# IMPRECISE PROBABILISTIC BELIEFS IN THE PRESENCE OF AMBIGUITY

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## INTRODUCTION

- In economic decision theory, widespread recognition that assumption of unique ("precise") subjective probability measure is very demanding
  - "Ambiguity"
  - Ellsberg paradox
    - deeper than other departures from SEU since it challenges very notion of probabilistic belief



**Example (2-color Ellsberg paradox).**

- 1 ball drawn from each of two urns
  - both contain only red and black balls
  - composition of one urn known to be 50:50, of the other unknown
  - associated events (draws) $R_{kn}$ and $B_{kn}$ – $R_{un}$ and $B_{un}$
- $X = \{0, 1\}$ with $1 \succ 0$, and

$$[1 \text{ on } R_{kn}, 0 \text{ on } B_{kn}] \sim [1 \text{ on } B_{kn}, 0 \text{ on } R_{kn}] \succ$$
$$[1 \text{ on } R_{un}, 0 \text{ on } B_{un}] \sim [1 \text{ on } B_{un}, 0 \text{ on } R_{un}].$$

- Behavior cannot be rationalized by unique subjective probability $\mu$
  (cannot be "probabilistically sophisticated")
  (a) By first $\sim$, $\mu(R_{kn}) = \frac{1}{2}$;
  (b) By second $\sim$, $\mu(R_{un}) = \frac{1}{2}$;
  (c) By $\succ$, $\mu(R_{kn}) > \mu(R_{un})$.



- Psychologically plausible explanation in terms of aversion to "ambiguity"
  - "ambiguity" as "probabilistic ignorance" of sorts
    - Keynes, Levi, Seidenfeld, Dempster, Shafer, Berger, Walley et al.
- Hence Ellsberg choices *rationally motivated;* not just "mere behavior"
- Are Ellsberg choices rational?
  - consistent with strong notions of rationality ...
  - but arguably not fully rational
    (Nehring 1991, 1992, 2000, 2009)
    - yet full rationality may be super-human
- Here, want to bracket issue of full rationality; sufficiently rational to be taken seriously
  - deliberate choices of many sophisticated DMs

What do Ellsbergian ("*ambiguity-sensitive*") DMs believe?

- Basic observation: probabilistic beliefs not canonically/directly revealed by behavior
  (a) Betting preference does not reveal corresponding likelihood comparison $\trianglerighteq_{rev}$ in meaningful sense
    * would imply that $R_{kn} \vartriangleright_{rev} R_{un}$ and $R^c_{kn} \vartriangleright_{rev} R^c_{un}$
    * reason: betting preferences determined by probabilistic beliefs plus ambiguity attitudes

$\rightsquigarrow$ gap between beliefs – behavior; beliefs need to be introduced as independent primitives.



PART 1: BELIEFS AS INDEPENDENT PRIMITIVE

- First-person (DM's) point of view:
  "I prefer to bet on event $A$ over $B$ *because* I believe that $A$ is more likely than $B$"

- Third-person (analyst's) point of view:
  "I expect the DM to bet on event $A$ over $B$ *because* it is reasonable for her to believe that $A$ is more likely than $B$"
  – this, rather than *direct* attribution of preferences, dominant plausibility consideration for economic modelling

- Yes, there are deep philosophical issues
  – subjective probability as *irreducible;*
  not reducible to behavior, information, frequency



*Beliefs*

- subjective (judgment) $\rightsquigarrow$
  qualitative primitive;
  any numeric/algebraic structure must be result of representation theorem

- *Judgements of comparative likelihood* $\trianglerighteq$ .

  $A \trianglerighteq B$ : "$A$ is believed/judged to be more likely than $B$"

  – Keynes (1921), De Finetti (1931), Koopman (1940), Savage (1954)



IMPRECISE QUALTITATIVE PROBABILITY

Comparative likelihood relation $\trianglerighteq$: partial order on $\Sigma$

Goal: find conditions that ensure existence and uniqueness of multiprior representation:

$$A \trianglerighteq B \text{ if and only if } \pi(A) \geq \pi(B) \text{ for all } \pi \in \Pi,$$

where $\Pi$ is (weak*-)closed, convex set of finitely additive probability measures $\pi$.

- such $\trianglerighteq$ **coherent**
  – real-valued part of representation mathematical 'heuristic'
- **Existence:**
  ensures that $\trianglerighteq$ incorporates "logic of subjective probability"
- **Uniqueness:**
  ensures that $\trianglerighteq$ fully captures imprecise probabilistic beliefs

AXIOMS

**Partial Order** $\trianglerighteq$ is transitive and reflexive.

**Nondegeneracy** $\Omega \vartriangleright \emptyset$.

**Nonnegativity** $A \trianglerighteq \emptyset$ for all $A \in \Sigma$.

**Additivity** $A \cap C = B \cap C = \emptyset \implies$
$A \trianglerighteq B$ if and only if $A + C \trianglerighteq B + C$.

**Splitting**
If $A + A' \trianglerighteq B + B',\ A \trianglerighteq A'$ and $B \trianglerighteq B'$ then $A \trianglerighteq B'$.

**Equidivisibility**
For any $A \in \Sigma$, there exists $B \subseteq A$ such that $B \equiv A \backslash B$.

- $A$ may be ambiguous;
  $B$ has unambiguous probability $\frac{1}{2}$ given $A$

$A$ is a $\frac{1}{K}-$*event* if there exist at $K - 1$ mutually disjoint events $A_i$, disjoint from $A$, such that $A \trianglelefteq A_i$ for all $i$.

**Continuity** If not $A \trianglerighteq B$, then there exists $K < \infty$ such that, for any $\frac{1}{K}-$events $C, D$, it is not the case that $A \cup C \trianglerighteq B \backslash D$.

- entailed by multi-prior representation.



- $\Pi$ is **convex-ranged** if, for any event $A$ and any $\alpha \in (0,1)$, there exists an event $B \subseteq A$ such that $\pi(B) = \alpha\pi(A)$ for all $\pi \in \Pi$.
  – convex-rangedness of $\Pi$ much stronger than convex-rangedness of every $\pi \in \Pi$.

**THEOREM:** A relation $\trianglerighteq$ has a multi-prior representation with a convex-ranged set of priors $\Pi$ if and only if it satisfies the seven axioms Partial Order, ..., Continuity.

The representing $\Pi$ is unique.



**<u>Foundational Value of Axioms</u>.**

**<u>Additivity</u>** $A \cap C = B \cap C = \emptyset \implies$
$A \trianglerighteq B$ if and only if $A + C \trianglerighteq B + C$.

- justification: the two likelihood comparisons $A\ vs.B$ and $A + C\ vs.B + C$ *equivalent*, since have same 'differential realizations' ($A \backslash B$ vs. $B \backslash A$).
  – note that equivalence is not just ordinal, but *entirely qualitative*, does not appeal to any notion of combining probabilities quantitatively
  – this is *as primitive as it gets;*
  if any axiom is found, not made, this is it!
  – compare to: **Strong Additivity** $A \cap C = B \cap D = \emptyset \implies A \trianglerighteq B$ and $C \trianglerighteq D$ implies $A + C \trianglerighteq B + D$
    ∗ Strong Additivity ordinal but 'quantitative'
      · why should likelihood be like that?
    ∗ Strong Additivity implied by Additivity (3x) and Transitivity (2x); this lemma at birth of quantitative, later cardinal probability



**<u>Splitting</u>**
If $A + A' \trianglerighteq B + B',\ A \trianglerighteq A'$ and $B \trianglerighteq B'$ then $A \trianglerighteq B'$.

- New axiom; needed under incompleteness, not under completeness.

- Worries:
  (a) Splitting already appeals to quantitative intuitions.
  (b) If genuinely distinct from Additivity, (coherent) likelihood composite; how then "irreducible", "canonical" character

- But Splitting can be deduced from Additivity (via Strong Additivity) by "*necessitation argument*":

**Lemma:** Additivity and Transitivity imply Pre-Splitting.

**Pre-Splitting**
If $A + A' \trianglerighteq B + B',\ A \trianglerighteq A'$ and $B \trianglerighteq B'$
<u>**and** $A \trianglerighteq B'$ **or** $B' \trianglerighteq A$</u>,
then $A \trianglerighteq B'$.

- Thus, given the premises of Splitting, however You, the DM, compare the likelihood of $A\ vs.\ B'$, You must judge $A \trianglerighteq B'$. Hence, there is no room for withholding this judgment, and You should thus assert it outright.

**Equidivisibility**

For any $A \in \Sigma$, there exists $B \subseteq A$ such that $B \equiv A \backslash B$.

- Richness assumption to bring out full implications of logical axioms
- Not empirically restrictive, since can obtain from postulating independent continuous random device: $\Omega = \Omega_0 \times [0,1]$, $\Sigma = \Sigma_0 \times \Sigma_{[0,1]}$, with $\trianglerighteq_{RAND}$ capturing random device on $\Sigma$
  - arguably, any 'truly coherent' likelihood relation on $\Sigma_1$ must be coherently mergeable with $\trianglerighteq_{RAND}$
    - the merged relation satisfies Equidivisibility by construction.



**Continuity**

- not logical, but mathematico-pragmatic to get multi-prior representation
  - Open question: can one drop continuity and get meaningful generalized representation, e.g. in terms of sets of non-standard probability measures?



**Uniqueness**

- $\trianglerighteq_\Pi = \trianglerighteq_{\Pi'}$ if and only if $\overline{co}\Pi = \overline{co}\Pi'$.
  - $\rightsquigarrow$uniqueness *among closed convex sets* of priors
    - does nothing to justify convexity,
      - · – could get unique representation alternatively in terms of sets of extreme points.

- Important here: comparative likelihood orderings **expressively as rich** as closed convex sets of priors
  - for this Equidivisibility nearly indispensable
    - in particular, state space must be infinite



**Proof Idea: Event Space as Mixture Space**

Using convex-rangedness, extend $\trianglerighteq$ to partial order $\widehat{\trianglerighteq}$ on *mixture-space* $B(\Sigma, [0,1])$ of finite-valued functions $Z : \Omega \to [0,1]$ as follows:

(1) For each $Z = \sum z_i 1_{E_i}$, define $[Z]$ as the family of all events $A \in \Sigma$ such that,

for all $i \in I$ and $\pi \in \Pi, \pi(A \cap E_i) = z_i \pi(E_i)$.

- For any $Z$, $[Z] \neq \emptyset$ by convex-rangedness.

(2) Define $\widehat{\trianglerighteq}$ by by setting

$$Y \widehat{\trianglerighteq} Z \text{ if } A \trianglerighteq B \text{ for some } A \in [Y] \text{ and } B \in [Z].$$

- Well-defined since for any two $A, B \in [Z]$ : $\pi(A) = \pi(B)$ for all $\pi \in \Pi$, and thus $A \equiv B$.

(3) $\widehat{\trianglerighteq}$ is monotone, continuous and satisfies

**(Additivity)** $Y \widehat{\trianglerighteq} Z$ if and only if $Y + X \widehat{\trianglerighteq} Z + X$ for any $X, Y, Z$ , (1)

and

**(Homogeneity)** $Y \widehat{\trianglerighteq} Z$ if and only if $\alpha Y \widehat{\trianglerighteq} \alpha Z$ for any $Y, Z$,$\alpha > 0$.

(4) By Walley (1991) or Bewley (1986, for finite state-spaces), there exists unique $\Pi \in \mathcal{K}(\Delta(\Omega))$ such that, for all $X, Y \in B(\Sigma, [0,1])$,

$X \widehat{\trianglerighteq} Y$ if and only if $E_\pi X \geq E_\pi Y$ for all $\pi \in \Pi$.

(5) Evidently, $\Pi$ is multiprior representation of $\trianglerighteq$ .

(6) Uniqueness of $\Pi$ by (2).

- Difficulties of proof:
  (a) mixture-space construction *without* availability of $\Pi$
  (b) $\widehat{\trianglerighteq}$ in proof only defined on dense subset of $B(\Sigma, [0,1])$; essential difficulty if $\Sigma$ is merely algebra.



PART 2: RATIONALITY RESTRICTIONS ON PREFERENCES

- Which restrictions on preferences/choice are rationally entailed by probabilistic beliefs *as such*?
  ⇝behavioral generality:
  in particular, do not want to impose here EU ('Bernoullian') rationality towards probabilistic beliefs

- Bernoullian rationality norms may be valid, but do not follow from having of probabilistic beliefs as such
  – have studied in companion paper (Nehring 2007: "Bernoulli without Bayes: Utility Sophisticated Preference under Ambiguity")
  – Behavioral generality ensures robust applicability.

*Preferences.*

- $X = \{x, y, ..\}$ set of **consequences**
- (Savage) **act** $f$ maps states to consequences, $f : \Omega \to X$
  – $\mathcal{F}$ = class of simple (finite-valued, $\Sigma$-measurable) acts
- A **preference relation** is a weak order $\succsim$ over $\mathcal{F}$.



**AXIOM (LIKELIHOOD COMPATIBILITY)**

For all $f \in \mathcal{F}$, $x, y \in X$ and events $A, B \in \Sigma$ :

i) $A \trianglerighteq B$ and $x \succsim y$ imply
$[x$ on $A \backslash B; y$ on $B \backslash A$; $f(\omega)$ elsewhere$] \succsim$
$[x$ on $B \backslash A; y$ on $A \backslash B$; $f(\omega)$ elsewhere$]$, and

ii) $A \rhd\rhd B$ and $x \succ y$ imply
$[x$ on $A \backslash B; y$ on $B \backslash A$; $f(\omega)$ elsewhere$] \succ$
$[x$ on $B \backslash A; y$ on $A \backslash B$; $f(\omega)$ elsewhere$]$.

- $A \rhd\rhd B$ :"$A$ is uniformly more likely than $B$"
  – in representation: $A \rhd\rhd B$ implies $\min_{\pi \in \Pi} [\pi(A) - \pi(B)] > 0$

- Idea: if two acts differ only in the states in which two particular consequences are realized, then the likelihood comparison of the corresponding events (*if available*) is a *decisive* criterion for their preference comparison.

- Simple instances of LC:
  $A \trianglerighteq B$ and $x \succsim y$ imply
  $[x$ on $A$, $y$ on $A^c] \succsim [x$ on $B$, $y$ on $B^c]$, and
  $[x$ on $B^c$, $y$ on $B] \succsim [x$ on $A^c$, $y$ on $A]$
  – betting "on $A$" better than betting on $B$;
  betting against $B$ better than betting against $A$



- Acceptance of LC: **Pragmatic Rationalism**
  – Compelling? Banale?

- Sources of skepticism?
  (a) comparative likelihood judgments meaningless, inscrutable
  (b) comparative likelihood judgments meaningful, but not decisive
    ∗ other conceivable factors such as familiarity, felt competence may play legitimate role, too
    ("source preference" position)
      · strong Humean flavor

"**Source Preference**"

- prefer to bet on $B$ rather than $A$ while $A \trianglerighteq B$.
  – "$\{B, B^c\}$ "more attractive" source of uncertainty than $\{A, A^c\}$
    ∗ frequent psychological explanation (Heath-Tversky and others),
    recently very popular with economists
      · non-credal factors of felt competence, familiarity, comfort of knowing, etc.
    ∗ e.g. hometown weather more attractive than roulette wheel more attractive than foreign town

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  - both contain only red and black balls
  - composition of one urn known to be 50:50, of the other unknown
  - associated events (draws) $R_{kn}$ and $B_{kn}$ – $R_{un}$ and $B_{un}$
- $X = \{0,1\}$ with $1 \succ 0$, and

$$[1 \text{ on } R_{kn}, 0 \text{ on } B_{kn}] \sim [1 \text{ on } B_{kn}, 0 \text{ on } R_{kn}] \succ$$
$$[1 \text{ on } R_{un}, 0 \text{ on } B_{un}] \sim [1 \text{ on } B_{un}, 0 \text{ on } R_{un}].$$

**Basic Observation.** Suppose that $\trianglerighteq$ is a coherent likelihood relation such that that

$$R_{kn} \equiv B_{kn} \text{ and } R_{un} \equiv B_{un}.$$

Then $\succsim$ is not compatible with $\trianglerighteq$ .

**Proof.** By coherence (Splitting axiom), $R_{kn} \equiv B_{kn}$ and $R_{un} \equiv B_{un}$ implies

$$R_{kn} \equiv R_{un}.$$

But then by Likelihood Compatibility,

$$[1 \text{ on } R_{kn}, 0 \text{ on } B_{kn}] \sim [1 \text{ on } R_{un}, 0 \text{ on } B_{kn}].$$

$$R_{kn} \sim R_{un}.$$



**Trilemma:** Joint inconsistency

(1) completeness of beliefs

(2) coherence

(3) epistemic rationalizability (likelihood compatibility)

- On proposed *pragmatic rationalism*, coherence (2) and rationalizability (3) categorical, give up completeness (without much regret)
- Source preferentists want to maintain completeness and give up rationalizability
  - hence $R_{kn} \succ R_{un}$ while $R_{kn} \equiv R_{un}$.



- Is this overly indulgent? Irrational?
  - Also, why remain attached to completeness?
- More general case for source preference: source dependent risk-attitudes
  - this seems very natural on view of risk attitude as matter of psychological disposition distinct from decreasing marginal 'real' utility
- source-dependent risk-attitudes precluded by LC



PROBABILISTIC SOPHISTICATION

- Family of **unambiguous events** $\Lambda = \{A \in \Sigma : \pi(A) = \pi'(A)$ for all $\pi, \pi' \in \Pi\}$.
  - with associated unambiguous probability $\overline{\pi}$, where $\overline{\pi}(A) = \pi(A)$ for any $\pi \in \Pi$, $A \in \Lambda$.
- $f$ is unambiguous if it is $f$-measurable.
- $\succsim$ is **probabibilistically sophisticated over unambiguous acts** if, for all $f, g \in \mathcal{F}$, $f \succsim g$ whenever $\pi(\{\omega : f(\omega) \succsim x\}) \geq \pi(\{\omega : g(\omega) \succsim x\})$ for all $x \in X$.
  - unambiguous acts are evaluated according to induced probability distribution over consequences $\rightsquigarrow$ i.e. there exist uniform, source-independent risk-preferences.

PROPOSITION. If the weak order $\succsim$ is compatible with the convex-ranged, coherent likelihood relation $\trianglerighteq$, $\succsim$ is probabilistically sophisticated over unambiguous acts.

COROLLARY. If $\succsim$ is in addition complete, it is probabilistically sophisticated a la Machina-Schmeidler.

ARGUING FOR PRAGMATIC RATIONALISM

- Are there non-question begging arguments supporting LC?
  - Humean concedes relevance of likelihood comparisons, simply denies decisiveness.
  - You, the rationalist, may be happy to take these to be decisive, but why should everyone do so?

  $\rightsquigarrow$ direct defense seems difficult
- Further defense *ex negativo*:
  consequences of giving up LC drastic
- In particular, **is moderate LC skepticism possible?**
  - Why not doubt LC for objective probabilities?
    - ∗ why even accept Reduction of Compound Lotteries?
    - ∗ Such skepticism has been articulated by CS Peirce and H Putnam
      - · “why knowledge of probabilities decisive for single events?”
      - · HP: “this is were my spade is turned”
        – Wittgensteinian humanism as “august Humeanism”
- Bottom line: LC seems necessary to maintain minimal normative connection between between beliefs/likelihood judgments and choices
  - importantly, under LC, this connection is not holistic but ‘modular’



- **Modularity.** Given LC, each likelihood judgment entails committment to family of choice judgments
  - Given weak order $\succsim_{const}$ on $X$, 1-1 relation between $\trianglerighteq$ and induced $\succsim_{\trianglerighteq}$
- $\succsim_{\trianglerighteq}$ *more than* ‘mere’ preferences:
  grounding in lkh judgment
- $\succsim_{\trianglerighteq}$ *less than* preferences:
  actual choice disposition $\succsim$ may contradict $\succsim_{\trianglerighteq}$
  - e.g. weakness of will, motivated irrationality
    sheepishness, wishful thinking, self-deception ...;
- Big Philosophical Question:
  can lkh judgment be *identified with choice committments* $\succsim_{\trianglerighteq}$?
  - this would allow *reduction* of subjective probability to behavior,
    w/o identifying it with behavior.



- Modularity privileges comparative likelihood orderings $\trianglerighteq$ vis-a-vis other candidate representations of subjective uncertainty
  - E.g. Lower Probability Orderings $A \geq B$ iff $\min_{\pi\in\Pi} \pi(A) \geq \min_{\pi\in\Pi} \pi(B)$
    - ∗ with sufficient structure (convex-rangedness of $\Pi$),
      holistic 1-1 relation between lpo.s and clo.s;
      but is there modular counterpart to LC ??
  - A fortiori, with less structure, there does not seem to exist modular epistemic rationalizability axiom if credal state described as $\Pi$ or as general Complete Non-Addiditive Probability Ordering $A \geq B$ iff $\nu(A) \geq \nu(B)$.