129 (2004) MATHEMATICA BOHEMICA No. 3, 255–271

NOTES ON MONADIC n-VALUED ŁUKASIEWICZ ALGEBRAS

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(Received August 27, 2003)

Abstract. A topological duality for monadic n-valued Łukasiewicz algebras introduced by M. Abad (Abad, M.: Estructuras cíclica y monádica de un álgebra de Łukasiewicz n-valente. Notas de Lógica Matemática 36. Instituto de Matemática. Universidad Nacional del Sur, 1988) is determined. When restricted to the category of Q-distributive lattices and Q-homomorphims, it coincides with the duality obtained by R. Cignoli in 1991. A new characterization of congruences by means of certain closed and involutive subsets of the associated space is also obtained. This allowed us to describe subdirectly irreducible algebras in this variety, arriving by a different method at the results established by Abad.

Keywords: n-valued Łukasiewicz algebras, Priestley spaces, congruences, subdirectly irreducible algebras

MSC 2000: 06D30, 03G20

Introduction

In 1941, G. Moisil ([17]) introduced n-valued Łukasiewicz algebras. From that moment on, many articles have been published about this class of algebras. Many of the results obtained have been reproduced in the important book by C. Boicescu, A. Filipoiu, G. Georgescu and S. Rudeanu ([4]) which can be consulted by any reader interested in broadening his knowledge on these algebras.

In 1988, M. Abad ([1]) began the research in monadic n-valued Łukasiewicz algebras. Among other results, using certain families of deductive systems of a monadic n-valued Łukasiewicz algebra, this author provided a method for determining congruences on these algebras and subdirectly irreducible algebras in this variety.

This paper centres around a duality theory for monadic n-valued Łukasiewicz algebras. In order to do this, we combine the duality for n-valued Łukasiewicz algebras, to be first described, and Cignoli's duality for Q-distributive lattices ([8]). The duality for monadic n-valued Łukasiewicz algebras is later used to determine congruences on these algebras and characterize subdirectly irreducible algebras in this variety, which are obviously obtained by a different method from that indicated in [1].

The main definitions and results needed in this paper are summarized in Section 1. In Section 2, we describe a topological duality for n-valued Łukasiewicz algebras, extending the one obtained by W. Cornish and P. Fowler for de Morgan algebras. The main part of this section is the development of a duality theory for monadic n-valued Łukasiewicz algebras. In Section 3, which is the core of the paper, the results of Section 2 are applied. We prove that the dual of an n-valued Łukasiewicz algebra is a Priestley space that is a disjoint union of its maximum chains having at most $n-1$ elements, and any maximun chain C is represented by an increasing surjective $(n-1)$-elements sequence $C = \{c_1 \leqslant c_2 \leqslant \ldots \leqslant c_{n-1}\}$ which does not have to be injective, and that the morphisms are continuous order preserving mappings sending the i-th element of the sequence representing a maximum chain onto the i-th element of the target sequence for all i, $1 \leqslant i \leqslant n-1$. We also prove that the dual of a monadic n-valued Łukasiewicz algebra is a Priestley space representing an n-valued Łukasiewicz algebra with an equivalence E such that if $x = c_i$ for a maximum chain $C = \{c_1 \leqslant c_2 \leqslant \ldots \leqslant c_{n-1}\}$ containing x and $y = d_j$ for a maximum chain $C = \{d_1 \leqslant d_2 \leqslant \ldots \leqslant d_{n-1}\}$ containing y, then from $(x, y) \in E$ it follows that $c_i = c_j$, $d_i = d_j$ and $(c_k, d_k) \in E$ for all k, $1 \leqslant k \leqslant n-1$. Also, the relation E satisfies that if U is a closed and open increasing subset of the Priestley space, then the set $\{y\colon$ there is $x \in U, (y, x) \in E\}$ is also closed, open and increasing. The morphisms dual to the morphisms f of monadic n-valued Łukasiewicz algebras satisfy $\{y\colon$ there is $x \in f^{-1}(U)$, $(y, x) \in E\} = f^{-1}\{y\colon$ there is $x \in U$, $(y, x) \in E\}$ for every increasing closed and open set U.

Furthermore, in this section we characterize congruences on n-valued Łukasiewicz algebras by means of closed subsets of the space associated with them, which are unions of maximum chains. This enables us to obtain a new characterization of congruences on monadic n-valued Łukasiewicz algebras. This result allows us to describe subdirectly irreducible algebras in this variety which are algebras such that the maximum chains of their associated space are equivalent.

It seems worth mentioning that the dualities obtained are potentially applicable to other problems. Specifically, in a forthcoming paper, we generalize them to the case of monadic θ-valued Łukasiewicz algebras with and without negation ([4]).

1. Preliminaries

In this paper we take for granted the concepts and results on distributive lattices, category theory and universal algebra. To obtain more information on this topics, we direct the reader to the bibliography indicated in [2], [5] and [15]. However, in order to simplify reading, in this section we summarize the fundamental concepts we use.

If X is a poset (i.e. partially ordered set) and $Y \subseteq X$, then we shall denote by $(Y]$ ($[Y)$) the set of all $x \in X$ such that $x \leqslant y$ ($y \leqslant x$) for some $y \in Y$, and we shall say that Y is increasing (decreasing) if $Y = [Y)$ ($Y = (Y]$). Furthermore, we shall denote by $\max Y$ the set of maximum elements of Y.

If A is a bounded distributive lattice, we shall denote by $\mathrm{Con}\,(A)$ and A/θ the set of all congruences on A and the quotient algebra of A by θ for each $\theta \in \mathrm{Con}\,(A)$, respectively.

Recall that if A is a bounded distributive lattice and $X(A)$ denotes the set of all prime filters of A, then the following conditions are satisfied:

(F1) If $\theta \in \mathrm{Con}\,(A)$, $h\colon A \longrightarrow A/\theta$ is the natural epimorphism and $Y = \{h^{-1}(Q)\colon Q \in X(A/\theta)\}$, then for all $P \in X(A) \setminus Y$ there are $a, b \in A$ such that $a \in P$, $b \notin P$ and $(a, b) \in \theta$ ([20]).

(F2) If $j\colon A \longrightarrow A$ is a join homomorphism, $P \in X(A)$ and F is a filter of A such that $F \subseteq j^{-1}(P)$, then there is $Q \in X(A)$ which satisfies $F \subseteq Q \subseteq j^{-1}(P)$ ([9]).

Although the theory of Priestley spaces and its relation to bounded distributive lattices is well-known (see [19], [20], [21]), we shall announce some results with the aim of fixing the notation used in this paper.

If X is a Priestley space (or P-space), we shall denote by $D(X)$ the family of increasing closed and open subsets of X. Then $\langle D(X), \cap, \cup, \emptyset, X\rangle$ is a bounded distributive lattice and the mapping $\varepsilon_X\colon X \longrightarrow X(D(X))$ defined by the prescription

(A1) $$\varepsilon_X(x) = \{U \in D(X)\colon x \in U\}$$

is a homeomorphism and an order isomorphism.

If A is a bounded distributive lattice, then $X(A)$ ordered by inclusion and with the topology having as a sub-basis the sets $\sigma_A(a) = \{P \in X(A)\colon a \in P\}$ and $X(A) \setminus \sigma_A(a)$ for each $a \in A$ is the Priestley space (or P-space) associated with A. Besides, the map

(A2) $$\sigma_A\colon A \longrightarrow D(X(A))$$

is a lattice isomorphism.

On the other hand, H. A. Priestley ([19], [20, Section 6], [21]) proved that if Y is a closed subset of $X(A)$, then

$$\text{(A3)} \qquad \Theta(Y) = \{(a,b) \in A \times A\colon\ \sigma_A(a) \cap Y = \sigma_A(b) \cap Y\}$$

is a congruence on A and that the correspondence $Y \longmapsto \Theta(Y)$ establishes an anti-isomorphism from the lattice of closed sets of $X(A)$ onto the congruence lattice of A.

Recall that a de Morgan algebra is an algebra $\langle A, \vee, \wedge, \sim, 0, 1\rangle$ of type $(2,2,1,0,0)$ such that $\langle A, \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice and the unary operation $\sim$ possesses the following properties: $\sim(x \vee y) = \sim x \wedge \sim y$ and $\sim\sim x = x$ (see [3], [14], [18]).

In what follows a de Morgan algebra $\langle A, \vee, \wedge, \sim, 0, 1\rangle$ will be denoted for symplicity by $(A, \sim)$.

In 1977, W. H. Cornish and P. R. Fowler ([10]) extended Priestley duality to de Morgan algebras (see [3], [14] and [18]) considering de Morgan spaces (or mP-spaces) as pairs (X, g), where X is a P-space and g is an involutive homeomorphism of X and an anti-isomorphism. They also defined the mP-functions from an mP-space (X, g) into another one, (X', g'), as continuous and increasing functions (P-functions) f from X into X' which satisfy the additional condition $f \circ g = g' \circ f$.

In order to extend the Priestley duality to the case of de Morgan algebras, they defined the operation $\sim$ on $D(X)$ by means of the formula

$$\text{(B1)} \qquad \sim U = X \setminus g^{-1}(U) \text{ for every } U \in D(X),$$

and the homeomorphism g_A from $X(A)$ onto $X(A)$ by

$$\text{(B2)} \qquad g_A(P) = A \setminus \{\sim x\colon\ x \in P\}.$$

In addition, these authors introduced the notion of an involutive set in an mP-space (X, g) as a subset Y of X such that $Y = g(Y)$; and they characterized the congruences of a de Morgan algebra A by means of the family $\mathcal{C}_I(X(A))$ of involutive closed subsets of $X(A)$. To achieve this result, they proved that

(B3) the function Θ_I from $\mathcal{C}_I(X(A))$ onto the family $\mathrm{Con}_M(A)$ of congruences on A, defined as in (A3), is a lattice anti-isomorphism.

R. Cignoli, in [8], introduced the category $\mathcal{Q}$ of Q-distributive lattices and Q-homomorphisms, where a Q-distributive lattice is an algebra $\langle A, \vee, \wedge, \nabla, 0, 1\rangle$ of type $(2,2,1,0,0)$ such that $\langle A, \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice and ∇ is a unary operator on A which satisfies the following equalities:

$$\nabla 0 = 0,\ x \wedge \nabla x = x,\ \nabla(x \wedge \nabla y) = \nabla x \wedge \nabla y,\ \nabla(x \vee y) = \nabla x \vee \nabla y.$$

For the sake of simplicity, we shall denote the objects in $\mathcal{Q}$ by A or (A, ∇) in case we want to specify the quantifier.

In [8], Priestley duality is extended to the category $\mathcal{Q}$. To this aim, the category $\mathbf{q}\mathcal{P}$ is considered whose objects are qP-spaces and whose morphisms are qP-functions. Specifically, a qP-space is a pair (X, E) such that X is a P-space and E is an equivalence relation on X which satisfies the following conditions:

(E1) $EU \in D(X)$ for each $U \in D(X)$ where $EU = \{y \in X \colon (x, y) \in E$ for some $x \in U\}$,

(E2) the equivalence classes for E are closed in X,

and a qP-function from a qP-space (X, E) into another one (X', E') is a P-function $f \colon X \longrightarrow X'$ such that $Ef^{-1}(U) = f^{-1}(E'U)$ for all $U \in D(X')$.

Besides, it is proved:

(E3) If A is an object in $\mathcal{Q}$ and $X(A)$ is the Priestley space associated with A, then $(X(A), E_\nabla)$ is a qP-space where E_∇ is the relation defined by $E_\nabla = \{(P, R) \in X(A) \times X(A) \colon P \cap \nabla(A) = R \cap \nabla(A)\}$. Moreover, $\sigma_A \colon A \longrightarrow D(X(A))$ is an isomorphism in $\mathcal{Q}$.

(E4) If (X, E) is a qP-space, then $(D(X), \nabla_E)$ is a Q-distributive lattice where $\nabla_E U = EU$ for every $U \in D(X)$ and $\varepsilon_X \colon X \longrightarrow X(D(X))$ is an isomorphism in $\mathbf{q}\mathcal{P}$.

Moreover, the isomorphisms σ_A and ε_X define a dual equivalence between the categories $\mathcal{Q}$ and $\mathbf{q}\mathcal{P}$.

On the other hand, in [8] it is also shown how some Q-congruences on A can be obtained from the qP-space associated with A. Indeed, it is proved:

(E5) If A is an object in $\mathcal{Q}$ and $Y \subseteq X(A)$ is closed and saturated (i.e. $\nabla_{E_\nabla} Y = Y$) then the lattice congruence $\Theta(Y)$ defined as in (A3) preserves the operation ∇.

In 1969, R. Cignoli in [7] (see also [4]) defined n-valued Łukasiewicz algebras (or Lk_n-algebras) where n is an integer, $n \geqslant 2$, as algebras $(A, \sim, \varphi_1, \ldots, \varphi_{n-1}, 0, 1)$ such that $(A, \sim)$ is a de Morgan algebra and φ_i, with $1 \leqslant i \leqslant n-1$, are unary operations on A which satisfy the following conditions:

(L1) $\varphi_i(x \vee y) = \varphi_i x \vee \varphi_i y$,

(L2) $\varphi_i x \vee \sim \varphi_i x = 1$,

(L3) $\varphi_i \varphi_j x = \varphi_j x$,

(L4) $\varphi_i \sim x = \sim \varphi_{n-i} x$,

(L5) $i \leqslant j$ implies $\varphi_i x \leqslant \varphi_j x$,

(L6) $\varphi_i x = \varphi_i y$ for all i, $1 \leqslant i \leqslant n-1$, implies $x = y$.

An example of an Lk_n-algebra is the chain of n rational fractions $C_n = \{\frac{j}{n-1}, 1 \leqslant j \leqslant n-1\}$ endowed with the natural lattice structure and the unary operations $\sim$ and φ_i, defined as follows: $\sim(\frac{j}{n-1}) = 1 - \frac{j}{n-1}$ and $\varphi_i(\frac{j}{n-1}) = 0$ if $i + j < n$ or

$\varphi_i(\frac{j}{n-1}) = 1$ in the other cases. Its importance is seen in the following statement proved in [7]:

(L7) Let A be a non trivial Lk_n-algebra. Then the following conditions are equivalent:
 (i) A is subdirectly irreducible,
 (ii) A is simple,
 (iii) A is isomorphic to a subalgebra of C_n.

In 1988, M. Abad ([1]) introduced the notion of a monadic n-valued Łukasiewicz algebra (or qLk_n-algebra) as an algebra $(A, \sim, \varphi_1, \dots, \varphi_{n-1}, \nabla)$ such that the following conditions are fulfilled:

(qL1) $(A, \sim, \varphi_1, \dots, \varphi_{n-1})$ is an Lk_n-algebra,
(qL2) $(A, \nabla) \in \mathcal{Q}$,
(qL3) $\varphi_i \nabla x = \nabla \varphi_i x$ for all i, $1 \leqslant i \leqslant n-1$.

The category whose objects are qLk_n-algebras and whose morphisms are qLk_n-homomorphisms will be denoted by $\mathbf{q\mathfrak{L}k_n}$.

Let X be a non empty set and C_n^X the set of all functions from X into C_n. We shall denote by $C_{n,X}^*$ the monadic functional algebra $(C_n^X, \sim, \varphi_1, \dots, \varphi_{n-1}, \nabla)$ where the operations of the Lk_n-algebra $(C_n^X, \sim, \varphi_1, \dots, \varphi_{n-1})$ are defined componentwise as usual and the unary operation ∇ by means of the formula $(\nabla f)(x) = \bigvee f(X)$ where $\bigvee f(X)$ is the supremum of $f(X) = \{f(y) \colon y \in X\}$ ([1, page 65]).

2. A duality for qLk_n-algebras

In this section we first describe how the Cornish and Fowler duality can be extended to the case of Lk_n-algebras.

Definition 2.1. $(X, g, f_1, \dots, f_{n-1})$ is an n-valued Łukasiewicz space (or l_nP-space) if it has the following properties:

(LP1) (X, g) is an mP-space,
(LP2) $f_i \colon X \longrightarrow X$ is continuous,
(LP3) $x \leqslant y$ implies $f_i(x) = f_i(y)$,
(LP4) $i \leqslant j$ implies $f_i(x) \leqslant f_j(x)$,
(LP5) $f_i \circ f_j = f_i$,
(LP6) $f_i \circ g = f_i$,
(LP7) $g \circ f_i = f_{n-i}$,
(LP8) $X = \bigcup\limits_{i=1}^{n-1} f_i(X)$.

If $(X, g, f_1, \dots, f_{n-1})$ and $(X', g', f_1', \dots, f_{n-1}')$ are l_nP-spaces, then an l_nP-function f from X into X' is an mP-function such that $f_i' \circ f = f \circ f_i$ for all i, $1 \leqslant i \leqslant n-1$.

R e m a r k 2.1. It is routine to prove that condition (LP8) is equivalent to any of the following conditions:

(LP9) for each $x \in X$ there is an index i, $1 \leqslant i \leqslant n-1$, such that $x = f_i(x)$,

(LP10) if Y, Z are subsets of X and $f_i^{-1}(Y) = f_i^{-1}(Z)$ for all i, $1 \leqslant i \leqslant n-1$, then $Y = Z$.

Next we shall show that the category $\mathbf{l_n}\mathcal{P}$ of l_nP-spaces and l_nP-functions is dually equivalent to the category $\mathfrak{L}\mathbf{k_n}$ of Lk_n-algebras and Lk_n-homomorphisms.

Lemma 2.1. *If $(X, g, f_1, \ldots, f_{n-1})$ is an l_nP-space, then $\mathbb{L}_n(X) = (D(X), \sim, \varphi_X^1, \ldots, \varphi_X^{n-1})$ is an Lk_n-algebra where for every $U \in D(X)$, $\sim U$ is defined as in* (B1) *and $\varphi_X^i(U) = f_i^{-1}(U)$ for all i, $1 \leqslant i \leqslant n-1$.*

P r o o f. It is routine. $\square$

Lemma 2.2. *If A is an Lk_n-algebra, then $\text{Ł}(A) = (X(A), g_A, f_A^1, \ldots, f_A^{n-1})$ is an l_nP-space where g_A is as indicated in* (B2) *and $f_A^i(P) = \varphi_i^{-1}(P)$ for every $P \in X(A)$. Besides, σ_A defined as in* (A2) *is an Lk_n-isomorphism from A onto $D(X(A))$.*

P r o o f. By virtue of the results obtained in [10] and Lemma 8.1 in [6] only condition (LP2) in Definition 2.1 remains to be proved. From the definitions of σ_A and f_A^i we deduce that for each $a \in A$, ${f_A^i}^{-1}(\sigma_A(a)) = \sigma_A(\varphi_i a)$ for all i, $1 \leqslant i \leqslant n-1$. Then taking into account that the topology has as a sub-basis the sets of the form $\sigma_A(a)$ and $X(A) \setminus \sigma_A(a)$ for each $a \in A$, we infer that the functions f_A^i are continuous for all i, $1 \leqslant i \leqslant n-1$. $\square$

Lemma 2.3. *Let $(X, g, f_1, \ldots, f_{n-1})$ and $(X', g', f'_1, \ldots, f'_{n-1},)$ be l_nP-spaces and let f be an l_nP-function from X into X'. Then the application $\mathbb{L}_n(f)$ defined by the prescription $\mathbb{L}_n(f)(U) = f^{-1}(U)$ for every $U \in D(X')$ is an Lk_n-homomorphism from $\mathbb{L}_n(X')$ into $\mathbb{L}_n(X)$.*

P r o o f. We only prove that for each $U \in D(X')$, $\mathbb{L}_n(f)({\varphi_{X'}}^iU) = {\varphi_X}^i(\mathbb{L}_n(f)(U))$ holds true for all i, $1 \leqslant i \leqslant n-1$. Indeed, $\mathbb{L}_n(f)({\varphi_{X'}}^iU) = (f'_i \circ f)^{-1}(U) = (f \circ f_i)^{-1}(U) = {\varphi_X}^i(\mathbb{L}_n(f)(U))$. $\square$

Lemma 2.4. *Let $(A, \sim, \varphi_1, \ldots, \varphi_{n-1})$ and $(A', \sim', \varphi'_1, \ldots, \varphi'_{n-1})$ be Lk_n-algebras and let h be an Lk_n-homomorphism from A into A'. Then the application $\text{Ł}(h)$ defined by the prescription $\text{Ł}(h)(P) = h^{-1}(P)$ for all $P \in X(A')$ is an l_nP-function.*

P r o o f. For each $P \in X(A')$, $(\text{Ł}(h) \circ f_{A'}^i)(P) = (f_A^i \circ \text{Ł}(h))(P)$ holds true for all i, $1 \leqslant i \leqslant n-1$. Indeed, $(\text{Ł}(h) \circ f_{A'}^i)(P) = ({\varphi_i}' \circ h)^{-1}(P) = (h \circ \varphi_i)^{-1}(P) = (f_A^i \circ \text{Ł}(h))(P)$. Since $\text{Ł}(h)$ is an mP-function, the proof is complete. $\square$

Taking into account the above mentioned lemmas, it is routine to prove the following theorem:

Theorem 2.1. *The category $\mathbf{l_n}\mathcal{P}$ is naturally equivalent to the dual of the category $\mathfrak{L}\mathbf{k_n}$.*

A. Filipoiu, in [4, pages 341–348] (see also [11], [12]), obtained a topological duality for ϑ-valued Łukasiewicz algebras with negation ([4, page 110]). In the particular case that $\vartheta = n$ with n positive integer, this duality is essentially equivalent to the one indicated above.

Now, we shall describe a topological duality for monadic n-valued Łukasiewicz algebras which extends the already mentioned dualities for Q-distributive lattices and Lk_n-algebras.

Definition 2.2. $(X, g, f_1, \ldots, f_{n-1}, E)$ is a ql_nP-space if it has the following properties:

(qLP1) $(X, E) \in \mathbf{q}\mathcal{P}$,
(qLP2) $(X, g, f_1, \ldots, f_{n-1}) \in \mathbf{l_n}\mathcal{P}$,
(qLP3) $f_i^{-1}(EU) = Ef_i^{-1}(U)$ for all $U \in D(X)$ and for all i, $1 \leqslant i \leqslant n-1$.

If $(X, g, f_1, \ldots, f_{n-1}, E)$ and $(X', g', f'_1, \ldots, f'_{n-1}, E')$ are ql_nP-spaces, then $f\colon X \longrightarrow X'$ is a ql_nP-function if f is both an l_nP and a qP-function.

We shall denote by $\mathbf{ql_n}\mathcal{P}$ the category of ql_nP-spaces and ql_nP-functions.

From properties (qLP2) and (qLP3), it follows immediately that f_i are qP-functions for all i, $1 \leqslant i \leqslant n-1$.

We shall now show a characterization of qP-functions which will be useful later on.

Proposition 2.1. *Let (X, E) and (X', E') be qP-spaces and let f be a P-function from X into X'. Then the following conditions are equivalent:*

(i) *f is a qP-function,*
(ii) *f satisfies the conditions*
  (a) *if $(x, y) \in E$, then $(f(x), f(y)) \in E'$,*
  (b) *if $(f(x), z) \in E'$, then there is $y \in X$ such that $(x, y) \in E$ and $z \leqslant f(y)$.*

Proof. (i) $\Rightarrow$ (ii): From the hypothesis and Lemma 2.8 in [8], property (a) follows immediately. Then, (b) remains to be proved. Let $x \in X$ and $z \in X'$ be such that (1) $(f(x), z) \in E'$. If $z \leqslant f(x)$, choosing $y = x$, (b) is obtained. Otherwise, there is $U \in D(X')$ such that $z \in U$ and $f(x) \notin U$, then by (1) and (i) we obtain that $x \in Ef^{-1}(U)$. From this last assertion, it follows that $f^{-1}(U) \cap E(x) \neq \emptyset$. Besides, as f is continuous and closed, $K = f(f^{-1}(U) \cap E(x))$ is a closed subset of X' and therefore a compact one. If we assume that $z \not\leqslant f(y)$ for all $y \in f^{-1}(U) \cap E(x)$, then $z \not\leqslant k$ for all $k \in K$. Taking into account the compactness of K, we can assert that there are sets $V_i \in D(X')$ with $1 \leqslant i \leqslant m$ such that $z \in V_i$ and $K \subseteq \bigcup_{i=1}^{m} X' \setminus V_i$. If $V = \bigcap_{i=1}^{m} V_i$, then (2) $V \cap K = \emptyset$. On the other hand, if $W = U \cap V$, it follows that $z \in W$. From (1) and (i) we obtain that $f^{-1}(W) \cap E(x) \neq \emptyset$, then we conclude that $V \cap K \neq \emptyset$, which contradicts (2). Hence, there is $y \in X$ such that $(x, y) \in E$ and $z \leqslant f(y)$.

(ii) $\Rightarrow$ (i): From the hypothesis and the results in [8], the proof is simple to obtain. $\square$

Corollary 2.1. *$(X, g, f_1, \ldots, f_{n-1}, E)$ is a ql_nP-space if and only if it satisfies* (qLP1), (qLP2) *and the following conditions:*

(qLP4) *if $(x, y) \in E$, then $(f_i(x), f_i(y)) \in E$ for all i, $1 \leqslant i \leqslant n-1$,*

(qLP5) *for each i, $1 \leqslant i \leqslant n-1$, if $(f_i(x), z) \in E$, then there is $y \in X$ such that $(x, y) \in E$ and $z \leqslant f_i(y)$.*

P r o o f. It is a direct consequence of Definition 2.2 and Proposition 2.1. $\square$

R e m a r k 2.2. Let $(X, g, f_1, \ldots, f_{n-1}, E)$ be a ql_nP-space and $x, y \in X$. Taking into account (qLP4) and (LP5), we deduce that the following conditions are equivalent:

(i) there is i, $1 \leqslant i \leqslant n-1$, such that $(f_i(x), f_i(y)) \in E$,

(ii) $(f_j(x), f_j(y)) \in E$ for all j, $1 \leqslant j \leqslant n-1$.

The following list presents the necessary results to prove the duality between the categories $\mathbf{ql_n}\mathcal{P}$ and $\mathbf{q}\mathfrak{L}\mathbf{k_n}$.

Lemma 2.5. *If $(X, g, f_1, \ldots, f_{n-1}, E)$ is a ql_nP-space, then $q\mathbb{L}_n(X) = (D(X), \sim, \varphi_X^1, \ldots, \varphi_X^{n-1}, \nabla_E)$ is a qLk_n-algebra where for all $U \in D(X)$, $\sim U$, $\nabla_E(U)$ and $\varphi_X^i(U)$ are defined as in* (B1), (E4) *and Lemma* 2.1, *respectively.*

P r o o f. It follows from Lemma 2.1, (E4) and (qLP3). $\square$

Lemma 2.6. *If $(A, \sim, \varphi_1, \ldots, \varphi_{n-1}, \nabla)$ is a qLk_n-algebra, then $q\text{Ł}(A) = (X(A), g_A, f_A^1, \ldots, f_A^{n-1}, E_\nabla)$ is a ql_nP-space and σ_A is a qLk_n-isomorphism from A onto $D(X(A))$, where g_A, f_A^i, E_∇ and σ_A are those indicated in* (B2), *Lemma* 2.2, (E3) *and* (A2), *respectively.*

P r o o f. From (E3) and Lemma 2.2 we obtain that conditions (qLP1), (qLP2) in Definition 2.2 hold true and that σ_A is a qLk_n-isomorphism. Besides, from the hypothesis it is easy to check that $\nabla_{E_\nabla} \varphi_{X(A)}^i \sigma_A(a) = \varphi_{X(A)}^i \nabla_{E_\nabla} \sigma_A(a)$ for all $a \in A$ and this implies that $E_\nabla f_A^{i^{-1}}(U) = f_A^{i^{-1}}(E_\nabla U)$ for all $U \in D(X(A))$. $\square$

Lemmas 2.7 and 2.8 are immediate consequences of Lemmas 2.3 and 2.4, respectively, and of the results obtained in [8].

Lemma 2.7. *Let* $(X, g, f_1, \ldots, f_{n-1}, E)$ *and* $(X', g', f'_1, \ldots, f'_{n-1}, E')$ *be* ql_nP*-spaces and let* f *be a* ql_nP*-function from* X *into* X'*. Then the application* $q\mathbb{L}_n(f) = \mathbb{L}_n(f)$ *defined as in Lemma 2.3 is a* qLk_n*-homomorphism from* $q\mathbb{L}_n(X')$ *into* $q\mathbb{L}_n(X)$*.*

Lemma 2.8. *Let* $(A, \sim, \varphi_1, \ldots, \varphi_{n-1}, \nabla)$ *and* $(A', \sim', \varphi'_1, \ldots, \varphi'_{n-1}, \nabla')$ *be* qLk_n*-algebras and let* h *be a* qLk_n*-homomorphism from* A *into* A'*. Then the application* $q\text{Ł}(h) = \text{Ł}(h)$ *defined as in Lemma 2.4 is a* ql_nP*-function from* $q\text{Ł}(A')$ *into* $q\text{Ł}(A)$*.*

From the above mentioned lemmas and using the usual procedures we conclude:

Theorem 2.2. *The category* $\mathbf{ql_n}\mathcal{P}$ *is naturally equivalent to the dual of the category* $\mathbf{q\mathfrak{L}k_n}$*.*

3. Subdirectly irreducible qLk_n-algebras

In this section, our first objective is the characterization of the congruence lattice on an n-valued Łukasiewicz algebra by means of certain closed subsets of its associated l_nP-space which allows us to describe the congruences on monadic Lk_n-algebras. Later, this result will be taken into account to obtain subdirectly irreducible qLk_n-algebras. With this purpose, we shall start by studying some properties of l_nP and ql_nP-spaces.

Proposition 3.1. *Let* $X \in \mathbf{l_n}\mathcal{P}$*. Then* X *is the cardinal sum of a family of chains, each of which has at most* $n-1$ *elements.*

P r o o f. Let $x \in X$. By (LP9) and (LP4) we obtain $f_1(x) \leqslant \ldots \leqslant x = f_i(x) \leqslant \ldots \leqslant f_{n-1}(x)$. If $y \in X$ and $y \leqslant x$, by (LP3) we have (1) $f_j(y) = f_j(x)$ for all j, $1 \leqslant j \leqslant n-1$. On the other hand, by (LP9) there is k, $1 \leqslant k \leqslant n-1$, such that $y = f_k(y)$; therefore, $y = f_k(x)$ by (1). Then we conclude that $f_1(x) \leqslant \ldots \leqslant y = f_k(x) \leqslant \ldots \leqslant x = f_i(x) \leqslant \ldots \leqslant f_{n-1}(x)$. If $y \in X$ and $x < y$, the proof is similar. $\square$

R e m a r k 3.1. If $(X, g, f_1, \ldots, f_{n-1})$ and $(X', g', f'_1, \ldots, f'_{n-1})$ are l_nP-spaces and f is an l_nP-function from X into X' then from Proposition 3.1 and Definition 2.1 we deduce that:

(i) for each $x \in X$, $f(C_x) = C_{f(x)}$ where C_y denotes the unique maximum chain containing y;

(ii) f sends the i-th element of the sequence representing C_x onto the i-th element of the sequence representing $C_{f(x)}$ for all i, $1 \leqslant i \leqslant n-1$.

Lemma 3.1. *Let $(X, g, f_1, \ldots, f_{n-1}, E)$ be a ql_nP-space and $x, y \in X$. If $(f_i(x), f_j(y)) \in E$, for some pair i, j with $1 \leqslant i \leqslant n-1$, $1 \leqslant j \leqslant n-1$, then $f_i(x) \leqslant f_j(x)$ and $f_j(y) \leqslant f_i(y)$.*

Proof. If $i \leqslant j$ then by (LP4) we have $f_i(x) \leqslant f_j(x)$. Furthermore, since $(f_i(x), f_j(y)) \in E$, by (qLP5) there is $z \in X$ such that $f_j(y) \leqslant f_i(z)$. Then by (LP3) and (LP5) it follows that $f_i(y) = f_i(z)$ and therefore, $f_j(y) \leqslant f_i(y)$. If $j < i$, the proof is similar. □

Proposition 3.2. *Let $X \in \mathbf{ql_n}\mathcal{P}$ and $x, y \in X$. Then the following condition is fulfilled:*

(qLP6) *if $(f_i(x), f_j(y)) \in E$, for some pair i, j with $1 \leqslant i \leqslant n-1$, $1 \leqslant j \leqslant n-1$, then $f_i(x) = f_j(x)$ and $f_i(y) = f_j(y)$.*

Proof. From the hypothesis and Lemma 3.1 we have $f_i(x) \leqslant f_j(x)$ and $f_j(y) \leqslant f_i(y)$. On the other hand, from the hypothesis, (qLP4) and (LP5) we have $(f_i(x), f_i(y)) \in E$. Then by Remark 2.2 we obtain that $(f_j(x), f_j(y)) \in E$ and taking into account the fact that E is an equivalence relation, it follows that $(f_j(x), f_i(y)) \in E$. Then by Lemma 3.1 we have $f_j(x) \leqslant f_i(x)$ and $f_i(y) \leqslant f_j(y)$, which completes the proof. □

Corollary 3.1. *$(X, g, f_1, \ldots, f_{n-1}, E)$ is a ql_nP-space if and only if the conditions* (qLP1), (qLP2), (qLP4) *and* (qLP6) *are satisfied.*

Proof. It is routine. □

Proposition 3.3. *Let $X \in \mathbf{ql_n}\mathcal{P}$. Then for every $x \in X$, the set $E(x)$ is order discrete, where $E(x)$ is an equivalence class of x.*

Proof. Let $y \in E(x)$ and suppose that $y \leqslant z$ for some $z \in E(x)$. Taking into account the proof of Proposition 3.1, we have that there is j, $1 \leqslant j \leqslant n-1$ such that $z = f_j(y)$. Furthermore, by (LP9), there is i, $1 \leqslant i \leqslant n-1$ such that $y = f_i(y)$. Then $(f_i(y), f_j(y)) \in E$ which by (qLP6) entails $f_i(y) = f_j(y)$ which implies that $y = z$. □

Definition 3.1. Let $X \in \mathbf{l_n}\mathcal{P}$. A subset Y of X is modal if $f_i^{-1}(Y) = Y$ for all i, $1 \leqslant i \leqslant n-1$.

Proposition 3.4. *Let $X \in \mathbf{l_n}\mathcal{P}$ and let Y be a subset of X. Then the following conditions are equivalent:*
(i) *Y is modal,*
(ii) *for each $y \in Y$ we have $f_i(y) \in Y$ for all i, $1 \leqslant i \leqslant n-1$.*

Proof. We only prove (ii) $\Rightarrow$ (i). Suppose $f_i(z) \in Y$. Since by (LP9) $f_{i_0}(z) = z$ for some i_0, $1 \leqslant i_0 \leqslant n-1$, from (ii) and (LP5) we obtain that $z = f_{i_0}(f_i(z)) \in Y$. Therefore, $f_i^{-1}(Y) \subseteq Y$. The other inclusion follows immediately. □

Corollary 3.2. *Every maximum chain in an l_nP-space is modal.*

Proof. If C is a maximum chain in X, then taking into account the proof of Proposition 3.1, there is $x \in X$ such that $C = \{f_i(x)\colon\ 1 \leqslant i \leqslant n-1\}$. Then by (LP5) and Proposition 3.4 we have that C is modal. □

Lemma 3.2. *Let $X \in \mathbf{l_n}\mathcal{P}$ and let Y be an involutive subset of X. Then the following conditions are equivalent:*
(i) *Y is increasing,*
(ii) *Y is decreasing.*

Proof. (i) $\Rightarrow$ (ii): Let $x \in X$ and $y \in Y$ such that $x \leqslant y$. Then (1) $g(y) \leqslant g(x)$ and taking into account that Y is involutive we have that $g(y) \in Y$. From this statement, (1) and (i) we deduce that $g(x) \in Y$ and therefore $x = g(g(x)) \in Y$. The converse implication is similar. □

Proposition 3.5. *Let $X \in \mathbf{l_n}\mathcal{P}$ and let Y be a non empty subset of X. Then the following conditions are equivalent:*
(i) *Y is modal,*
(ii) *Y is involutive and increasing,*
(iii) *Y is a cardinal sum of maximum chains in X.*

Proof. (i) $\Rightarrow$ (ii): In order to prove that Y is involutive, it is sufficient to check that $g(Y) \subseteq Y$. Let $z = g(y)$ with $y \in Y$. Then by (LP9) $z = f_i(z)$ for some i, $1 \leqslant i \leqslant n-1$ and by (LP6) $z = f_i(y)$. Hence, by (i) and Proposition 3.4 we obtain that $z \in Y$.

Moreover, Y is increasing. Indeed, let $y \in Y$ and $z \in X$ be such that $y \leqslant z$. By (LP9) $z = f_i(z)$ for some i, $1 \leqslant i \leqslant n-1$. Then, by (LP3) and Proposition 3.3 we conclude that $z \in Y$.

(ii) $\Rightarrow$ (iii): From the hypothesis and Lemma 3.2 we have that Y is decreasing. Then for each $y \in Y$, $C_y = \{f_i(y)\colon\ 1 \leqslant i \leqslant n-1\} \subseteq Y$. Therefore, $Y = \bigcup\limits_{y \in Y} C_y$ and from Proposition 3.1 we conclude that Y is the cardinal sum of maximum chains in X.

(iii) $\Rightarrow$ (i): It is a direct consequence of Corollary 3.2. $\square$

Theorem 3.1. *Let $A \in \mathfrak{Lk}_\mathbf{n}$ and let $\text{Ł}(A)$ be the l_nP-space associated with A. Then the lattice $\mathcal{C}_M(\text{Ł}(A))$ of modal and closed subsets of $X(A)$ is isomorphic to the dual lattice $\mathrm{Con}_{Lk_n}(A)$ of Lk_n-congruences on A, and the isomorphism is the function Θ_M defined by the same prescription as in* (A3).

Proof. Let $Y \in \mathcal{C}_M(\text{Ł}(A))$. Then by condition (ii) in Proposition 3.5, Y is involutive and by (B3) we have that $\Theta_M(Y)$ is a de Morgan congruence. Since σ_A is an Lk_n-homomorphism and Y is a modal subset of X(A), it follows that $\sigma_A(\varphi_i a) \cap Y = f_A^{i^{-1}}(\sigma_A(a) \cap Y)$ for all $a \in A$, which implies that $\Theta_M \in \mathrm{Con}_{Lk_n}(A)$.

Conversely, let $\theta \in \mathrm{Con}_{Lk_n}(A)$ and let $h\colon\ A \longrightarrow A/\theta$ be the natural epimorphism. Since $\mathrm{Con}_{Lk_n}(A)$ is a sublattice of $\mathrm{Con}_M(A)$, we have that $Y = \{h^{-1}(Q)\colon\ Q \in X(A/\theta)\} = \{\text{Ł}(h)(Q)\colon\ Q \in X(A/\theta)\}$ is an involutive closed subset of $X(A)$ and $\theta = \Theta_M(Y)$. In addition, Y is modal. Indeed, let $P = \text{Ł}(h)(Q)$ with $Q \in X(A/\theta)$. Taking into account that $\text{Ł}(h)$ is an l_nP-function, we obtain that $f_A^i(P) = \text{Ł}(h)(f_{A/\theta}^i(Q))$. Therefore, $f_A^i(P) \in Y$ for all i, $1 \leqslant i \leqslant n-1$ and by Proposition 3.4 the proof is concluded. $\square$

Theorem 3.2. *Let $A \in \mathbf{q}\mathfrak{Lk}_\mathbf{n}$ and let $q\text{Ł}(A)$ be the ql_nP-space associated with A. Then the lattice $\mathcal{C}_{MS}(q\text{Ł}(A))$ of saturated, modal and closed subsets of $X(A)$ is isomorphic to the dual lattice $\mathrm{Con}_{qLk_n}(A)$ of qLk_n-congruences on A, and the isomorphism is the function Θ_{MS} defined by the same prescription as in* (A3).

Proof. If Y is a saturated, modal and closed subset of $X(A)$, then by Theorem 3.1 and (E5) it follows that $\Theta_{MS}(Y)$ is a qLk_n-congruence.

Conversely, if $\theta \in \mathrm{Con}_{qLk_n}(A)$, then by Theorem 3.1 it only remains to prove that $Y = \{h^{-1}(Q)\colon\ Q \in X(A/\theta)\}$ is saturated. Suppose that it is not, then there are $P \in Y$ and (1) $Q \in E_\nabla(P)$ such that $Q \notin Y$. Hence, from this last assertion we have by (F1) that there are $a, b \in A$ such that $a \in Q$, $b \notin Q$ and (2) $(a, b) \in \theta$. On the other hand, if F is the filter of A generated by $Q \cup \{b\}$, from (1), (F2) and Proposition 3.4 we infer that $P \cap \nabla(A) \subset F \cap \nabla(A)$. This statement means that there is $q \in Q$ such that (3) $\nabla(q \wedge b) \notin P$. By (2) we have that (4) $(\nabla(a \wedge q), \nabla(b \wedge q)) \in \theta$. Since $\nabla(a \wedge q) \in Q \cap \nabla(A)$, from (1) it follows that $\nabla(a \wedge q) \in P \cap \nabla(A)$. Hence by (4), taking into account that $\theta = \Theta_{MS}(Y)$, we conclude that $\nabla(b \wedge q) \in P$ which contradicts (3). $\square$

Next we shall use the results just obtained in order to determine the subdirectly irreducible qLk_n-algebras.

Proposition 3.6. *Let $(X, g, f_1, \ldots, f_{n-1}, E)$ be a ql_nP-space. If $x \in X$ and $x = f_i(x)$ then $f_i^{-1}(E(x))$ is a saturated, modal and closed subset of X such that $x \in f_i^{-1}(E(x))$.*

P r o o f. It follows immediately from the hypothesis that $x \in f_i^{-1}(E(x))$. On the other hand, from (E2), (LP2) and (LP5) we have that $f_i^{-1}(E(x))$ is a closed and modal subset of X. In addition, it is also saturated since, if $y \in \nabla_E f_i^{-1}(E(x))$, then there is $z \in X$ such that $(y, z) \in E$ and $(f_i(z), x) \in E$. Hence, by (qLP4) we infer that $(f_i(y), f_i(z)) \in E$ and therefore, $(f_i(y), x) \in E$. $\square$

Proposition 3.7. *Let $(X, g, f_1, \ldots, f_{n-1}, E)$ be a ql_nP-space such that $q\mathbb{L}_n(X)$ is a subdirectly irreducible qLk_n-algebra. If Y is a non-empty closed and modal subset of X, then the following conditions are equivalent:*

(i) *Y is saturated,*

(ii) $\max X \subseteq Y$.

P r o o f. (i) $\Rightarrow$ (ii): Suppose that $\max X \not\subseteq Y$. By Proposition 3.6, it follows that for each $x \in \max X \setminus Y$, $W_x = f_{n-1}^{-1}(E(x))$ is a closed, modal and saturated subset of X such that $x \in W_x$. Moreover, $W_x \cap Y = \emptyset$ for all $x \in \max X \setminus Y$. Indeed, if $z \in W_x \cap Y$ for some $x \in \max X \setminus Y$, then $(f_{n-1}(z), x) \in E$. Since Y is modal, $f_{n-1}(z) \in Y$ and taking into account that Y is saturated we obtain that $x \in Y$, which is a contradiction. Hence, there are at least two non-trivial closed, modal and saturated subsets in X. Since $X = Y \cup \bigcup\limits_{x \in \max X \setminus Y} W_x$, we infer that a maximum non-trivial closed, modal and saturated subset does not exist. Therefore, by Theorem 3.2, $q\mathbb{L}_n(X)$ is not a subdirectly irreducible qLk_n-algebra.

(ii) $\Rightarrow$ (i): As Y is modal and $\max X \subseteq Y$, we conclude from Proposition 3.5 that $Y = X$. $\square$

Proposition 3.8. *Let $(X, g, f_1, \ldots, f_{n-1}, E)$ be a ql_nP-space and $q\mathbb{L}_n(X)$ the qLk_n-algebra associated with X. Then the following conditions are equivalent:*

(i) *$q\mathbb{L}_n(X)$ is subdirectly irreducible,*

(ii) *$q\mathbb{L}_n(X)$ is simple.*

P r o o f. It follows immediately from Theorem 3.2, Proposition 3.5 and Proposition 3.7. $\square$

Proposition 3.9. *Let $(X, g, f_1, \ldots, f_{n-1}, E)$ be a ql_nP-space. Then the following conditions are equivalent:*

(i) *$(f_i(x), f_i(y)) \in E$ for all $x, y \in X$ and for all i, $1 \leqslant i \leqslant n-1$,*

(ii) *$\nabla_E(D(X))$ is a simple Lk_n-algebra.*

Proof. (i) $\Rightarrow$ (ii): By Lemma 2.5, $\nabla_E D(X)$ is an Lk_n-algebra. If we suppose that it is not a simple one, by (L7) there are U, $V \in D(X)$ such that $\nabla_E U \nsubseteq \nabla_E V$ and $\nabla_E V \nsubseteq \nabla_E U$. Then there are $x \in V$, $y \in U$ such that $(x, u) \notin E$ for all $u \in U$ and $(y, v) \notin E$ for all $v \in V$. By (LP9) there are i, j, $1 \leqslant i \leqslant n-1$, $1 \leqslant j \leqslant n-1$ such that $x = f_i(x)$ and $y = f_j(y)$. If $i \leqslant j$, by (LP4) we have that $f_j(x) \in V$ and therefore $(f_j(x), f_j(y)) \notin E$, which contradicts (i). If $j < i$, the proof is similar.

(ii) $\Rightarrow$ (i): Suppose that there are $x, y \in X$ such that $(f_i(x), f_i(y)) \notin E$ for some i, $1 \leqslant i \leqslant n-1$. Then, by Lemma 2.5 in [8], we can assume that there is $U \in D(X)$ such that (1) $f_i(x) \in \nabla_E U$ and (2) $f_i(y) \notin \nabla_E U$. On the other hand, from the hypothesis and (L7) we have $\varphi^i_X(\nabla_E U) = \emptyset$ or $\varphi^i_X(\nabla_E U) = X$; that is, $f_i^{-1}(\nabla_E U) = \emptyset$ or $f_i^{-1}(\nabla_E U) = X$. From these last statements and (1) we have that $f_i^{-1}(\nabla_E U) = X$ and therefore $f_i(y) \in \nabla_E U$, which contradicts (2). $\square$

Theorem 3.3. *Let $(X, g, f_1, \ldots, f_{n-1}, E)$ be a ql_nP-space and $q\mathbb{L}_n(X)$ the associated qLk_n-algebra. Then the following conditions are equivalent:*

(i) *$q\mathbb{L}_n(X)$ is a simple qLk_n-algebra,*

(ii) *$\nabla_E D(X)$ is a simple Lk_n-algebra.*

Proof. (i) $\Rightarrow$ (ii): If we suppose that $\nabla_E D(X)$ is not a simple Lk_n-algebra, then from Proposition 3.9 and Remark 2.2 we have that there are $x, y \in X$ such that (1) $(f_i(x), f_i(y)) \notin E$ for all $i, 1 \leqslant i \leqslant n-1$. On the other hand, by (LP9) $y = f_j(y)$ for some j, $1 \leqslant j \leqslant n-1$. Then, from Proposition 3.6 and (1) it follows that $f_j^{-1}(E(y))$ is a non-empty closed, modal and saturated subset of X such that $x \notin f_j^{-1}(E(y))$. Therefore, by Theorem 3.2 we conclude that $q\mathbb{L}_n(X)$ is not a simple qLk_n-algebra.

(ii) $\Rightarrow$ (i): If Y is a non-trivial saturated, modal and closed subset of X, then there are $y \in Y$ and $x \in X$ such that $x \notin Y$. Since Y is modal, from Proposition 3.4 we have that $f_i(y) \in Y$ and $f_i(x) \notin Y$ for all $i, 1 \leqslant i \leqslant n-1$. Then, as Y is saturated, $(f_i(x), f_i(y)) \notin E$ for all $i, 1 \leqslant i \leqslant n-1$, which implies by Proposition 3.9 that $\nabla_E D(X)$ is not a simple Lk_n-algebra. $\square$

Corollary 3.3. *Let $A \in \mathbf{q\mathfrak{L}k_n}$. Then the following conditions are equivalent:*

(i) *A is subdirectly irreducible,*

(ii) *A is simple,*

(iii) *∇A is a simple Lk_n-algebra.*

Proof. It is a direct consequence of Proposition 3.8 and Theorem 3.3. $\square$

Corollary 3.4. *The monadic functional algebra $C^*_{n,X}$ is simple.*

Lemma 3.3. *Let* $(X, g, f_1, \ldots, f_{n-1}, E)$ *and* $(X', g', f'_1, \ldots, f'_{n-1}, E')$ *be* ql_nP*-spaces such that* $q\mathbb{L}_n(X)$ *and* $q\mathbb{L}_n(X')$ *are simple algebras. If* f *is an* l_nP*-function from* X *onto* X' *then* f *is a* ql_nP*-function.*

P r o o f. Since f is an l_nP-function, hence $f_i^{-1}(f^{-1}(\nabla_{E'}U)) = f^{-1}(f_i'^{-1}(\nabla_{E'}U))$ for all $U \in D(X')$. Furthermore, since $f_i'^{-1}(\nabla_{E'}U)$ is a closed, modal and saturated subset of X', by Theorem 3.2 it follows that $f_i^{-1}(f^{-1}(\nabla_{E'}U)) = \emptyset$ or $f_i^{-1}(f^{-1}(\nabla_{E'}U)) = X$.

On the other hand, following an analogous reasoning we can prove that $f_i^{-1}(\nabla_E f^{-1}(U)) = \emptyset$ or $f_i^{-1}(\nabla_E f^{-1}(U)) = X$. Taking into account that f is surjective, the following conditions are equivalent:

(a) $f_i^{-1}(f^{-1}(\nabla_{E'}U)) = \emptyset$,

(b) $f_i'^{-1}(U) = \emptyset$,

(c) $f_i^{-1}(\nabla_E f^{-1}(U)) = \emptyset$.

Then we infer that $f_i^{-1}(f^{-1}(\nabla_{E'}U)) = f_i^{-1}(\nabla_E f^{-1}(U))$ for all i, $1 \leqslant i \leqslant n-1$ and by (LP10), $f^{-1}(\nabla_{E'}U) = \nabla_E f^{-1}(U)$. □

Theorem 3.4 ([1]). *Let* A *be a* qLk_n*-algebra. Then the following conditions are equivalent:*

(i) *A is simple,*

(ii) *A is isomorphic to a qLk_n-subalgebra of $C^*_{n,X}$.*

P r o o f. (i) ⇒ (ii): By the Moisil representation theorem ([16], [7]) we know that there exists an injective Lk_n-homorphism h from A into $C^*_{n,X}$. Then, by Lemma 2.4, $\text{Ł}(h)$ is a surjective l_nP-function from $\text{Ł}(C^*_{n,X})$ onto $\text{Ł}(A)$. Since A and $C^*_{n,X}$ are simple qLk_n-algebras and $q\text{Ł}(h) = \text{Ł}(h)$, we have by Lemma 3.3 that $q\text{Ł}(h)$ is a ql_nP-function. Therefore, by Lemma 2.7 and Theorem 2.2 we obtain that $h = q\mathbb{L}_n(q\text{Ł}(h))$ is an injective qLk_n-homomorphism.

(ii) ⇒ (i): From the hypothesis we deduce that ∇A is an Lk_n-algebra isomorphic to a subalgebra of $\nabla C^*_{n,X}$. Therefore, ∇A is isomorphic to a subalgebra of C_n and this implies by (L7) that ∇A is a simple Lk_n-algebra. Hence, by Corollary 3.3 we complete the proof. □

A c k n o w l e d g m e n t. The authors are truly thankful to the referee for his helpful comments.

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