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A NOTE ON SURFACES WITH RADIALLY SYMMETRIC NONPOSITIVE GAUSSIAN CURVATURE

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Abstract. It is easily seen that the graphs of harmonic conjugate functions (the real and imaginary parts of a holomorphic function) have the same nonpositive Gaussian curvature. The converse to this statement is not as simple. Given two graphs with the same nonpositive Gaussian curvature, when can we conclude that the functions generating their graphs are harmonic? In this paper, we show that given a graph with radially symmetric nonpositive Gaussian curvature in a certain form, there are (up to) four families of harmonic functions whose graphs have this curvature. Moreover, the graphs obtained from these functions are not isometric in general.

Keywords: Gaussian curvature, holomorphic function

MSC 2000: 53A05, 35C05

1. Introduction

We are concerned with the graphs of harmonic conjugate functions into $\mathbb{R}^3$ where their charts are determined by the usual rectangular-coordinate transformation. We will discuss the geometry of such charts, in particular, our focus will be the Gaussian curvature of these charts: $Ku = (u_{xx}u_{yy} - u_{xy}^2)/(1 + u_x^2 + u_y^2)^2$. In Sect. 2 we note that if u and v are harmonic conjugate functions (the real and imaginary part of a holomorphic function), then the Gaussian curvature of the graphs of u and v is the same. We also investigate a further property concerning the invariance of the Gaussian curvature on graphs obtained from harmonic functions. Specifically, the action of the unit circle, $\mathbb{S}^1$, on the set of holomorphic functions preserves the Gaussian curvature of the graphs of the real and imaginary parts. In general, this property is not isometric, so Gauss' Theorema Egregium does not apply. In Sect. 3 we observe that the conjugate functions of a given complex function possesses a chart with radially symmetric nonpositive Gaussian curvature; that is, level curves of the graph of the Gaussian curvature are circles. With this, we show the existence of (up to) two families of holomorphic functions whose real and imaginary parts have a prescribed, radially symmetric, nonpositive Gaussian curvature. This is done by "solving" a fully nonlinear PDE which represents the Gauss curvature equation and an appropriate radially symmetric function. Thus, given the Gaussian curvature of a graph has an appropriate radially symmetric form, there are (up to) four families of harmonic functions (the real and imaginary parts of each holomorphic function) which have the given curvature (to restate this from a PDE point of view, for appropriate Gauss curvature functions, we find (up to) four families of harmonic functions satisfying the equation of prescribed Gauss curvature). We note that the graphs of the functions from distinct families are not isometric, and furthermore, graphs from the same family are not isometric in general. Whether these families of solutions are unique is not yet determined.

Throughout the paper, Ω is assumed to be a region of $\mathbb{R}^2$ or $\mathbb{C}$ where appropriate, unless noted otherwise, u is (at least) a C^2 function on Ω, and Ku will denote the Gaussian curvature of the graph of u. Finally, by graph of a function, say u, we mean (at least) a C^2 chart parametrized by the (x, y)-coordinate system: $\Gamma_u(x, y) = (x, y, u)$.

2. The invariance of Gaussian curvature

The claims in this section will be of much use in Sect. 3.

The following fact is immediate with the use of Laplace's equation and the Cauchy-Riemann (C-R) equations.

Lemma 2.1. *If $f = u + \mathrm{i}v$ is holomorphic on Ω, then $Ku = Kv \leqslant 0$ on Ω.*

The converse is not true via Gauss' Theorem. See Example 3.4. The following claim is for notational convenience.

Lemma 2.2. *Let $f = u + \mathrm{i}v$ be a complex function on Ω and define*

$$Kf := \frac{-|f''|^2}{(1 + |f'|^2)^2}. \tag{1}$$

If f is holomorphic on Ω, then $Kf = Ku$ on Ω.

Proof. With the C-R equations and

$$f' = \begin{cases} u_x + \mathrm{i}v_x, \\ u_y + \mathrm{i}v_y, \end{cases} \tag{2}$$

$$f'' = \begin{cases} u_{xx} + \mathrm{i}v_{xx}, \\ u_{yy} + \mathrm{i}v_{yy}, \end{cases} \tag{3}$$

we can write the curvature equation on Ω as

$$Ku = -\frac{u_{xx}^2 + u_{xy}^2}{(1 + u_x^2 + u_y^2)^2} = Kf. \tag{4}$$

□

The following is a direct consequence of the lemmas.

Corollary 2.3. *Let $f = u + \mathrm{i}v$ be a holomorphic function on Ω. Let $\sigma \in \mathbb{C}$ where $|\sigma| = 1$ and $\sigma f = \hat{u} + \mathrm{i}\hat{v}$. Then*

$$K(\sigma f) = Kf = K\hat{u} = Ku = Kv = K\hat{v} \tag{5}$$

on Ω.

Remark 2.4. We note that the first-fundamental forms on the graphs of $\hat{u}$, u, v, and $\hat{v}$ are not the same. In fact, if g_{ij} denotes the matrix form of the Riemannian metric, g^{ij} denotes its inverse, and if g denotes the determinant of g_{ij}, then $g_{ij}^v = g^u g_u^{ij}$. Hence a diffeomorphism between any u and v is not an isometry [2, Proposition 10.5 p. 148], [3, Theorem 5.1 pp. 101–102]. Moreover, whenever $\sigma \neq \{\pm 1, \pm \mathrm{i}\}$, then the graphs of u and $\hat{u}$ or v and $\hat{v}$ are not isometric.

Finally,

Lemma 2.5. *Let $f = u + \mathrm{i}v$ be a holomorphic function on Ω and let $\sigma \in \mathbb{C}$. Suppose $f'' \neq 0$ on Ω. Then*

$$K(\sigma f) = Kf \tag{6}$$

iff $|\sigma| = 1$.

Proof. The nontrivial direction reduces to the algebraic equation

$$|f''|^2(1 - |\sigma|^2|f'|^4)(1 - |\sigma|^2) = 0. \tag{7}$$

□

Lemma 3.1. *Let $g(z) = cz^n$, where $c \in \mathbb{C}$, and $n \in \mathbb{Q}$. Set*

$$\Omega = \{z \in \mathbb{C}\colon\ z = \varrho e^{i\theta} \text{ for } \varrho \in (0, \infty),\ \theta \in (-\pi, \pi)\}. \tag{8}$$

Let $s = |z|$, and $w(s) = |g'(z)|$. Then for all $z \in \Omega$

$$(w^{(k)}(s))^2 = |g^{(k+1)}(z)|^2. \tag{9}$$

Proof. Write $c = re^{i\varphi}$ where $r = |c|$ and $\varphi = \arg c$. Notice that $|g'| = r|n|s^{n-1}$ as $|z^a| = |z|^a$ for $a \in \mathbb{R}$. Now

$$(w'(s))^2 = \Big(\frac{d}{ds} rns^{n-1}\Big)^2 = r^2n^2(n-1)^2s^{2(n-2)} = |g''(z)|^2. \tag{10}$$

Suppose $k = m - 1$, and

$$(w^{(m-1)}(s))^2 = r^2n^2(n-1)^2 \ldots (n-(m-1))^2 s^{2(n-m)} = |g^{(m)}(z)|^2. \tag{11}$$

Then

$$\begin{aligned}(w^{(m)}(s))^2 &= \Big(\frac{d}{ds} w^{(m-1)}(s)\Big)^2 \\ &= r^2n^2(n-1)^2 \ldots (n-(m-1))^2(n-m)^2 s^{2(n-(m+1))} \\ &= |g^{(m+1)}(z)|^2.\end{aligned} \tag{12}$$

□

Remark 3.2. Notice that the notation in the conclusion of Lemma 3.1 begins with the first derivative. This is because the term $|f| = \sqrt{u^2 + v^2}$ has no equivalent "nice" expression in terms of u alone. Rather $|f'| = \sqrt{u_x^2 + v_x^2} = \sqrt{u_x^2 + u_y^2}$ by the C-R equations.

Lemma 3.3. *Let Ω, $g(z) = cz^n$, $s = |z|$, and $w(s) = |g'(z)|$ be as in Lemma 3.1. Then*

$$Kg = -\Big[\frac{d}{ds} \arctan(w(s))\Big]^2 \tag{13}$$

on Ω.

Proof. This is an application of Lemma 3.1. □

Now we turn our attention to the converse of Lemma 2.1; that is, given two graphs with the same nonpositive Gaussian curvature, when can we conclude that the functions generating their graphs are harmonic? There are nonharmonic functions whose Gaussian curvature is nonpositive. After mapping this by an isometry we can obtain another nonharmonic function with the same Gaussian curvature by Gauss' Theorema Egregium. We give an illustration.

Example 3.4. Let $u(x,y) = -2x^2 + y^2$ be our nonharmonic function. Then

$$Ku = \frac{-8}{(1+16x^2+4y^2)^2} < 0 \tag{14}$$

for all $(x,y) \in \mathbb{R}^2$. Furthermore, notice that

$$Ku = -2\Big[\frac{\mathrm{d}}{\mathrm{d}s}\arctan(2s)\Big]^2, \tag{15}$$

where $s = \sqrt{4x^2+y^2} = |2x + \mathrm{i}y|$. Hence, the graph of Ku is not radially symmetric; rather, its level curves are elliptical. See Figure 1.

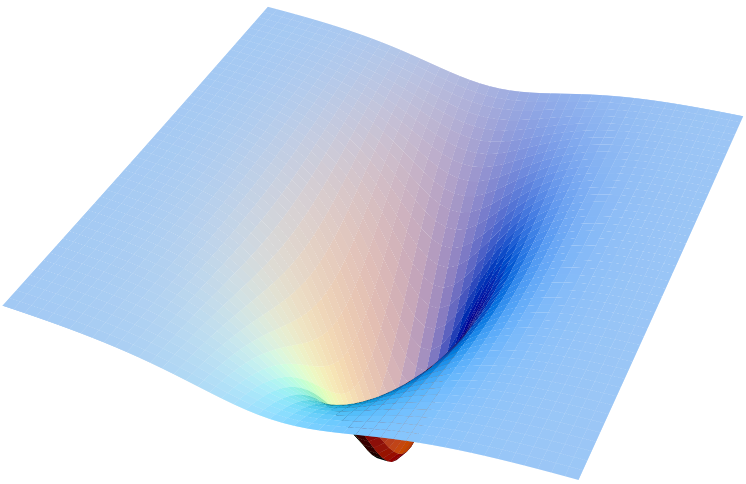

Figure 1. The graph of Ku in Example 3.4.

We consider another example, but where u is harmonic.

Example 3.5. Suppose Ω is a simply-connected open subset of $\mathbb{R}^2$, and $u(x,y) = x^2 - y^2$. Since u is harmonic on a simply-connected open set Ω, there is a harmonic conjugate function v on Ω s.t. $g = u + \mathrm{i}v$ is holomorphic on Ω [1, Theorem 2.2(j),

p. 202]. The harmonic conjugate v is given by $v(x,y) = 2xy + C$ (we set $C = 0$ for simplicity), so now $g(z) = z^2 = (x^2 - y^2) + \mathrm{i}(2xy)$. Then

$$K\cdot = \frac{-4}{(1+4x^2+4y^2)^2}, \tag{16}$$

or by Lemma 3.3 this can be written as

$$Kg = \frac{-4}{(1+4|z|^2)^2} = -\left[\frac{\mathrm{d}}{\mathrm{d}s}\arctan(2s)\right]^2 \tag{17}$$

where $s = |z| = \sqrt{x^2+y^2}$. Hence, the graph of Kg is radially symmetric. See Figure 2.

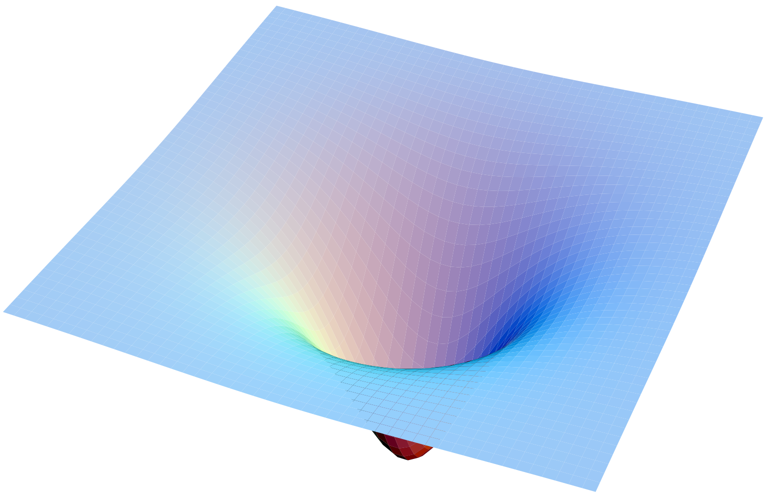

Figure 2. The graph of Kg in Example 3.5.

In light of Corollary 2.3, we know that if $f_1(z) = \sigma z^2 + C_2$, where C_2 is a complex constant, and $\sigma \in \mathbb{C}$ where $|\sigma| = 1$, then $Kf_1 = Kg$. By this, we mean the graphs of the real and imaginary parts of f_1 have the same Gaussian curvature as the graph of the given u. Also, one can show that the function

$$f_2(z) = -\frac{z}{C_1} + \frac{\sigma(C_1^2+1)}{2C_1^2}\ln(2C_1 z + \sigma) + C_2, \tag{18}$$

(here C_1 is a real constant) also satisfies equation (17); that is $Kf_2 = Kg$.

As f_1 is a polynomial, it's entire. Thus $u_1 = \Re(f_1)$ and $v_1 = \Im(f_1)$ are harmonic conjugates whose Gaussian curvature is given by equation (16). For f_2, we use a the principal branch of the logarithm: $\mathbb{C} \sim \{z \in \mathbb{C} \colon \Re(2C_1 z + \sigma) \leqslant 0\}$. Hence f_2 is holomorphic on $\mathbb{C}$ less the branch cut. So there is another set of harmonic conjugates, u_2 and v_2, whose graphs have the curvature given by equation (16). Moreover, these are not the same graphs. Say $\sigma = 1$, $C_1 = 1$, and $C_2 = 0$. Then

$$\left.\begin{aligned} u_2 &= -x + \frac{1}{2}\ln((2x+1)^2 + 4y^2) \\ v_2 &= -y + \arctan\Big(\frac{2y}{2x+1}\Big) \end{aligned}\right\} \forall (x,y) \in \mathbb{R}^2 \sim \Big\{(x,0) \colon\ x \leqslant -\frac{1}{2}\Big\}.$$

This example shows that there are two families of holomorphic functions whose real and imaginary parts share the same Gaussian curvature on a subset of $\mathbb{R}^2$. Of course their real and imaginary parts are harmonic conjugates, but they are not all conjugate to each other (e.g. u_1 is not conjugate to u_2 or v_2). So in these first two examples, we have seen that when given two graphs determined by functions, say u and v, then it may be the case that either u and v are nonharmonic, or u and v are the real or imaginary part of some holomorphic function; hence they are harmonic functions, though not necessarily conjugate. In contrast, recall the nonharmonic functions in Example 3.4 did *not* have radially symmetric Gaussian curvature, whereas the graphs of the harmonic functions in Example 3.5 did have radially symmetric Gaussian curvature.

We will present a general method to obtain the families f_1 and f_2 as seen in Example 3.5. Before we proceed with this, we note that not all harmonic functions have radially symmetric Gaussian curvature by a counterexample.

Example 3.6. Let $f(z) = \mathrm{e}^z = \mathrm{e}^x(\cos y + \mathrm{i}\sin y)$. Then

$$Kf = -\frac{1}{4}\operatorname{sech}^2(x), \tag{20}$$

and hence Kf is not radially symmetric even though f is entire, and every term in its power series has radially symmetric Gaussian curvature.

Theorem 3.7. *Let Ω be as in Lemma 3.1. Suppose u is a C^2 function on Ω with*

$$Ku = -\Big[\frac{\mathrm{d}}{\mathrm{d}s}\arctan(rns^{n-1})\Big]^2, \tag{21}$$

where $n \in \mathbb{Q}$, $r \in (0,\infty)$ and $s = |z|$. Then for

$$f_1(z) = \sigma c z^n + C_2 \tag{22}$$

defined on Ω where $\sigma, c, C_2 \in \mathbb{C}$ with $|\sigma| = 1$ and $|c| = r$, and

$$f_2(z) = \int \frac{C_1 + \sigma c n z^{n-1}}{1 - \sigma C_1 c n z^{n-1}}\,\mathrm{d}z \tag{23}$$

defined on $D = \{z \in \mathbb{C}\colon\ |z| < (|C_1|r|n|)^{1/(1-n)}\}$*, the functions* $u_j = \Re(f_j)$ *and* $v_j = \Im(f_j)$*,* $j = 1, 2$*, are four families of harmonic functions s.t.* $Ku_j = Kv_j = Ku$ *on* $\Omega \cap D$*.*

Proof. Choose $c \in \mathbb{C}$ s.t. $|c| = r$. Then there is a holomorphic function of the form

$$g(z) = cz^n \tag{24}$$

where $Kg = Ku$ on Ω. Now it suffices to show that there are two holomorphic functions, f_1 and f_2, on a simply-connected subset of Ω where $Kf_1 = Kf_2 = Kg$. If $Kh = Ku$, $|h'| = m(s)$, and $|h''| = m''(s)$ where $s = |z|$, then the equation,

$$-\Big[\frac{\mathrm{d}}{\mathrm{d}s}\arctan(m(s))\Big]^2 = -\Big[\frac{\mathrm{d}}{\mathrm{d}s}\arctan(rns^{n-1})\Big]^2 \tag{25}$$

holds iff

$$\begin{cases} m_1(s) = \pm rns^{n-1} & \text{for } C \equiv 0 \bmod \pi, \\ m_2(s) = \dfrac{C_1 \pm rns^{n-1}}{1 \mp C_1 rns^{n-1}} & \text{for } C_1 = \tan C \ \& \\ & C \neq \{\arctan(\pm 1/rns^{n-1}), \tfrac{1}{2}\pi + k\pi\colon\ k \in \mathbb{Z}\}, \\ \text{undefined} & \text{otherwise.} \end{cases} \tag{26}$$

The distinction between m_1 and m_2 by $C \equiv 0$ and $C \not\equiv 0$ is important. By this, we obtain two distinct families.

Let $\sigma \in \mathbb{C}$ where $|\sigma| = 1$, and choose $f_1'(z) = \sigma cnz^{n-1}$. Hence $f_1(z) = \sigma g(z) + C_2$ and $Kf_1 = Kg$ by Corollary 2.3. For f_2 choose

$$f_2'(z) = \frac{C_1 + \sigma cnz^{n-1}}{1 - \sigma C_1 cnz^{n-1}} \tag{27}$$

for appropriate C_1. Now $(m_2(s))^2 \neq |f_2'|^2$, but we can verify that $Kf_2 = Kg$ independently of m_2. We explicitly show $Kf_2 = Kg$ in the Appendix. This is just a tedious calculation.

As g is holomorphic on Ω, then f_1 is. For f_2, we need the antiderivative of the right hand side of equation (27). As n is arbitrary, this is nontrivial. It is given by the Lerch-Phi function [4, Search "lerchphi"]). When $n \neq 1$ the antiderivative is given by

$$f_2(z) = \frac{z}{C_1(n-1)}\Big[1 - n + (1 + C_1^2)\,\mathrm{LerchPhi}\Big(\sigma C_1 cnz^{n-1}, 1, \frac{1}{n-1}\Big)\Big] + C_2 \tag{28}$$

where

$$\text{LerchPhi}\Big(\sigma C_1 cnz^{n-1}, 1, \frac{1}{n-1}\Big) = \sum_{k=0}^{\infty} \frac{(n-1)\left(\sigma C_1 cnz^{n-1}\right)^k}{k(n-1)+1} \tag{29}$$

is defined on the disk $D := \{z \in \mathbb{C}\colon\ |z| < (|C_1|r|n|)^{1/(1-n)}\}$.

As both Ω and D are simply-connected and open, so is $\Omega \cap D$. Also set $u_j = \Re(f_j)$ and $v_j = \Im(f_j)$ for $j = 1, 2$. Then u_j and v_j are clearly harmonic on $\Omega \cap D$. □

Remark 3.8. For $n = 2$ and appropriate constants, we obtain f_2 as in Example 3.5. The principal branch cut, $x \leqslant -1/2$, lies outside the disk of radius $1/2$. Hence, in this case, the domain of f_2 can be extended beyond D to $\mathbb{C}$ less some branch cut.

Additionally, recall that the graphs of harmonic conjugate functions are not isometric (Remark 2.4). Also, any two functions obtained from the same family do not have isometric graphs in general. Let $f = z^2$ as in Example 3.5. For $\sigma = 1$ and $\tilde{\sigma} = 1/\sqrt{2} + \mathrm{i}/\sqrt{2}$, then $u_1 = x^2 - y^2$ and $\tilde{u}_1 = 1/\sqrt{2}\,(x^2 - 2xy - y^2)$ are of the type given by $\Re(f_1)$. Thus, Theorem 3.7 gives us (up to) four continua of (mostly) non-isometric graphs with the same Gaussian curvature. Observe that this is generated by the unit circle, $\mathbb{S}^1$, acting on the set of holomorphic functions.

4. Conclusion

We have seen that if a function is nonharmonic, then the Gaussian curvature of its graph is not necessarily radially symmetric. Also, there are harmonic functions that do not have radially symmetric Gaussian curvature. Theorem 3.7 has given us the existence of (up to) four families of harmonic functions whose graphs have a given radially symmetric Gaussian curvature. We have not shown uniqueness in the sense that these are the only functions whose graphs have this Gaussian curvature. Also, we have not shown that there is a nonharmonic function with radially symmetric Gaussian curvature of the form described by equation (21). With uniqueness, we'd have a definite "converse" to Lemma 2.1. We end with that conjecture.

Conjecture 4.1. *Suppose u is C^2 function on some region Ω in $\mathbb{R}^2$ with*

$$Ku = -\Big[\frac{\mathrm{d}}{\mathrm{d}s}\arctan(rns^{n-1})\Big]^2 < 0, \tag{30}$$

where $n \in \mathbb{Q}$, $r \in (0, \infty)$ and $s = |z|$. Then is u harmonic?

Appendix

In the proof of Theorem 3.7 we needed to explicitly show that $Kf_2 = Kg$. We see that

$$f_2{}' = \frac{C_1 + \sigma c n z^{n-1}}{1 - \sigma C_1 c n z^{n-1}}, \tag{31}$$

$$|f_2{}'|^2 = \frac{C_1^2 + \overline{\sigma} C_1 \overline{c} n \overline{z}^{n-1} + \sigma C_1 c n z^{n-1} + r^2 n^2 |z|^{2(n-1)}}{1 - \overline{\sigma} C_1 \overline{c} n \overline{z}^{n-1} - \sigma C_1 c n z^{n-1} + C_1^2 r^2 n^2 |z|^{2(n-1)}}, \tag{32}$$

and

$$f_2{}'' = \frac{\sigma c n (1 + C_1^2) z^{n-2}}{(1 - \sigma C_1 c n z^{n-1})^2}, \tag{33}$$

$$|f_2{}''|^2 = \frac{r^2 n^2 (n-1)^2 (1 + C_1^2)^2 |z|^{2(n-2)}}{\left(1 - \overline{\sigma} C_1 \overline{c} n \overline{z}^{n-1} - \sigma C_1 c n z^{n-1} + C_1^2 r^2 n^2 |z|^{2(n-1)}\right)^2}. \tag{34}$$

Then

$$\begin{aligned} Kf_2 &= \frac{-|f_2{}''|^2}{\left(1 + |f_2{}'|^2\right)^2} \\ &= -\frac{\dfrac{r^2 n^2 (n-1)^2 (1 + C_1^2)^2 |z|^{2(n-2)}}{(1 - \overline{\sigma} C_1 \overline{c} n \overline{z}^{n-1} - \sigma C_1 c n z^{n-1} + C_1^2 r^2 n^2 |z|^{2(n-1)})^2}}{\left(1 + \dfrac{C_1^2 + \overline{\sigma} C_1 \overline{c} n \overline{z}^{n-1} + \sigma C_1 c n z^{n-1} + r^2 n^2 |z|^{2(n-1)}}{1 - \overline{\sigma} C_1 \overline{c} n \overline{z}^{n-1} - \sigma C_1 c n z^{n-1} + C_1^2 r^2 n^2 |z|^{2(n-1)}}\right)^2} \\ &= -\frac{r^2 n^2 (n-1)^2 |z|^{2(n-2)}}{(1 + r^2 n^2 |z|^{2(n-1)})^2} = Kg. \end{aligned} \tag{35}$$

This completes the proof.

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