GRAPH AUTOMORPHISMS OF SEMIMODULAR LATTICES

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Abstract. This paper deals with the relations between graph automorphisms and direct factors of a semimodular lattice of locally finite length.

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1. Introduction

Each lattice dealt with in the present paper is assumed to be of locally finite length (i.e., all its bounded chains are finite).

For a lattice L let $G(L)$ be the corresponding unoriented graph.

An automorphism of the graph $G(L)$ is called also a graph automorphism of the lattice L. The graph isomorphism of lattices is defined analogously.

We denote by $\mathcal{C}$ the class of all finite lattices L such that each automorphism of $G(L)$ turns out to be a lattice automorphism.

In connection with Birkhoff's problem 6 from [1], the following result has been proved in [5] (by using the results of [2] and [6]):

($*$) Let L be a finite modular lattice. Then the following conditions are equivalent:

(i) L belongs to $\mathcal{C}$.

(ii) No direct factor of L having more than one element is self-dual.

The natural question arises whether in ($*$) the assumption of modularity can be replaced by the assumption that L is semimodular.

In Section 3 we show by an example that the answer is "No".

We define the notions of an interval of type (C) in L and of a graph automorphism of type (C) (cf. Definitions 2.1 and 2.2).

Let A be a direct factor of a lattice L and $\emptyset \neq X \subseteq L$. We say that A is orthogonal to X if for any $x_1, x_2 \in X$, the components of x_1 and x_2 in the direct factor A are equal.

Let $\mathcal{C}_1$ be the class of all lattices L such that each graph automorphism of type (C) of L is a lattice automorphism.

We prove (by applying the results and the methods of [3], [5] and [6]):

($*_1$) Let L be a semimodular lattice. Then the following conditions are equivalent:

(i) L belongs to $\mathcal{C}_1$.

(ii) If A is a direct factor of L such that A is self-dual and orthogonal to each interval of type (C) in L, then A is trivial (i.e., $\operatorname{card} A = 1$).

2. Preliminaries

In what follows, L is a lattice. For the notion of the unoriented graph $G(L)$ of L cf., e.g. [1], [2].

If $x, y \in L$, $x < y$ and if the interval $[x, y]$ of L is a two-element set, then we write $x \prec y$ or $y \succ x$.

Hence a graph automorphism of L is a one-to-one mapping φ of L onto L such that, whenever $x, y \in L$ and $x \prec y$, then

(i) either $\varphi(x) \prec \varphi(y)$ or $\varphi(y) \prec \varphi(x)$,

(ii) either $\varphi^{-1}(x) \prec \varphi^{-1}(y)$ or $\varphi^{-1}(y) \prec \varphi^{-1}(x)$.

2.1. Definition. Let L_0 be a sublattice of L such that L_0 is isomorphic to the lattice in Fig. 1; then the convex closure $\overline{L_0}$ of L_0 in L is said to be an interval of type (C) in L.

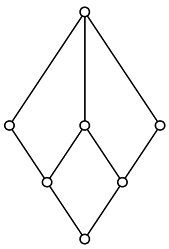

Fig. 1

2.2. Definition. A graph automorphism φ of L is said to be of type (C) if, whenever L_1 is an interval of type (C) in L and $x, y \in L_1$, $x \prec y$, then $\varphi(x) \prec \varphi(y)$ and $\varphi^{-1}(x) \prec \varphi^{-1}(y)$.

It is easy to verify that if L is modular, then it has no sublattice of type (C); consequently, in this case each graph automorphism of L is of type (C). Therefore $(*)$ is a corollary of $(*_1)$.

We denote by $L^{\sim}$ the lattice dual to L. If L and $L^{\sim}$ are isomorphic, then L is said to be self-dual.

3. An example

Let us recall that if L can be expressed as a direct product $L_1 \times L_2$ and if $x = (x_1, x_2) \in L$, $y = (y_1, y_2) \in L$, then $x \prec y$ if and only if either $x_1 \prec y_1$ and $x_2 = y_2$, or $x_1 = x_2$ and $y_1 \prec y_2$.

From this we immediately obtain

3.1. Lemma. *Let L_1, L_2 be lattices and let φ be a graph isomorphism of L_1 onto L_2. Put $L = L_1 \times L_2$. For each $x = (x_1, x_2) \in L$ we set*

$$\varphi(x) = (\varphi^{-1}(x_2), \varphi(x_1)).$$

Then ψ is a graph automorphism of L.

Consider the lattices L_1 and L_2 in Fig. 2 or Fig. 3, respectively. Both L_1 and L_2 are semimodular.

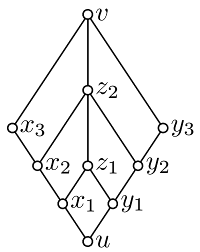

Fig. 2

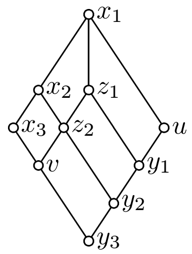

Fig. 3

3.2. Lemma. *Both L_1 and L_2 are directly indecomposable.*

P r o o f. The assertion for L_1 was proved in [5], pp. 164–165. The proof for L_2 is similar. $\square$

3.3. Lemma. *Let $i \in \{1, 2\}$. Then the lattice L_i fails to be self-dual.*

P r o o f. It is easy to verify that $L_i^{\sim}$ fails to be semimodular. Therefore $L_i^{\sim}$ is not isomorphic to L_i. $\square$

Put $L = L_1 \times L_2$.

Since any two direct product decompositions of L have a common refinement and since L_1, L_2 are directly indecomposable by 3.3, we conclude

3.4. Lemma. *Let A be a direct factor of L having more than one element. Then the lattice A is isomorphic to some of the lattices L, L_1, L_2.*

By the same argument as in 3.3 we obtain

3.5. Lemma. *The lattice L is not self-dual.*

Now, 3.3, 3.4 and 3.5 yield

3.6. Corollary. *The lattice L satisfies the condition* (ii) *from* $(*)$.

It is easy to verify that there exists a graph isomorphism φ of L_1 onto L_2 such that φ fails to be a lattice isomorphism. Hence there are x_1, y_1 in L_1 such that $x_1 \prec y_1$ and $\varphi(x_1) \succ \varphi(y_1)$. Consequently, if ψ is defined as above, then ψ is not a lattice automorphism of L.

In view of 3.1 we conclude that in $(*)$, the assumption of modularity cannot be replaced by the assumption of semimodularity of the lattice L.

We also remark that ψ is an example of a graph automorphism on a semimodular lattice such that ψ is not of type (C).

4. Proof of $(*_1)$

In this section we assume that the lattice L is semimodular.

4.1. Lemma. *Suppose that B is a direct factor of L such that*
(i) *B is self-dual;*
(ii) *B is orthogonal to each interval of type* (C) *in L;*
(iii) $\operatorname{card} B > 1$.
Then L does not belong to $\mathcal{C}_1$.

Proof. There is a lattice A such that there exists an isomorphism ψ of L onto $A \times B$. Further, in view of (i), there is an isomorphism χ of the lattice B onto $B^{\sim}$. For each $x \in L$ we put $\varphi(x) = y$, where

$$\psi(x) = (a, b), \quad y = \psi^{-1}((a, \chi(b))).$$

Then φ is a graph automorphism of the lattice L (cf. [5], Lemma 1.1). Moreover, (ii) yields that φ is of type (C). By applying Lemma 1.2 of [5] we conclude that φ fails to be a lattice automorphism. Therefore L does not belong to $\mathcal{C}_1$. $\square$

Let L_1 and L_2 be semimodular lattices. Suppose that φ is a graph isomorphism of L_1 onto L_2 such that

(a) if X is an interval of type (C) in L_1 and $x_1, x_2 \in X$, $x_1 \prec x_2$, then $\varphi(x_1) \prec \varphi(x_2)$;

(b) if Y is an interval of type (C) in L_2 and $y_1, y_2 \in Y$, $y_1 \prec y_2$, then $\varphi^{-1}(y_1) \prec \varphi^{-1}(y_2)$.

We apply similar steps as in Section 2 of [5]. For the sake of completenes, we recall the corresponding notation.

Let $\mathcal{A}_1$ be the set of all intervals $[x, y]$ of L_1 such that

$$x \prec y \quad \text{and} \quad \varphi(x) \prec \varphi(y).$$

Further, let $\mathcal{B}_1$ be the set of all intervals $[u, v]$ of L_1 such that

$$u \prec v \quad \text{and} \quad \varphi(u) \succ \varphi(v).$$

Similarly we define the sets $\mathcal{A}_2$ and $\mathcal{B}_2$ of intervals of L_2 (with φ^{-1} instead of φ).

Choose $x_1^0 \in L_1$, $x_2^0 \in L_2$. We denote by A_1^0 the set of all elements $x \in L_1$ such that either $x = x_1^0$, or there exist $y_1, y_2, \dots, y_n \in L_1$ such that

(i) $y_1 = x_1^0$, $y_n = x$,

(ii) for each $i \in \{1, 2, \dots, n-1\}$, the elements y_i, y_{i+1} are comparable and the corresponding interval belongs to $\mathcal{A}_1$.

Similarly we define the set B_1^0 (taking $\mathcal{B}_1$ instead of $\mathcal{A}_1$). The subsets A_2^0 and B_2^0 are defined analogously (taking x_2^0 and φ^{-1} instead of x_1^0 and φ).

We apply the notion of the internal direct product decomposition of a lattice L with the central element x^0 in the same sense as in [5] (cf. also [6]). By using this notion and by applying the assumption given above we conclude that the results of [3] (cf. Theorem 2 in [3] and the lemmas applied for proving this Theorem) yield

4.2. Proposition. *Under the assumptions as above, there exist internal direct product decompositions*

$$\psi_1\colon\ L_1 \to A_1^0 \times B_1^0 \quad \textit{(with the central element } x_1^0\textit{)},$$
$$\psi_2\colon\ L_2 \to A_2^0 \times B_2^0 \quad \textit{(with the central element } x_2^0\textit{)}$$

such that

(i) *the lattices A_1^0 and A_2^0 are isomorphic,*

(ii) *the lattice B_1^0 is isomorphic to $(B_2^0)^\sim$.*

Now suppose that the lattice L satisfies the condition (ii) of $(*_1)$.

Let φ be a graph automorphism of type (C) of the lattice L.

Choose $x^0 \in L$. We put $L = L_1 = L_2$ and $x^0 = x_1^0 = x_2^0$. The fact that φ is of type (C) yields that the conditions (a) and (b) are satisfied. Hence we can apply Proposition 4.2.

The further steps are the same as in Part 3 of [5]. By using them we obtain

4.3. Lemma. *Let L be a semimodular lattice satisfying the condition* (ii) *of* $(*_1)$. *Then the condition* (i) *of* $(*_1)$ *is valid.*

In view of 4.1 and 4.3, we infer that $(*_1)$ holds.

If L_1 is a sublattice of L and $a, b \in L_1$, $a < b$, then we denote by $[a, b]_1$ the corresponding interval of L_1. We put $a \prec_1 b$ if $[a, b]_1$ is a two-element set.

We say that L_1 is a c-sublattice of L if, whenever $a, b \in L_1$ and $a \prec_1 b$, then $a \prec b$.

We remark that Theorem 2 in the paper [7] by Ratanaprasert and Davey (this theorem solved a problem proposed in [4]) implies that in Definition 2.1 above it suffices to consider only those sublattices L_0 of L which are c-sublattices of L.

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