{\rho_2'}(X), L_{\rho_1'}(X))}{\lambda t}\right) dt \leq 1\right\},$$

*$g \in L_{\rho_2'}(X) + L_{\rho_1'}(X)$. In particular, $g \in L_{\rho_2'}(X) + L_{\rho_1'}(X)$ belongs, to $L_{\rho'}(X)$ if and only if there exists a constant $\lambda_g \in \mathbb{R}_+$ such that*

$$\int_{\mathbb{R}_+} C\left(\frac{K(t, g; L_{\rho_1}(X), L_{\rho_2}(X))}{\lambda_g t}\right) dt < \infty.$$

**Remark 4.3.** Theorem 4.2 is essential to the characterization of the optimal r.i. imbedding space of an Orlicz-Sobolev space found in [4, Theorem 6.3].

## References


[1] C. Bennett, R. Sharpley, *Interpolation of operators*, Pure and Applied Mathematics, vol. 129, Academic Press. Inc., Boston, MA, 1988

[2] Yu.A. Brudnyi, N.Ya. Krugljak, *Interpolation functors and interpolation spaces, Volume I*, North-Holland Mathematical Library, vol. 47, North-Holland Publishing Co., Amsterdam, 1991.

[3] R. Kerman, M. Milman and G. Sinnamon, *On the Brudnyi-Krugljak duality theorem of spaces formed by the $\mathcal{K}$-method of interpolation*, Rev. Mat. Complut., **20**(2007), no. 2. 367-389.

[4] R. Kerman and L. Pick, *Explicit formulas for optimal rearrangement-invariant norms in Sobolev imbedding inequalities*, Preprint No. MATH-KMA-2008/270. Department of Mathematical Analysis, Faculty of Mathematics and Physics, Charles University. 18 pages, http://www.karlin.mff.cuni.cz/kma-preprints/.

[5] M.M. Rao and Z.D. Ren, *Theory of Orlicz spaces*, Pure and Applied Mathematics, vol. 140, Marcel and Deker Inc., New York, NY, 1991.

[6] E. Stein, *Singular integrals and differentiability properties of functions*, Princeton University Press, Princeton, NJ, 1970.



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