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SOME RECENT RESULTS ON THE EXISTENCE OF GLOBAL-IN-TIME WEAK SOLUTIONS TO THE NAVIER-STOKES EQUATIONS OF A GENERAL BAROTROPIC FLUID

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Abstract. This is a survey of some recent results on the existence of globally defined weak solutions to the Navier-Stokes equations of a viscous compressible fluid with a general barotropic pressure-density relation.

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1. Introduction

This is a survey of some recent results on the existence and qualitative properties of the global-in-time weak solutions to the Navier-Stokes system

$$\partial_t \varrho + \operatorname{div}(\varrho \vec{u}) = 0, \tag{1.1}$$

$$\partial_t(\varrho \vec{u}) + \operatorname{div}(\varrho \vec{u} \otimes \vec{u}) + \nabla p = \mu \Delta \vec{u} + (\lambda + \mu) \nabla \operatorname{div} \vec{u} + \varrho \vec{f}. \tag{1.2}$$

The system describes the time evolution of the density $\varrho = \varrho(t, x)$ and the velocity $\vec{u} = \vec{u}(t, x)$ of a viscous compressible fluid occupying a spatial domain $\Omega \subset \mathbb{R}^N$. Though the problem makes sense for any positive integer N, the physically interesting cases are $N = 1, 2, 3$.

The viscosity coefficients are assumed to be constants satisfying

$$\mu > 0, \ \lambda + \mu \geqslant 0.$$

The symbol $\vec{f}$ stands for a given external volumic force, for instance the gravity, which is allowed to depend on both t and x. For the sake of simplicity, we will

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assume that $\vec{f}$ is a bounded measurable function of t and x though much more general hypotheses could be treated by the same method.

We concentrate on the so-called barotropic case where p is a given function of the density ϱ, and, consequently, (1.1), (1.2) represent, at least formally, a closed system of equations. The typical situation we have in mind is the isentropic regime where

$$p = a\varrho^\gamma,\ a > 0,\ \gamma > 1.$$

As we shall see, the adiabatic constant γ plays the role of a critical exponent for the problem in question.

For the sake of definiteness, the system (1.1), (1.2) is supplemented by no-slip boundary conditions for the velocity $\vec{u}$ as well as initial conditions for both the density ϱ and the momentum $\varrho\vec{u}$

$$\vec{u}|_{\partial\Omega} = 0, \tag{1.3}$$

$$\varrho(0) = \varrho_0,\ (\varrho\vec{u})(0) = \vec{q}. \tag{1.4}$$

Clearly, the function $\vec{q}$ must satisfy the compatibility conditions

$$\vec{q} = 0 \text{ a.e. on the set } \{\varrho_0 = 0\}.$$

Multiplying (formally) the equations (1.2) by $\vec{u}$, integrating by parts and making use of (1.1), we arrive at the energy inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}E[\varrho,(\varrho\vec{u})](t) + \int_\Omega \mu|\nabla\vec{u}(t)|^2 + (\lambda+\mu)|\operatorname{div}\vec{u}(t)|^2\,\mathrm{d}x \leqslant \int_\Omega \varrho\vec{f}\cdot\vec{u}\,\mathrm{d}x \tag{1.5}$$

where the total energy E is given by the formula

$$E = E[\varrho,(\varrho\vec{u})] = \int_\Omega \frac{1}{2}\varrho|\vec{u}|^2 + P(\varrho)\,\mathrm{d}x,\ P(\varrho) = \varrho\int_1^\varrho \frac{p(z)}{z^2}\,\mathrm{d}z.$$

As a matter of fact, the function P satisfies

$$P'(z)z - P(z) = p(z)$$

and, consequently, it is uniquely determined up to an affine function of ϱ. In the isentropic case, one typically takes

$$P(\varrho) = \frac{a}{\gamma-1}\varrho^\gamma,$$

in particular, the behaviour of the pressure p and the "potential" P is the same for large values of the density.

To give a weak formulation of the problem (1.1)–(1.3), we consider the space $D_0^{1,2}(\Omega)$—the completion of the space $\mathcal{D}(\Omega)$ of all compactly supported smooth functions with respect to the (semi-)norm

$$\|v\|^2_{D^{1,2}(\Omega)} = \int_\Omega |\nabla v|^2 \,\mathrm{d}x. \tag{1.6}$$

Note that the quantity defined in (1.6) is a norm on the space $D_0^{1,2}(\Omega)$ provided $N = 3$ or when Ω is a bounded domain with a sufficiently smooth boundary. In the latter case, $D_0^{1,2}(\Omega)$ coincides with the Sobolev space $W_0^{1,2}(\Omega)$. Here "sufficiently smooth boundary" means that the Poincaré inequality is satisfied.

Following [6], we shall say that ϱ, $\vec{u}$ is a finite energy weak solution to the problem (1.1)–(1.3) on the set $(0,T)\times\Omega$ if the following conditions hold:

- the density ϱ is a non-negative function,

$$\varrho \in L^\infty(0,T;L^1(\Omega)),\ \ P(\varrho) \in L^\infty(0,T;L^1(\Omega)),\ \ \vec{u} \in L^2(0,T;D_0^{1,2}(\Omega));$$

- the total energy E is locally integrable, and the energy inequality (1.5) holds in $\mathcal{D}'(0,T)$ (in the sense of distributions);
- the continuity equation (1.1) is satisfied in $\mathcal{D}'((0,T)\times\mathbb{R}^N)$ provided ϱ, $\vec{u}$ are extended to be zero outside Ω; moreover, the functions ϱ, $\vec{u}$ represent a renormalized solution of the equation (1.1), i.e., one has

$$\partial_t b(\varrho) + \operatorname{div}(b(\varrho)\vec{u}) + (b'(\varrho)\varrho - b(\varrho))\operatorname{div} u = 0 \text{ in } \mathcal{D}'((0,T)\times\mathbb{R}^N) \tag{1.7}$$

  for any function $b \in C^1(R)$ such that

$$b'(z) \equiv 0 \text{ for all } z \text{ large enough, say, } z \geqslant M;$$

- the pressure p is locally integrable and the equations (1.2) are satisfied in $\mathcal{D}'((0,T)\times\Omega)$.

As we will see, under some "reasonable hypotheses" concerning the domain Ω and the pressure-density constitutive relation, the finite energy weak solutions belong to the class

$$\varrho \in C([0,T];L^1(\Omega)),\ \ (\varrho\vec{u}) \in C([0,T];L^1_{\mathrm{weak}}(\Omega)),$$

so the initial conditions (1.4) make sense. Following this philosophy, one can redefine the energy (on a set of zero Lebesgue measure in $(0,T)$) as

$$E = E[\varrho,(\varrho\vec{u})] = \int_{\Omega\cap\{\varrho>0\}} \frac{1}{2}\frac{|\varrho\vec{u}|^2}{\varrho}\,\mathrm{d}x + \int_\Omega P(\varrho)\,\mathrm{d}x$$

to obtain a quantity defined for any $t \in [0,T]$ which is lower semi-continuous in t (see [4]).

It is worthwhile to note that there seems to be a large qualitative gap between the existence theory available for $N=1$ and $N=2,3$. Here, we concentrate on the more difficult case $N=2,3$ leaving the reader to consult the monograph of Antontsev, Kazhikhov and Monakhov [1] for the former case.

2. Basic existence result

We start with the isentropic case

$$p(\varrho) = a\varrho^{\gamma},\ a>0, \gamma>1. \tag{2.1}$$

The main result we want to present here reads as follows:

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^N$, $N=2,3$ be a bounded spatial domain with a boundary of the class $C^{2+\nu}$, $\nu>0$. Let the pressure p be given by the constitutive relation* (2.1) *with*

$$\gamma > \frac{N}{2}.$$

Let the initial data ϱ_0, $\vec{q}$ satisfy the compatibility conditions

$$\varrho_0 \geqslant 0,\ \varrho_0 \in L^{\gamma}(\Omega),\ \frac{|\vec{q}|^2}{\varrho_0} \in L^1(\Omega). \tag{2.2}$$

Finally, let $T>0$ be given and let $\vec{f}$ be a bounded measurable function on the set $(0,T)\times\Omega$.

Then there exists a finite energy weak solution ϱ, $\vec{u}$ of the problem (1.1)–(1.3) *on $(0,T)\times\Omega$ satisfying the initial conditions* (1.4).

Lions [9] proved Theorem 2.1 for the critical values $\gamma \geqslant 3/2$ for $N=2$, and $\gamma \geqslant 9/5$ if $N=3$. The present result was obtained in [5], [7]. As was already indicated in the introduction, the value of the adiabatic constant γ plays the role of the critical exponent here. As a matter of fact, the critical values treated in [9] are related to the pressure estimates of the form

$$p(\varrho)\varrho^{\theta} \text{ bounded in } L^1((0,T)\times\Omega) \text{ for } \theta = \frac{2}{N}\gamma - 1 \tag{2.3}$$

(cf. Lions [9], [10], and [8]). For both $\gamma \geqslant 3/2$ if $N=2$ and $\gamma \geqslant 9/5$ for $N=3$, the relation (2.3) yields

$$\varrho \text{ bounded in } L^2((0,T)\times\Omega).$$

The square integrability of the density can be used to show the following result. Assume that $\varrho \in L^2((0,T)\times\Omega)$, $\vec{u} \in L^2(0,T;W^{1,2}(\Omega))$ solve the continuity equation (1.1) in the sense of distributions. Then (1.1) is also satisfied in the sense of renormalized solutions in the spirit of DiPerna and Lions [2] (cf. (1.7)). This fact in turn plays the crucial role in the existence proof presented in [9].

The main contribution of [5], [7] to the existence theory lies in the observation that one can replace the square integrability of the density by a different condition. Specifically, assume that ϱ_n, $\vec{u}_n$ is a sequence of renormalized solutions to the equation (1.1) such that

$$\left\{\begin{array}{l} \varrho_n \to \varrho \text{ weakly star in } L^\infty(0,T;L^\gamma(\Omega)), \\ \vec{u}_n \to \vec{u} \text{ weakly in } L^2(0,T;W^{1,2}(\Omega)). \end{array}\right\}$$

Suppose, in addition, that the quantity

$$\mathbf{osc}_p[\varrho_n - \varrho] = \sup_{k\geqslant 1}\bigl(\limsup_{n\to\infty} \|T_k(\varrho_n) - T_k(\varrho)\|_{L^p((0,T)\times\Omega)}\bigr) \tag{2.4}$$

is bounded for a certain $p > 2$. Here $T_k(z) = \min\{z,k\}$ are the cut-off functions. Then the limit functions ϱ, $\vec{u}$ represent a renormalized solution of (1.1).

Boundedness of the quantity $\mathbf{osc}_p[\varrho_n - \varrho]$ called the oscillation defect measure is an essential ingredient of the existence theory presented in [7]. In fact, one can show that it is bounded for $p = \gamma + 1$. This might indicate the proof should work for any $\gamma > 1$ though there are, of course, some unsurmountabe difficulties connected with a priori estimates when $N = 3$.

3. General barotropic pressure laws

The first possible generalization of the above existence results addresses a general barotropic pressure-density constitutive law $p = p(\varrho)$. More specifically, we will assume

$$p = p(\cdot) \in C^1[0,\infty),\ p(0) = 0,\ \frac{1}{a}\varrho^{\gamma-1} - b \leqslant p'(\varrho) \leqslant a\varrho^{\gamma-1} + b \text{ for all } \varrho \geqslant 0 \tag{3.1}$$

for certain positive constants a, b.

Observe that p need be neither convex nor even a monotone function of the density. The non-monotone pressure-density constitutive laws occur, for example, in astrophysics, nuclear astrophysics, low energy nuclear physics etc (cf. [3]).

The following result can be found in [3]:

Theorem 3.1. *Theorem* 2.1 *remains valid in the case when the isentropic pressure-density relation* (2.1) *is replaced by a general barotropic constitutive law satisfying* (3.1).

The general monotone pressure density relations are also discussed by Lions [9].

4. Unbounded and/or irregular domains

The last question we want to discuss here is to which extent the existence results presented above depend on the regularity of the spatial domain Ω. The first result is proved in [3].

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^3$ be a domain (not necessarily bounded) with a compact boundary of the class $C^{2+\nu}$, $\nu > 0$. Let the data ϱ_0, $\vec{q}$, $\vec{f}$ satisfy the hypotheses of Theorem* 2.1, *and, in addition, let $\varrho_0 \in L^1(\Omega)$. Finally, let the pressure p be given by a constitutive law obeying* (3.1) *with $\gamma > 3/2$.*

Then there exists a finite energy weak solution ϱ, $\vec{u}$ of the problem (1.1)–(1.3) *satisfying the initial conditions* (1.4).

Now, assume the boundary of Ω is not regular, say, not even Lipschitz. In that case, we have to "give up" the differential form (1.5) of the energy inequality. Integrating (1.5) with respect to t, we obtain

$$\begin{aligned} E[\varrho, (\varrho\vec{u})](\tau) &+ \int_0^\tau \int_\Omega \mu|\nabla\vec{u}|^2 + (\lambda+\mu)|\operatorname{div}\vec{u}|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &\leqslant E_0 + \int_0^\tau \int_\Omega \varrho\vec{f}\cdot\vec{u}\,\mathrm{d}x\,\mathrm{d}t \text{ for a.e. } \tau \in (0,T) \end{aligned} \tag{4.1}$$

where

$$E_0 = \int_\Omega \frac{1}{2}\frac{|\vec{q}|^2}{\varrho_0} + P(\varrho_0)\,\mathrm{d}x.$$

Replacing (1.5) by (4.1) in the definition of the finite energy weak solutions (cf. Section 1), we will speak about the bounded energy weak solutions of the problem (1.1)–(1.3) for which we report the following rather general result (see [6]):

Theorem 4.2. *Let $\Omega \subset \mathbb{R}^3$ be an arbitrary open set. Let the pressure p be given by a general constitutive law obeying* (3.1) *with*

$$\gamma > 3/2.$$

Let the initial data satisfy

$$\varrho_0 \geqslant 0,\ \varrho_0,\ P(\varrho_0) \in L^1(\Omega),\ \vec{q} \in L^1(\Omega),\ \frac{|\vec{q}|^2}{\varrho_0} \in L^1(\Omega).$$

Finally, let $\vec{f} = \vec{f}(t,x)$ *be a given bounded measurable function.*

Then the problem (1.1)–(1.3) *supplemented by the initial conditions* (1.4) *admits a bounded energy weak solution* ϱ, $\vec{u}$ *on* $(0,T) \times \Omega$, $T > 0$ *arbitrary.*

References

[1] *S. N. Antontsev, A. V. Kazhikhov, V. N. Monakhov*: Boundary Value Problems of Mechanics of Non-Homogeneous Fluids. Novosibirsk, 1983. (In Russian.)

[2] *R. J. DiPerna, P.-L. Lions*: Ordinary differential equations, transport theory and Sobolev spaces. Invent. Math. *98* (1989), 511–547.

[3] *B. Ducomet, E. Feireisl, H. Petzeltová, I. Straškraba*: Global in time weak solutions for compressible barotropic self-gravitating fluids. Adv. Appl. Math. Submitted (2001).

[4] *E. Feireisl*: The dynamical systems approach to the Navier-Stokes equations of compressible fluid. Advances in Mathematical Fluid Mechanics (J. Málek, J. Nečas, M. Rokyta, eds.). Springer, Berlin, 2000.

[5] *E. Feireisl*: On compactness of solutions to the compressible isentropic Navier-Stokes equations when the density is not square integrable. Comment. Math. Univ. Carolin. *42* (2001), 83–98.

[6] *E. Feireisl, A. Novotný, H. Petzeltová*: On the domain dependence of solutions to the compressible Navier-Stokes equations of a barotropic fluid. Math. Meth. Appl. Sci. To appear.

[7] *E. Feireisl, A. Novotný, H. Petzeltová*: On the existence of globally defined weak solutions to the Navier-Stokes equations of compressible isentropic fluids. J. Math. Fluid Dynamics *3* (2001), 358–392.

[8] *E. Feireisl, H. Petzeltová*: On integrability up to the boundary of the weak solutions of the Navier-Stokes equations of compressible flow. Commun. Partial Differential Equations *25* (2000), 755–767.

[9] *P.-L. Lions*: Mathematical Topics in Fluid Dynamics, Vol. 2, Compressible Models. Oxford Science Publication, Oxford, 1998.

[10] *P.-L. Lions*: Bornes sur la densité pour les équations de Navier-Stokes compressible isentropiques avec conditions aux limites de Dirichlet. C. R. Acad. Sci. Paris, Sér I. *328* (1999), 659–662.

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