Preprint, Institute of Mathematics, AS CR, Prague. 2008-10-11

SMOOTH APPROXIMATIONS

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ABSTRACT. We prove that a Lipschitz (or uniformly continuous) mapping $f\colon X \to Y$ can be approximated by smooth Lipschitz (resp. uniformly continuous) mapping, if X is a separable Banach space admitting a smooth Lipschitz bump and either X or Y is a $C(K)$ space (resp. super-reflexive space). As a corollary we obtain also smooth approximation of C^1-smooth mappings together with their first derivatives.

1. Introduction

The theory of approximation of continuous mappings between infinite dimensional Banach spaces by smooth mappings, which goes back to Kurzweil [K] and Bonic and Frampton [BF], is nowadays well understood and provides satisfactory results, see for example [DGZ].

The related problem, whether the smooth approximation of Lipschitz (or uniformly continuous) mappings can retain the Lipschitz (or uniform continuity) property is much less studied, and so far the result available are not very general. One of the reasons is that most of the results on approximation of continuous mappings use the notion of smooth partition of unity, and it is very difficult, if not impossible, to keep some uniformity in the partition.

The main results of the present paper are that a Lipschitz (or uniformly continuous) mapping $f\colon X \to Y$ can be approximated by smooth Lipschitz mapping (Corollary 8), resp. uniformly continuous mapping (Corollary 10), if X is a separable Banach space admitting a smooth Lipschitz bump and either X or Y is a $C(K)$ space (resp. super-reflexive space). To this end, we develop some more general theorems and apply the results of Lindenstrauss on absolute retracts (see e.g. [BL, Theorem I.1.6, Theorem I.1.26]). These two results complement the presently known theorems (see below), for example we remove the assumption on X having a basis from Theorem H but unfortunately we have to restrict the type of the target space.

Further, we show that smooth approximation of Lipschitz mappings is closely related to a smooth approximation of C^1-smooth mappings together with their first derivatives, namely we generalise the result of Moulis (Theorem C) into arbitrary (non-separable) spaces (Theorem 13). As a corollary we obtain also a result on approximation of C^1-smooth mappings (Corollary 14).

To put our results into perspective, we summarise the current state of the theory below.

But first, we need to fix some notation. Let B_X (U_X) denote a closed (open) unit ball of a normed linear space X. Further, for a metric space (P,ρ), we denote $B(x,r) = \{y \in P\colon \rho(x,y) \le r\}$ and $U(x,r) = \{y \in P\colon \rho(x,y) < r\}$ the closed and open ball in P centred at $x \in P$ with radius $r \ge 0$. Let $A \subset P$. A neighbourhood $U \subset P$ of A is called an *r-uniform neighbourhood* if there is $r > 0$ such that $\bigcup_{x\in A} U(x,r) \subset U$. A neighbourhood is called a *uniform neighbourhood* if it is r-uniform for some $r > 0$. For a set $M \in P$ and $\varepsilon > 0$ we denote $M_\varepsilon = \{x \in M\colon \operatorname{dist}(x, P\setminus M) > \varepsilon\}$.

Now we list the known results, in the order as they appeared in the literature:

Theorem A (Moulis)**.** *Let X be a Banach space with an unconditional Schauder basis that admits a C^k-smooth Lipschitz bump function, $k \in \mathbb{N}\cup\{\infty\}$, and Y be a Banach space. For any open $\Omega \subset X$, any mapping $f \in C^1(\Omega, Y)$ and any continuous function $\varepsilon\colon \Omega \to (0,+\infty)$ there is $g \in C^k(\Omega, Y)$, such that $\|f(x) - g(x)\| < \varepsilon(x)$ and $\|f'(x) - g'(x)\| < \varepsilon(x)$ for all $x \in \Omega$.*

This theorem immediately follows from the following two results:

Theorem B (Moulis)**.** *Let X be a Banach space with a monotone unconditional Schauder basis that admits a C^k-smooth Lipschitz bump function. There is a constant $C > 0$ such that if Y is a Banach space, $M \subset X$ such that $P_n M \subset M$ for all $n \in \mathbb{N}$, Ω a uniform open neighbourhood of M, $f\colon \Omega \to Y$ an L-Lipschitz Gâteaux differentiable mapping such that for each $n \in \mathbb{N}$ the mapping $x \mapsto f'(x)e_n$ is uniformly continuous on Ω, and $\varepsilon > 0$, then there is $g \in C^k(X,Y)$ such that $\|g'(x)\| \le C(1+\varepsilon)L$ for all $x \in M_\varepsilon$ and $\|f(x) - g(x)\| < \varepsilon$ for all $x \in M_\varepsilon$.*

Theorem C (Moulis)**.** *Let X, Y be normed linear spaces, X separable, and $k \in \mathbb{N}\cup\{\infty\}$. Suppose there is $C \in \mathbb{R}$ such that for any L-Lipschitz mapping $f \in C^1(2U_X, Y)$ and any $\varepsilon > 0$ there is a CL-Lipschitz mapping $g \in C^k(U_X, Y)$, such that $\sup_{x\in U_X} \|f(x) - g(x)\| \le \varepsilon$. Then for any open $\Omega \subset X$, any mapping $f \in C^1(\Omega, Y)$ and any continuous*

Date: September 2008.
2000 *Mathematics Subject Classification.* 46B03, 46B20, 46B26.
Key words and phrases. smoothness, approximation.
Supported by grants: the research project MSM 0021620839, Inst. Res. Plan AV0Z10190503, A100190801.

function $\varepsilon\colon \Omega \to (0,+\infty)$ *there is* $g \in C^k(\Omega, Y)$*, such that* $\|f(x)-g(x)\| < \varepsilon(x)$ *and* $\|f'(x)-g'(x)\| < \varepsilon(x)$ *for all* $x \in \Omega$.

We note, that Theorem B is actually formulated as [M, Lemme fondamental 1] under much stronger assumptions, namely for L_p spaces and mappings C^1-smooth on some ball. However, the proof in [M] works also for spaces with unconditional basis with only formal modifications. Denote $X_n = \operatorname{span}\{e_i\}_{i=1}^n$ and $f_n = f\restriction_{X_n}$. Then the assumptions of Theorem B imply that that f_n' are uniformly continuous on $\Omega \cap X_n$. Noticing this, the proof in [M] works also almost verbatim under the relaxed differentiability assumptions.

The next theorem uses the infimal convolution techniques, hence provides only C^1-smooth approximation of functions. Nevertheless, it is the first non-separable result.

Theorem D (Cepedello)**.** *Let* X *be a super-reflexive Banach space,* $f\colon X \to \mathbb{R}$ *a Lipschitz function and* $\varepsilon > 0$*. Then there is a function* $g \in C^1(X)$ *which is Lipschitz on bounded sets and such that* $\sup_{x\in X} |f(x)-g(x)| \le \varepsilon$*.*

This follows from [C, Corollary 3].

If we put no assumptions on the smoothness of the source space, we obtain only a uniformly Gâteaux differentiable approximation.

Theorem E (Johanis, [J])**.** *Let* X *be a separable Banach space,* Y *a Banach space,* $f\colon X \to Y$ *be an* L*-Lipschitz mapping and* $\varepsilon > 0$*. Then there is a mapping* $g\colon X \to Y$ *which is* L*-Lipschitz, uniformly Gâteaux differentiable, and* $\sup_{x\in X} \|f(x)-g(x)\| \le \varepsilon$*.*

The following theorem gives smooth approximations of bounded Lipschitz functions.

Theorem F (Fry)**.** *Let* X *be a separable normed linear space that admits a* C^k*-smooth Lipschitz bump function,* $k \in \mathbb{N}\cup\{\infty\}$*. For each* $\varepsilon > 0$ *there is a constant* $K \in \mathbb{R}$ *such that if* $f\colon X \to [0,1]$ *is* 1*-Lipschitz, then there is a* K*-Lipschitz function* $g \in C^k(X)$*, such that* $\sup_{x\in X} |f(x)-g(x)| \le \varepsilon$*.*

By obvious adjustments of the proof of [F1, Theorem 1] we obtain this more general Theorem F, see also the proof of Theorem 3, (i)⇒(ii). We note that the subsequent attempt to generalise Theorem F for WCG spaces in [F3] appears to be seriously flawed and it is unknown at present if the result holds.

Finally, there is a recent result on approximation of Lipschitz (or more generally uniformly continuous) mappings on $c_0(\Gamma)$.

Theorem G (Hájek-Johanis)**.** *Let* Γ *be an arbitrary set,* Y *be a Banach space,* $M \subset c_0(\Gamma)$*,* $U \subset c_0(\Gamma)$ *be a uniform neighbourhood of* M*,* $f\colon U \to Y$ *be a uniformly continuous mapping with modulus of continuity* ω *and let* $\varepsilon > 0$*. Then there is a mapping* $g \in C^\infty(c_0(\Gamma), Y)$ *which locally depends on finitely many coordinates such that* $\sup_M \|f(x)-g(x)\| \le \varepsilon$ *and* g *is uniformly continuous on* M *with modulus of continuity dominated by* ω*. In particular, if* f *is* L*-Lipschitz, then* g *is* L*-Lipschitz on* M*.*

This stronger version of [HJ, Theorem 1] follows by not very difficult modification of the proof.

If a uniformly continuous mapping $f\colon X \to Y$ is uniformly Gâteaux differentiable, then the mappings $x \mapsto f'(x)h$ are uniformly continuous on X (see e.g. [HJ, Lemma 4]). Thus combining Theorem E and Theorem B we immediately obtain the following corollary:

Theorem H. *Let* X *be a Banach space with an unconditional Schauder basis that admits a* C^k*-smooth Lipschitz bump function. There is a constant* $C > 0$ *such that if* Y *is a Banach space,* $f\colon X \to Y$ *an* L*-Lipschitz mapping, and* $\varepsilon > 0$*, then there is a* $C(1+\varepsilon)L$*-Lipschitz mapping* $g \in C^k(X,Y)$ *such that* $\|f(x)-g(x)\| < \varepsilon$ *for all* $x \in X$*.*

This result was first stated by R. Fry in [F2]. His proof (as well as the corrigendum) is however not correct.

2. Approximation of functions and embeddings into $c_0(\Gamma)$

First, although not directly related to our results, we show the following observation, which basically says that to approximate Lipschitz functions it only suffices to consider approximation of *bounded* functions, and moreover we gain control over the Lipschitz constant of the approximation.

Proposition 1. *Let* $k \in \mathbb{N}\cup\{\infty\}$ *and* X *be a normed linear space with the following property: There is a* $C \in \mathbb{R}$ *such that for each* $A \subset X$ *there is a* C*-Lipschitz function* $h_A \in C^k(X,[0,1])$ *satisfying* $h_A(x) = 0$ *for all* $x \in A$ *and* $h_A(x) = 1$ *for all* $x \in X$ *such that* $\operatorname{dist}(x,A) \ge 1$*.*

Then for each $\varepsilon > 0$ *and an arbitrary* L*-Lipschitz function* $f\colon X \to \mathbb{R}$ *there is a* CL*-Lipschitz function* $g \in C^k(X)$ *such that* $|g(x)-f(x)| \le \varepsilon$ *for each* $x \in X$*.*

Proof. Let us define a function $\tilde f\colon X \to \mathbb{R}$ by $\tilde f(x) = \frac{1}{\varepsilon} f(\frac{\varepsilon}{L}x)$. This function is obviously 1-Lipschitz. Next, let us define sets $A_n = \{x \in X\colon \tilde f(x) \ge n\}$ for $n \in \mathbb{Z}$. Clearly, $A_{n+1} \subset A_n$ for all $n \in \mathbb{Z}$, and using the 1-Lipschitz property of $\tilde f$ it is easy to check that

$$\operatorname{dist}(X \setminus A_n, A_{n+1}) \ge 1 \quad \text{for all } n \in \mathbb{Z}. \tag{1}$$

Further, denote $h_n(x) = 1 - h_{A_{n+1}}(x)$ for $n \in \mathbb{Z}$. For each $n \in \mathbb{Z}$, $h_n \in C^k(X, [0,1])$, h_n is C-Lipschitz, $h_n(x) = 1$ for all $x \in A_{n+1}$ and, by (1), $h_n(x) = 0$ for all $x \in X \setminus A_n$.

Now, put

$$h(x) = \sum_{n=0}^{\infty} h_n(x) - \sum_{n=-\infty}^{-1} \big(1 - h_n(x)\big). \tag{2}$$

Fix an arbitrary $x \in X$. Then there is $m \in \mathbb{Z}$ such that $x \in A_m \setminus A_{m+1}$. It follows, that $h_n(x) = 0$ for all $n > m$ and $h_n(x) = 1$ for all $n < m$. Hence (2) defines a function $h \colon X \to \mathbb{R}$. Moreover, by (1), the sums in (2) are even locally finite, therefore $h \in C^k(X)$. Further, it is easy to check that $h(x) = m + h_m(x)$. This implies that $h(x) \in [m, m+1]$, while $\tilde{f}(x) \in [m, m+1)$ and hence $\big|h(x) - \tilde{f}(x)\big| \leq 1$.

It remains to show that h is C-Lipschitz. To this end, choose $x, y \in X$ and find $n, l \in \mathbb{Z}$ such that $x \in A_n \setminus A_{n+1}$ and $y \in A_{n+l} \setminus A_{n+l+1}$. Without loss of generality we may assume that $l \geq 0$. If $l = 0$, then clearly $|h(x) - h(y)| = |n + h_n(x) - n - h_n(y)| \leq C\,\|x - y\|$.

We prove the case $l > 0$ by induction on l. As a first step of the induction assume that $l = 1$. Denote by $[x, y]$ the line segment between the points x and y. Since $[x, y]$ is connected, there is a point $z \in [x, y] \cap A_{n+1} \cap \overline{(X \setminus A_{n+1})}$. From the properties of h_n and h_{n+1} and from the continuity of h_{n+1} it follows that $h_n(z) = 1$ and $h_{n+1}(z) = 0$. Thus

$$\begin{aligned} |h(y) - h(x)| &= |n + 1 + h_{n+1}(y) - n - h_n(x)| = |h_{n+1}(y) + 1 - h_n(x)| = |h_{n+1}(y) - h_{n+1}(z) + h_n(z) - h_n(x)| \\ &\leq |h_{n+1}(y) - h_{n+1}(z)| + |h_n(z) - h_n(x)| \leq C\,\|y - z\| + C\,\|z - x\| = C\,\|y - x\|\,. \end{aligned}$$

To prove the general induction step assume that $l > 1$. By the continuity of $\tilde{f}$ there is $z \in [x, y]$ such that $z \in A_{n+1} \setminus A_{n+2}$. Using the induction hypothesis on the pair x, z and again on the pair z, y we obtain $|h(x) - h(y)| \leq |h(x) - h(z)| + |h(z) - h(y)| \leq C\,\|x - z\| + C\,\|z - y\| = C\,\|x - y\|$.

Finally, let $g(x) = \varepsilon h(\frac{L}{\varepsilon} x)$. It is straightforward to check that g satisfies the conclusion of our theorem.

□

Combining Proposition 1 and Theorem F we would obtain a smooth approximation of Lipschitz functions on smooth separable normed linear spaces. However, we skip the details, since we will show more, see Corollary 8.

In the sequel we will be using smooth bi-Lipschitz homeomorphisms into $c_0(\Gamma)$. The following two results show how they can be constructed and how they are related to smooth approximation of Lipschitz functions. First we define some notions useful in this context.

For a metric space P, we denote $\mathcal{U}(r) = \{U(x, r);\ x \in P\}$. For a real function f we denote $\operatorname{supp} f = f^{-1}(\mathbb{R} \setminus \{0\})$.

Let X be a set. A collection $\{\psi_\alpha\}_{\alpha \in \Lambda}$ of functions on X is called a *sup-partition of unity* if

- $\psi_\alpha \colon X \to [0, 1]$ for all $\alpha \in \Lambda$,
- for each $x \in X$ the set $\{\alpha \in \Lambda \colon \psi_\alpha(x) > 0\}$ is finite,
- for each $x \in X$ there is $\alpha \in \Lambda$ such that $\psi_\alpha(x) = 1$.

Let $\mathcal{U}$ be a covering of X. We say that the sup-partition of unity $\{\psi_\alpha\}_{\alpha \in \Lambda}$ is subordinated to $\mathcal{U}$ if $\{\operatorname{supp} \psi_\alpha\}_{\alpha \in \Lambda}$ refines $\mathcal{U}$.

Fact 2. *Let Γ be an infinite set, $r > 0$ and $0 < \delta < \frac{r}{2}$. There is an open point-finite uniform refinement $\mathcal{V} = \{V_\gamma\}_{\gamma \in \Gamma}$ of the uniform covering $\mathcal{U}(r)$ of $c_0(\Gamma)$ such that $\mathcal{U}(\frac{r}{2} - \delta)$ refines $\mathcal{V}$. Moreover, $\mathcal{V}$ is formed by the translates of the open ball $U(0, r - \delta)$. Further, there is a C^∞-smooth, locally dependent on finitely many coordinate functionals, and $(\frac{2}{r} + \delta)$-Lipschitz sup-partition of unity $\{\psi_\gamma\}_{\gamma \in \Gamma}$ on $c_0(\Gamma)$ subordinated to $\mathcal{U}(r)$.*

The first part of this fact was already shown in [P, Proposition 2.3], but with more complicated proof.

Proof. Notice that, by homogeneity, it suffices to prove all the statements only for $r = 1$.

Let $\{a_\gamma\}_{\gamma \in \Gamma}$ be the set of all vectors in $c_0(\Gamma)$ with coordinates in $\mathbb{Z}$. (Notice that the cardinality of such set is $|\Gamma|$ and so we may index its points by Γ.) We claim that $\mathcal{V} = \{U(a_\gamma, 1 - \delta)\}_{\gamma \in \Gamma}$ is the desired refinement.

Clearly, $\mathcal{V}$ is an open refinement of $\mathcal{U}(1)$. To see that it is point-finite, pick any $x \in c_0(\Gamma)$ and find a finite $F \subset \Gamma$ such that $|x(\gamma)| < \delta$ whenever $\gamma \in \Gamma \setminus F$. Suppose that $\alpha \in \Gamma$ is such that $x \in U(a_\alpha, 1 - \delta)$. Then for $\gamma \notin F$, $|a_\alpha(\gamma)| \leq |a_\alpha(\gamma) - x(\gamma)| + |x(\gamma)| < 1$ and so $a_\alpha(\gamma) = 0$. From $|x(\gamma) - a_\alpha(\gamma)| < 1 - \delta$ and $a_\alpha(\gamma) \in \mathbb{Z}$ it follows that there are at most two possibilities for $a_\alpha(\gamma)$ for each $\gamma \in F$. From this we can conclude that $\big|\{\alpha \colon x \in U(a_\alpha, 1 - \delta)\}\big| \leq 2^{|F|}$.

Finally, we show that $\mathcal{U}(\frac{1}{2} - \delta)$ refines $\mathcal{V}$. Choose any $x \in c_0(\Gamma)$ and find $\beta \in \Gamma$ such that $\|x - a_\beta\| \leq \frac{1}{2}$. This is always possible, since there is a finite $F \subset \Gamma$ such that $|x(\gamma)| < \frac{1}{2}$ whenever $\gamma \notin F$, and so $a_\beta(\gamma) = 0$ for such γ. Suppose $z \in U(x, \frac{1}{2} - \delta)$. Then $\|a_\beta - z\| \leq \|a_\beta - x\| + \|x - z\| < \frac{1}{2} + \frac{1}{2} - \delta = 1 - \delta$, which implies $U(x, \frac{1}{2} - \delta) \subset U(a_\beta, 1 - \delta)$.

To construct the sup-partition of unity subordinated to $\mathcal{U}(1)$, find $\varepsilon > 0$ and $0 < \eta < \frac{1}{2}$ such that $0 < 1/\big(1 - \eta - \frac{1+\varepsilon}{2}\big) < 2 + \frac{\delta}{4}$ and $(1 + \varepsilon)\big(2 + \frac{\delta}{2}\big) \leq 2 + \delta$. Let $\mathcal{W} = \{U(a_\gamma, 1 - \eta)\}_{\gamma \in \Gamma}$ be the point-finite refinement of $\mathcal{U}(1)$ from the first part of the proof such that $\mathcal{U}(\frac{1}{2} - \eta)$ refines $\mathcal{W}$. Further, let $\|\cdot\|$ be an equivalent C^∞-smooth norm $\|\cdot\|$ on $c_0(\Gamma)$ which locally depends on finitely many of the coordinate functionals $\{e^*_\gamma\}_{\gamma \in \Gamma}$ (away from the origin) and such that $\|x\|_\infty \leq \|x\| \leq (1+\varepsilon)\,\|x\|_\infty$ for all $x \in c_0(\Gamma)$. (To construct such a norm, take for example the Minkowski functional of the set $\{x \in c_0(\Gamma)\colon \sum_{\gamma\in\Gamma} \varphi(x_\gamma) \le 1\}$, where $\varphi \in C^\infty(\mathbb{R})$, φ is convex and even, $\varphi(1) = 1$, and $\varphi(t) = 0$ for $t \in [-\frac{1}{1+\varepsilon}, \frac{1}{1+\varepsilon}]$.)

For each $\gamma \in \Gamma$ we put $\psi_\gamma(x) = q\big(\|x - a_\gamma\|\big)$, where $q \in C^\infty(\mathbb{R}, [0,1])$, q is $\big(2 + \frac{\delta}{2}\big)$-Lipschitz, $q(t) = 0$ for $t \ge 1 - \eta$, and $q(t) = 1$ for $t \le \frac{1+\varepsilon}{2}$. The collection $\{\psi_\gamma\}_{\gamma\in\Gamma}$ is a sup-partition of unity. Indeed, it is easy to see, that $\operatorname{supp}\psi_\gamma \subset U(a_\gamma, 1-\eta)$ for each $\gamma \in \Gamma$, and consequently the set $\{\gamma \in \Gamma\colon \psi_\gamma(x) > 0\}$ is finite for each $x \in X$. Further, fix any $x \in X$. There is an $\alpha \in \Gamma$ such that $U(x, \frac{1}{2} - \eta) \subset U(a_\alpha, 1 - \eta)$, which gives $\|x - a_\alpha\|_\infty \le \frac{1}{2}$. Hence $\|x - a_\alpha\| \le (1+\varepsilon)\|x - a_\alpha\|_\infty \le \frac{1+\varepsilon}{2}$, which in turn implies $\psi_\alpha(x) = 1$.

As the function q is $\big(2 + \frac{\delta}{2}\big)$-Lipschitz and the function $\|\cdot\|$ is $(1+\varepsilon)$-Lipschitz (with respect to the norm $\|\cdot\|_\infty$), the functions ψ_γ are $(2+\delta)$-Lipschitz according to the choice of ε. The rest of the properties of the functions ψ_γ is obvious.

□

Theorem 3. *Let X be a normed linear space, Γ an infinite set, and $k \in \mathbb{N}\cup\{0,\infty\}$. Then the following are equivalent:*

(i) There is $M \in \mathbb{R}$ such that there is a C^k-smooth and M-Lipschitz sup-partition of unity $\{\phi_\gamma\}_{\gamma\in\Gamma}$ on X subordinated to $\mathcal{U}(1)$.

(ii) X is uniformly homeomorphic to a subset of $c_0(\Gamma)$ and for each $\varepsilon > 0$ there is $K > 0$ such that for each 1-Lipschitz function $f\colon X \to [0,1]$ there is a K-Lipschitz function $g \in C^k(X)$ such that $\sup_{x\in X} |g(x) - f(x)| \le \varepsilon$.

(iii) There is a bi-Lipschitz homeomorphism $\varphi\colon X \to c_0(\Gamma)$ such that the coordinate functions $e^*_\gamma \circ \varphi \in C^k(X)$ for every $\gamma \in \Gamma$.

Proof. First we show that (i) implies (iii). From the properties of the sup-partition of unity there is $\beta \in \Gamma$ such that $\phi_\beta(0) = 1$. By scaling and composing ϕ_β with a suitable function we construct a C-Lipschitz function $h \in C^k(X, [0,1])$ such that $h = 0$ on $B(0, r)$ and $h = 1$ outside $U(0,1)$ for some constants $C, r \in \mathbb{R}$, $r > 0$. (We may for example choose r such that $1 - 2Mr > 0$ and take $h(x) = q(\phi_\beta(2x))$, where $q \in C^k(\mathbb{R})$, q is Lipschitz, $q([0,1]) = [0,1]$, $q(0) = 1$, and $q(s) = 0$ for $s \ge 1 - 2Mr$.)

Choose $t > 1$ and for each $n \in \mathbb{Z}$ and $\gamma \in \Gamma$ define functions $\phi^n_\gamma \in C^k(X)$ by

$$\phi^n_\gamma(x) = t^n \phi_\gamma\left(\frac{x}{t^n}\right) h\left(\frac{x}{t^n}\right).$$

The properties of the functions ϕ_γ and h guarantee that each ϕ^n_γ is $(M+C)$-Lipschitz. Let $d\colon \mathbb{Z}\times\Gamma \to \Gamma$ be some one-to-one mapping and define $\varphi\colon X \to \mathbb{R}^\Gamma$ by $\varphi(x)_\alpha = \phi^n_\gamma(x)$ if $\alpha = d(n,\gamma)$ for some $n \in \mathbb{Z}$, $\gamma \in \Gamma$; $\varphi(x)_\alpha = 0$ otherwise.

We show that φ actually maps into $c_0(\Gamma)$. Choose an arbitrary $x \in X$ and $\varepsilon > 0$. There is $n_0 \in \mathbb{Z}$ such that $t^n < \varepsilon$ for all $n < n_0$ and $n_1 \in \mathbb{Z}$ such that $\|x\| \le rt^n$ for all $n > n_1$. It follows that $\left|\phi^n_\gamma(x)\right| < \varepsilon$ for all $n < n_0$ and $\gamma \in \Gamma$, and, by the properties of h, $\phi^n_\gamma(x) = 0$ for all $n > n_1$ and $\gamma \in \Gamma$. As for each $n_0 \le n \le n_1$, $\phi_\gamma(x/t^n) \ne 0$ only for finitely many $\gamma \in \Gamma$, we can conclude that $\varphi\colon X \to c_0(\Gamma)$.

Since each ϕ^n_γ is $(M+C)$-Lipschitz, the mapping φ is $(M+C)$-Lipschitz as well.

To prove that φ is one-to-one and φ^{-1} is Lipschitz too, choose any two points $x, y \in X$, $x \ne y$, and find $m \in Z$ such that $2t^m \le \|x - y\| < 2t^{m+1}$. Without loss of generality we may assume that $\|x\| \ge t^m$. Then $h(x/t^m) = 1$ and so there is $\gamma \in \Gamma$ such that $\phi^m_\gamma(x) = t^m$. Now suppose there is $z \in X$ such that $\phi^m_\gamma(z) > 0$. As $\operatorname{supp}\phi_\gamma \subset U(w,1)$ for some $w \in X$, $\left\|\frac{x}{t^m} - \frac{z}{t^m}\right\| < 2$ and consequently $\|x - z\| < 2t^m$. But this means that $\phi^m_\gamma(y) = 0$ and therefore

$$\|\varphi(x) - \varphi(y)\|_\infty \ge \left|\phi^m_\gamma(x) - \phi^m_\gamma(y)\right| = \phi^m_\gamma(x) = t^m > \frac{1}{2t}\|x - y\|.$$

(iii)⇒(i): Let $A, B \in \mathbb{R}$ are such that $A\|x - y\| \le \|\varphi(x) - \varphi(y)\|_\infty \le B\|x - y\|$. By Fact 2, there is a $C > 0$ and a C^∞-smooth, locally dependent on finitely many coordinate functionals, and C-Lipschitz sup-partition of unity $\{\psi_\gamma\}_{\gamma\in\Gamma}$ on $c_0(\Gamma)$ subordinated to $\mathcal{U}(A)$. Putting $\phi_\gamma = \psi_\gamma \circ \varphi$, $\{\phi_\gamma\}_{\gamma\in\Gamma}$ is a BC-Lipschitz sup-partition of unity subordinated to $\mathcal{U}(1)$. Fix $\gamma \in \Gamma$. To see that $\phi_\gamma \in C^k(X)$, pick any $x \in X$. There is a neighbourhood V of $\varphi(x)$ such that $\psi_\gamma(w) = G(f_1(w), \ldots, f_n(w))$ for each $w \in V$, where $f_1, \ldots, f_n \in \{e^*_\gamma\}_{\gamma\in\Gamma}$ and $G \in C^\infty(\Omega)$ for some $\Omega \subset \mathbb{R}^n$ open. Let U be an open neighbourhood of x such that $\varphi(U) \subset V$. Then $\phi_\gamma(y) = \psi_\gamma(\varphi(y)) = G(f_1(\varphi(y)), \ldots, f_n(\varphi(y)))$ for each $y \in U$. Since, by the assumption, $f_i \circ \varphi \in C^k(X)$ for each $i = 1, \ldots, n$, and $G \in C^\infty(\Omega)$, ϕ_γ is C^k-smooth on U.

(i)⇒(ii): We already know that (iii) holds and from this the first part of (ii) follows immediately. To prove the second part of (ii), let $\varepsilon > 0$. The basic idea of the proof is that Lipschitz functions are stable under the operation of pointwise supremum. To preserve the smoothness, we will use a "smoothened supremum", or an equivalent smooth norm on $c_0(\Gamma)$. Let $\|\cdot\|$ be an equivalent C^∞-smooth norm on $c_0(\Gamma)$ which locally depends on finitely many of the coordinate functionals $\{e^*_\gamma\}_{\gamma\in\Gamma}$ (away from the origin), and let $C > 0$ be such that $\|x\|_\infty \le \|x\| \le C\|x\|_\infty$ for all $x \in c_0(\Gamma)$ (see the proof of Fact 2). We will show, that $K = 4C^3M/\varepsilon$ satisfies our claim.

By adding the constant 1 we may and do assume that f maps into $[1,2]$. Put $\delta = \frac{\varepsilon}{C}$ and $\psi_\gamma(x) = \phi_\gamma(\frac{x}{\delta})$ for all $x \in X$, $\gamma \in \Gamma$. It follows, that $\{\psi_\gamma\}_{\gamma\in\Gamma}$ is a C^k-smooth and M/δ-Lipschitz sup-partition of unity subordinated to $\mathcal{U}(\delta)$. Since the sets $\{\gamma \in \Gamma\colon \psi_\gamma(x) > 0\}$ are finite, $(\psi_\gamma(x))_{\gamma\in\Gamma} \in c_0(\Gamma)$ for each $x \in X$. For each $\gamma \in \Gamma$ there is a point $x_\gamma \in X$

such that $\operatorname{supp}\psi_\gamma \subset U(x_\gamma,\delta)$. The boundedness of the function f guarantees that also $\big(f(x_\gamma)\psi_\gamma(x)\big)_{\gamma\in\Gamma} \in c_0(\Gamma)$ for each $x \in X$. Therefore we can define the function $g\colon X \to \mathbb{R}$ by

$$g(x) = \frac{\left\|\big(f(x_\gamma)\psi_\gamma(x)\big)_{\gamma\in\Gamma}\right\|}{\left\|\big(\psi_\gamma(x)\big)_{\gamma\in\Gamma}\right\|}.$$

As

$$\|(\psi_\gamma(x))\| \geq \|(\psi_\gamma(x))\|_\infty = \sup_{\gamma\in\Gamma}\psi_\gamma(x) = 1 \quad \text{for each } x \in X, \tag{3}$$

the function g is well defined on all of X.

The mapping $x \mapsto \big(\psi_\gamma(x)\big)$ and, by the boundedness of f, also the mapping $x \mapsto \big(f(x_\gamma)\psi_\gamma(x)\big)$ are Lipschitz mappings from X into $c_0(\Gamma)\setminus U(0,1)$. (Notice that for each $x\in X$ there is $\gamma\in\Gamma$ such that $\psi_\gamma(x)=1$ and $f(x_\gamma)\psi_\gamma(x)\geq 1$.) Since $\|\cdot\|$ is C^∞-smooth and depends locally on finitely many coordinates away from the origin, and since $\psi_\gamma \in C^k(X)$ and $f(x_\gamma)\psi_\gamma \in C^k(X)$ for each $\gamma\in\Gamma$, similarly as in the proof of (iii)⇒(i) we infer that $g \in C^k(X)$.

To see that the function g is K-Lipschitz, choose any two points $x, y \in X$. Then, using (3) and the facts that ψ_γ maps into $[0,1]$, f maps into $[1,2]$, and ψ_γ are M/δ-Lipschitz, we can estimate

$$\begin{aligned}
|g(x)-g(y)| &= \frac{\Big|\left\|\big(f(x_\gamma)\psi_\gamma(x)\big)\right\|\left\|\big(\psi_\gamma(y)\big)\right\| - \left\|\big(f(x_\gamma)\psi_\gamma(y)\big)\right\|\left\|\big(\psi_\gamma(x)\big)\right\|\Big|}{\left\|\big(\psi_\gamma(x)\big)\right\|\left\|\big(\psi_\gamma(y)\big)\right\|} \\
&\leq \Big|\left\|\big(f(x_\gamma)\psi_\gamma(x)\big)\right\|\left\|\big(\psi_\gamma(y)\big)\right\| - \left\|\big(f(x_\gamma)\psi_\gamma(y)\big)\right\|\left\|\big(\psi_\gamma(x)\big)\right\|\Big| \\
&\leq \left\|\big(\psi_\gamma(y)\big)\right\|\Big|\left\|\big(f(x_\gamma)\psi_\gamma(x)\big)\right\| - \left\|\big(f(x_\gamma)\psi_\gamma(y)\big)\right\|\Big| + \left\|\big(f(x_\gamma)\psi_\gamma(y)\big)\right\|\Big|\left\|\big(\psi_\gamma(y)\big)\right\| - \left\|\big(\psi_\gamma(x)\big)\right\|\Big| \\
&\leq C\left\|\big(f(x_\gamma)(\psi_\gamma(x)-\psi_\gamma(y))\big)\right\| + 2C\left\|\big(\psi_\gamma(y)-\psi_\gamma(x)\big)\right\| \\
&\leq C^2\left\|\big(f(x_\gamma)(\psi_\gamma(x)-\psi_\gamma(y))\big)\right\|_\infty + 2C^2\left\|\big(\psi_\gamma(y)-\psi_\gamma(x)\big)\right\|_\infty \\
&\leq 4C^2\sup_{\gamma\in\Gamma}|\psi_\gamma(y)-\psi_\gamma(x)| \leq 4C^2\frac{M}{\delta}\|x-y\| = K\|x-y\|.
\end{aligned}$$

Finally, to show that g approximates f, choose an arbitrary $x\in X$. Applying successively the inequality (3) and the facts that $\operatorname{supp}\psi_\gamma \subset U(x_\gamma,\delta)$ and f is 1-Lipschitz, we obtain

$$\begin{aligned}
|g(x)-f(x)| &= \left|\frac{\left\|\big(f(x_\gamma)\psi_\gamma(x)\big)\right\|}{\left\|\big(\psi_\gamma(x)\big)\right\|} - f(x)\frac{\left\|\big(\psi_\gamma(x)\big)\right\|}{\left\|\big(\psi_\gamma(x)\big)\right\|}\right| \leq \frac{\left\|\big((f(x_\gamma)-f(x))\psi_\gamma(x)\big)\right\|}{\left\|\big(\psi_\gamma(x)\big)\right\|} \leq C\left\|\big((f(x_\gamma)-f(x))\psi_\gamma(x)\big)\right\|_\infty \\
&= C\sup_{\gamma\in\Gamma}\big\{|f(x_\gamma)-f(x)|\,\psi_\gamma(x)\big\} = C\sup_{\substack{\gamma\in\Gamma\\ x\in U(x_\gamma,\delta)}}\big\{|f(x_\gamma)-f(x)|\,\psi_\gamma(x)\big\} \leq C\sup_{\substack{\gamma\in\Gamma\\ x\in U(x_\gamma,\delta)}}\big\{\|x_\gamma - x\|\big\} < C\delta = \varepsilon.
\end{aligned}$$

(ii)⇒(i): It is not difficult to construct a point-finite base of the uniform coverings of $c_0(\Gamma)$ and pull it back onto X via the uniform homeomorphism (cf. [P, Proposition 2.3]). So let $\mathcal{V} = \{V_\gamma\}_{\gamma\in\Gamma}$ be an open point-finite uniform refinement of the covering $\mathcal{U}(1)$ of X. (We note that such refinement can be chosen so that $|\mathcal{V}| = |\Gamma|$ and so we can indeed index it by Γ.) Let $0<\delta\leq 1$ be such that $\mathcal{U}(\delta)$ refines $\mathcal{V}$. For each $\gamma\in\Gamma$ we define the function $f_\gamma\colon X\to[0,1]$ by $f_\gamma(x) = \min\{\operatorname{dist}(x, X\setminus V_\gamma), \delta\}$.

Choose some $0<\theta<\frac{\delta}{2}$. For each $\gamma\in\Gamma$, the function f_γ is 1-Lipschitz and so, by (ii), there is a K-Lipschitz function $g_\gamma \in C^k(X)$ such that $\sup_{x\in X}|g_\gamma(x)-f_\gamma(x)| \leq \theta$. Let $q\in C^k(\mathbb{R},[0,1])$ be a C-Lipschitz function for some $C\in\mathbb{R}$, such that $q(t)=0$ for $t\leq\theta$ and $q(t)=1$ for $t\geq\delta-\theta$. Finally, we let $\phi_\gamma(x) = q(g_\gamma(x))$ for each $\gamma\in\Gamma$. Clearly, each function ϕ_γ belongs to $C^k(X,[0,1])$ and is M-Lipschitz, where $M = CK$. Further, for any $x\in X$ there is $\alpha\in\Gamma$ such that $U(x,\delta)\subset V_\alpha$, hence $f_\alpha(x)=\delta$ and consequently $\phi_\alpha(x)=1$. As $\operatorname{supp}\phi_\gamma\subset V_\gamma$ for all $\gamma\in\Gamma$ and $\mathcal{V}$ is point-finite, $\{\phi_\gamma\}_{\gamma\in\Gamma}$ is a sup-partition of unity subordinated to $\mathcal{U}(1)$.

□

We note, that the proof could be made considerably shorter by proving (iii)⇒(ii) directly using Theorem G (see the proof of Theorem 7) instead of (i)⇒(ii) and (iii)⇒(i). However, the reasons for our strategy of the proof were two: First, we do not need the full generality (and associated machinery) of Theorem G and second, the proof of (i)⇒(ii) shows an interesting technique for constructing smooth Lipschitz approximations (due to Fry, [F1]), and in fact shows the reason for the definition of the notion of sup-partition of unity.

Corollary 4. *Let X be a separable normed linear space that admits a C^k-smooth Lipschitz bump function, $k\in\mathbb{N}\cup\{\infty\}$. Then there is a bi-Lipschitz homeomorphism $\varphi\colon X\to c_0$ such that the coordinate functions $e_j^*\circ\varphi\in C^k(X)$ for every $j\in\mathbb{N}$.*

Proof. Fry in [F1] has constructed a C^k-smooth M-Lipschitz sup-partition of unity $\{\psi_j\}_{j=1}^\infty$ on X that is subordinated to $\mathcal{U}(1)$, so Theorem 3 applies.

□

3. Approximation of mappings

To be able to use Theorem G, we need to "extend" Lipschitz mappings from subsets of $c_0(\Gamma)$. To this end we introduce some additional notions.

Let (X,ρ) be a metric space, $A\subset X$. For $\varepsilon>0$, a mapping $r_\varepsilon\colon X\to A$ such that $\rho(r_\varepsilon(x),x)<\varepsilon$ for each $x\in A$ is called an *ε-retraction.*

A is called a *Lipschitz approximate retract* (LAR), if there is $K>0$ such that for any $\varepsilon>0$ there is a K-Lipschitz ε-retraction of X into A. A is called a *Lipschitz approximate uniform neighbourhood retract* (LAUNR), if there is $K>0$ such that for any $\varepsilon>0$ there is a uniform open neighbourhood $U\subset X$ of A and a K-Lipschitz ε-retraction of U into A.

A metric space is called an *absolute Lipschitz approximate uniform neighbourhood retract* (ALAUNR) if it is a LAUNR of every metric space containing it as a subspace.

Example. Let X be a Banach space with an unconditional basis $\{e_n\}_{n=1}^\infty$ and let $X_\infty=\operatorname{span}\{e_n\}_{n=1}^\infty$ be its linear subspace consisting of finitely supported vectors. Then X_∞ is a Lipschitz approximate retract of X.

Indeed, let $C=2\operatorname{ubc}\{e_n\}$ and $D=\operatorname{bc}\{e_n\}$, put $K=C(5+4D)$ and choose an arbitrary $\varepsilon>0$. Let $\varphi\colon\mathbb{R}\to[0,1]$ be defined as $\varphi(t)=0$ for $t\le\varepsilon/(2C)$, $\varphi(t)=1$ for $t\ge\varepsilon/C$, and φ is affine on $[\varepsilon/(2C),\varepsilon/C]$. Notice that φ is $2C/\varepsilon$-Lipschitz. Denote $R_n=I-P_n$. Define the ε-retraction $r\colon X\to X_\infty$ by $r(x)=\sum_{n=1}^\infty\varphi(\|R_{n-1}x\|)x_ne_n$. We claim that $\|x-r(x)\|<\varepsilon$ for all $x\in X$. To see this, fix $x\in X$ and find $n_0\in\mathbb{N}\cup\{0\}$ such that $\|R_{n_0}x\|<\varepsilon/C$ and $\|R_nx\|\ge\varepsilon/C$ for all $0\le n<n_0$. Then

$$\begin{aligned}\|x-r(x)\|&=\left\|\sum_{n=1}^\infty\big(1-\varphi(\|R_{n-1}x\|)\big)x_ne_n\right\|=\left\|\sum_{n>n_0}\big(1-\varphi(\|R_{n-1}x\|)\big)x_ne_n\right\|\\&\le C\left\|\sum_{n>n_0}x_ne_n\right\|=C\,\|R_{n_0}x\|<C\frac{\varepsilon}{C}=\varepsilon.\end{aligned}$$

To show that r is K-Lipschitz, choose any $x,y\in X$. We may without loss of generality assume that $\|x-y\|\le\varepsilon/(C(1+D))$. (It is an easy fact, that mappings on normed linear spaces that are Lipschitz on short distances are Lipschitz globally with the same Lipschitz constant.) Find $n_0\in\mathbb{N}\cup\{0\}$ such that $\|R_{n_0}y\|<2\varepsilon/C$ and $\|R_ny\|\ge2\varepsilon/C$ for all $0\le n<n_0$. Then $\|R_nx\|\ge\|R_ny\|-\|R_n(x-y)\|\ge2\varepsilon/C-(1+D)\varepsilon/(C(1+D))=\varepsilon/C$ for all $0\le n<n_0$. It follows that $\varphi(\|R_nx\|)=\varphi(\|R_ny\|)=1$ for all $0\le n<n_0$. Using this fact, we can estimate

$$\begin{aligned}\|r(x)-r(y)\|&=\left\|\sum_{n=1}^\infty\big(\varphi(\|R_{n-1}x\|)x_n-\varphi(\|R_{n-1}y\|)y_n\big)e_n\right\|\\&\le\left\|\sum_{n=1}^\infty\varphi(\|R_{n-1}x\|)(x_n-y_n)e_n\right\|+\left\|\sum_{n=1}^\infty\big(\varphi(\|R_{n-1}x\|)-\varphi(\|R_{n-1}y\|)\big)y_ne_n\right\|\\&\le C\,\|x-y\|+\left\|\sum_{n>n_0}\big(\varphi(\|R_{n-1}x\|)-\varphi(\|R_{n-1}y\|)\big)y_ne_n\right\|\\&\le C\,\|x-y\|+C\sup_{n>n_0}\big|\varphi(\|R_{n-1}x\|)-\varphi(\|R_{n-1}y\|)\big|\left\|\sum_{n>n_0}y_ne_n\right\|\\&\le C\,\|x-y\|+C\frac{2C}{\varepsilon}\sup_{n>n_0}\big|\|R_{n-1}x\|-\|R_{n-1}y\|\big|\,\|R_{n_0}y\|\\&<C\,\|x-y\|+C\frac{2C}{\varepsilon}(1+D)\,\|x-y\|\,\frac{2\varepsilon}{C}=K\,\|x-y\|\,.\end{aligned}$$

The following proposition shows how the notion of ALAUNR relates to "approximate extensions" of Lipschitz mappings.

Proposition 5. *Let (X,ρ) be a metric space. The following are equivalent:*

(i) X is an ALAUNR.
(ii) There is $K>0$ such that X is an absolute K-Lipschitz approximate uniform neighbourhood retract (i.e. the Lipschitz constant K does not depend on the metric space which X is a subspace of).
(iii) There is $K>0$ such that for each $\varepsilon>0$ there is $\delta>0$ such that for any metric spaces $Q\subset P$ and every L-Lipschitz mapping $f\colon Q\to X$ there is $U\subset P$ a δ/L-uniform open neighbourhood of Q and a KL-Lipschitz mapping $g\colon U\to X$ such that $\rho(f(x),g(x))<\varepsilon$ for all $x\in Q$.
(iv) For any metric spaces $Q\subset P$ and every L-Lipschitz mapping $f\colon Q\to X$ there is $K>0$ such that for any $\varepsilon>0$ there is $U\subset P$ a uniform open neighbourhood of Q and a KL-Lipschitz mapping $g\colon U\to X$ such that $\rho(f(x),g(x))<\varepsilon$ for all $x\in Q$.

(v) *There is $K > 0$ such that for each $\varepsilon > 0$ there is $\delta > 0$ such that for any metric space P, $X \subset P$, there is $U \subset P$ a δ-uniform open neighbourhood of X such that for any metric space (Q, σ) and every L-Lipschitz mapping $f\colon X \to Q$ there is a KL-Lipschitz mapping $g\colon U \to Q$ such that $\sigma(f(x), g(x)) < \varepsilon$ for all $x \in X$.*

(vi) *For any metric spaces P and (Q, σ), $X \subset P$, and every L-Lipschitz mapping $f\colon X \to Q$ there is $K > 0$ such that for any $\varepsilon > 0$ there is $U \subset P$ a uniform open neighbourhood of X and a KL-Lipschitz mapping $g\colon U \to Q$ such that $\sigma(f(x), g(x)) < \varepsilon$ for all $x \in X$.*

Proof. (ii)⇒(i), (iii)⇒(iv), and (v)⇒(vi) are trivial.

(i)⇒(iii): Embed X isometrically into $\ell_\infty(\Gamma)$. There is $K > 0$ such that X is a K-Lipschitz approximate neighbourhood retract of $\ell_\infty(\Gamma)$. Choose $\varepsilon > 0$ and let $\delta > 0$ be such that there is a K-Lipschitz ε-retraction $r\colon V \to X$ for some δ-uniform open neighbourhood V of X in $\ell_\infty(\Gamma)$. Let $Q \subset P$ be metric spaces and $f\colon Q \to X$ be an L-Lipschitz mapping. Since $\ell_\infty(\Gamma)$ is an absolute Lipschitz retract, there is an L-Lipschitz extension $h\colon P \to \ell_\infty(\Gamma)$ of $f\colon Q \to X \subset \ell_\infty(\Gamma)$. Put $U = h^{-1}(V)$. Then U is open in P, and it is a δ/L-uniform neighbourhood of Q. Indeed, if $y \in U(z, \delta/L)$ for some $z \in Q$, then $h(y) \in U(h(z), \delta)$, where $h(z) \in X$; hence $h(y) \in V$. Finally, put $g(x) = r(h(x))$ for any $x \in U$. Then $\rho(f(x), g(x)) = \rho\big(f(x), r(h(x))\big) = \rho\big(f(x), r(f(x))\big) < \varepsilon$ whenever $x \in Q$.

(iii)⇒(ii), (v)⇒(ii) (and (iv)⇒(i), (vi)⇒(i) similarly): Let X be a subspace of a metric space P, we put $Q = X$ and $f = \mathrm{id}$. For any $\varepsilon > 0$, the K-Lipschitz mapping g is the desired retraction r_ε.

(ii)⇒(v): Let $\varepsilon > 0$ and $r\colon U \to X$ be the K-Lipschitz ε/L-retraction from some δ-uniform neighbourhood U of X. Put $g = f \circ r$. Then $\sigma(f(x), g(x)) = \sigma\big(f(x), f(r(x))\big) \le L\rho(x, r(x)) < L\varepsilon/L = \varepsilon$ for any $x \in X$.

□

Corollary 6. *Let (X, ρ) be an ALAUNR.*

(a) If (Z, σ) is bi-Lipschitz homeomorphic to X, then Z is an ALAUNR.

(b) If Z is a LAUNR of X, then Z is an ALAUNR.

Proof. (a): Let $\varphi\colon Z \to X$ be a bi-Lipschitz homeomorphism such that $A\sigma(x, y) \le \rho(\varphi(x), \varphi(y)) \le B\sigma(x, y)$. We show that (iv) of Proposition 5 holds. Let $Q \subset P$ be metric spaces and $f\colon Q \to Z$ an L-Lipschitz mapping. Let $\tilde f\colon Q \to X$ be defined as $\tilde f = \varphi \circ f$ and let K_0 be the constant in Proposition 5(iv) for $\tilde f$. Put $K = K_0 B/A$. Choose any $\varepsilon > 0$. There is a uniform open neighbourhood $U \subset P$ of Q and a K_0BL-Lipschitz mapping $\tilde g\colon U \to X$ such that $\rho(\tilde f(x), \tilde g(x)) < A\varepsilon$ for all $x \in Q$. Then $g\colon U \to Z$, $g = \varphi^{-1} \circ \tilde g$ is a K_0BL/A-Lipschitz mapping such that $\sigma(f(x), g(x)) = \sigma\big(f(x), \varphi^{-1}(\tilde g(x))\big) = \sigma\big(\varphi^{-1}(\tilde f(x)), \varphi^{-1}(\tilde g(x))\big) \le (1/A)\rho(\tilde f(x), \tilde g(x)) < A\varepsilon/A = \varepsilon$ whenever $x \in Q$.

(b): Let K_0 be the Lipschitz constant of the ε-retractions into X (as X is ALAUNR) and K_1 be the Lipschitz constant of the ε-retractions from $U \subset X$ into Z. We show that (iv) of Proposition 5 holds. Let $Q \subset P$ be metric spaces and $f\colon Q \to Z \subset X$ an L-Lipschitz mapping. Put $K = K_1K_0$. Choose any $\varepsilon > 0$. There is a δ-uniform open neighbourhood $V \subset X$ of Z and a K_1-Lipschitz $(\varepsilon/2)$-retraction $r\colon V \to Z$. Further, there is an η-uniform open neighbourhood $W \subset P$ of Q and a K_0L-Lipschitz mapping $h\colon W \to X$ such that $\rho(f(x), h(x)) < \min\{\varepsilon/(2K_1), \delta/2\}$ for all $x \in Q$.

Let $U = h^{-1}(V)$. Then $U \subset W$ is open in W and hence in P, and it is a uniform neighbourhood of Q. Indeed, let $\zeta = \min\{\delta/(2K_0L), \eta\}$. If $y \in U(z, \zeta)$ for some $z \in Q$, then $y \in W$ and so $h(y) \in U(h(z), \delta/2)$. From this we obtain $h(y) \in U(f(z), \delta)$, and since $f(z) \in Z$, it follows that $h(y) \in V$.

Finally, put $g(x) = r(h(x))$ for any $x \in U$. Then $g\colon U \to Z$ is a K_1K_0L-Lipschitz mapping such that $\rho(f(x), g(x)) = \rho\big(f(x), r(h(x))\big) \le \rho\big(f(x), r(f(x))\big) + \rho\big(r(f(x), r(h(x))\big) < \varepsilon/2 + K_1\rho(f(x), h(x)) < \varepsilon$ whenever $x \in Q$.

□

Finally we can prove our main approximation theorem.

Theorem 7. *Let Y be a Banach space, $k \in \mathbb{N} \cup \{\infty\}$, and X be a normed linear space such that there is a set Γ and a bi-Lipschitz homeomorphism $\varphi\colon X \to c_0(\Gamma)$ such that the coordinate functions $e^*_\gamma \circ \varphi \in C^k(X)$ for every $\gamma \in \Gamma$. Assume further that X or Y is an ALAUNR. There is a constant $C \in \mathbb{R}$ such that if $f\colon X \to Y$ is L-Lipschitz and $\varepsilon > 0$, then there is a CL-Lipschitz mapping $g \in C^k(X, Y)$, such that $\sup_{x \in X} \|f(x) - g(x)\| \le \varepsilon$.*

Moreover, if $C_1, C_2 \in \mathbb{R}$ are such that φ is C_1-Lipschitz and φ^{-1} is C_2-Lipschitz, and if K is the Lipschitz constant of the ALAUNR, then $C = C_1C_2K$.

Proof. We define $\tilde f\colon \varphi(X) \to Y$ by $\tilde f(z) = f(\varphi^{-1}(z))$ for any $z \in \varphi(X)$. The mapping $\tilde f$ is C_2L-Lipschitz. If Y is a K-Lipschitz ALAUNR, then by Proposition 5(iii) there is a uniform open neighbourhood U of $\varphi(X)$ in $c_0(\Gamma)$ and a mapping $\hat f\colon U \to Y$ such that $\hat f$ is KC_2L-Lipschitz and $\big\|\hat f(z) - \tilde f(z)\big\| < \frac{\varepsilon}{2}$ for each $z \in \varphi(X)$. In case that X is a K-Lipschitz ALAUNR, we come to the same conclusion by using Proposition 5(iii) to a mapping φ^{-1} to obtain a uniform open neighbourhood U of $\varphi(X)$ in $c_0(\Gamma)$ and a KC_2 Lipschitz mapping $q\colon U \to X$ such that $\big\|q(z) - \varphi^{-1}(z)\big\| < \frac{\varepsilon}{2L}$ for all $z \in \varphi(X)$, and then putting $\hat f = f \circ q$. (Using Corollary 6 and Proposition 5(iii) to $\tilde f$ instead, we would arrive to a worse Lipschitz constant $KC_1C_2^2L$.)

By Theorem G there is a mapping $\hat g \in C^\infty(c_0(\Gamma), Y)$ locally dependent on finitely many coordinates and such that it is C_2KL-Lipschitz on $\varphi(X)$ and $\big\|\hat g(z) - \hat f(z)\big\| \le \frac{\varepsilon}{2}$ for all $z \in \varphi(X)$. We define the mapping $g\colon X \to Y$ by $g = \hat g \circ \varphi$.

Similarly as in the proof of Theorem 3, (iii)⇒(i), we obtain that $g \in C^k(X,Y)$. Clearly, g is C_1C_2KL-Lipschitz. To see that g approximates f, choose any $x \in X$. Then

$$\begin{aligned}\|g(x)-f(x)\| &= \left\|\hat{g}(\varphi(x)) - f\big(\varphi^{-1}(\varphi(x))\big)\right\| = \left\|\hat{g}(\varphi(x)) - \tilde{f}(\varphi(x))\right\| \\ &\le \left\|\hat{g}(\varphi(x)) - \hat{f}(\varphi(x))\right\| + \left\|\hat{f}(\varphi(x)) - \tilde{f}(\varphi(x))\right\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

□

We note that the notion of ALAUNR is necessary for our approach to Theorem 7 (at least in the case of the source space X): For any Banach space Y and any Lipschitz mapping $f\colon \varphi(X) \to Y$ we need to find a Lipschitz "approximate extension" to a uniform neighbourhood U of $\varphi(X)$. Now, consider $Y = X$ and a mapping $\varphi^{-1}\colon \varphi(X) \to X$, find an "approximate extension" $q\colon U \to X$ and put $r = \varphi \circ q$. Then r is a Lipschitz ε-retraction of U into $\varphi(X)$.

Let V be a topological space, let $v_0 \in V$. By $B_0(V)$ we denote the space of all bounded real-valued functions f on V for which $f(v) \to 0$ whenever $v \to v_0$, considered with the supremum norm. Let P be a metric space, by $C_u(P)$ we denote the space of all bounded, uniformly continuous, real-valued functions on P with the supremum norm. By the result of Lindenstrauss, [L, Theorem 6] (see also [BL]), both $B_0(V)$ and $C_u(P)$ are absolute Lipschitz retracts.

Now using Corollary 4 and Theorem 7 we obtain the following:

Corollary 8. *Let X be a separable normed linear space that admits a C^k-smooth Lipschitz bump function, $k \in \mathbb{N}\cup\{\infty\}$. Let Y be a Banach space. If at least one of the spaces X or Y is equal to either $B_0(V)$ for some topological space V, or $C_u(P)$ for some metric space P, then there is a constant $C \in \mathbb{R}$ such that for any L-Lipschitz mapping $f\colon X \to Y$ and any $\varepsilon > 0$ there is a CL-Lipschitz mapping $g \in C^k(X,Y)$ for which $\sup_{x\in X}\|f(x)-g(x)\| \le \varepsilon$.*

The above approach can be modified to deal with uniformly continuous mappings. However, we must be somewhat careful in the formulation of the result (notice the necessity of a sub-additive modulus of the embedding in Theorem 9). We skip the details, as the proofs are almost identical to the ones already given.

A modulus is a non-decreasing function $\omega\colon [0,+\infty) \to [0,+\infty)$ continuous at 0 such that $\omega(0) = 0$. The set of all moduli will be denoted by $\mathcal{M}$. The subset of $\mathcal{M}$ of all moduli that are sub-additive will be denoted by $\mathcal{M}_s \subset \mathcal{M}$. A modulus of continuity of a mapping f is denoted by ω_f.

Theorem 9. *Let Y be a Banach space, $k \in \mathbb{N}\cup\{\infty\}$, and X be a normed linear space such that there is a set Γ and a uniform homeomorphism $\varphi\colon X \to c_0(\Gamma)$ such that $\omega_{\varphi^{-1}} \le \omega_1 \in \mathcal{M}_s$ and the coordinate functions $e^*_\gamma \circ \varphi \in C^k(X)$ for every $\gamma \in \Gamma$. Assume further that X or Y is an absolute uniform approximate uniform neighbourhood retract. If $f\colon X \to Y$ is uniformly continuous and $\varepsilon > 0$, then there is a function $\omega \in \mathcal{M}$ and a mapping $g \in C^k(X,Y)$, such that $\omega_g \le \omega$ and $\sup_{x\in X}\|f(x)-g(x)\| \le \varepsilon$.*

Moreover, if X is AUAUNR with modulus ω_0, then $\omega = \omega_f \circ \omega_0 \circ \omega_1 \circ \omega_\varphi$. If Y is AUAUNR with modulus ω_0, then $\omega = \omega_0 \circ \omega_f \circ \omega_1 \circ \omega_\varphi$.

By the result of Lindenstrauss, [L, Theorem 8] (see also [BL]), super-reflexive Banach spaces are absolute uniform uniform (sic) neighbourhood retracts. Hence, using Corollary 4 and Theorem 9 we obtain the following:

Corollary 10. *Let X be a separable normed linear space that admits a C^k-smooth Lipschitz bump function, $k \in \mathbb{N}\cup\{\infty\}$. Let Y be a Banach space. If X or Y is a super-reflexive Banach space, then there is a constant $C \in \mathbb{R}$ and a modulus $\omega_0 \in \mathcal{M}$ such that for any uniformly continuous mapping $f\colon X \to Y$ and any $\varepsilon > 0$ there is a mapping $g \in C^k(X,Y)$ for which $\sup_{x\in X}\|f(x)-g(x)\| \le \varepsilon$ and $\omega_g(\delta) \le \omega_f(\omega_0(C\delta))$ (if X is super-reflexive) or $\omega_g(\delta) \le \omega_0(\omega_f(C\delta))$ (if Y is super-reflexive).*

4. Approximation of C^1-smooth mappings

First, we extend the result of Moulis about the relation of Lipschitz approximation and the approximation of derivatives to non-separable case. For this we need some finer information about refinements of open coverings.

Lemma 11 (M.E. Rudin, [R])**.** *Let P be a metric space, $\mathcal{U} = \{U_\alpha\}_{\alpha\in\Lambda}$ be an open covering of P. Then there are open refinements $\{V_{n\alpha}\}_{n\in\mathbb{N},\alpha\in\Lambda}$, $\{W_{n\alpha}\}_{n\in\mathbb{N},\alpha\in\Lambda}$ of $\mathcal{U}$ that satisfy the following:*

- *$V_{n\alpha} \subset W_{n\alpha} \subset U_\alpha$ for all $n \in \mathbb{N}$, $\alpha \in \Lambda$,*
- *$\operatorname{dist}(V_{n\alpha}, P \setminus W_{n\alpha}) \ge 2^{-n}$ for all $n \in \mathbb{N}$, $\alpha \in \Lambda$,*
- *$\operatorname{dist}(W_{n\alpha}, W_{n\beta}) \ge 2^{-n}$ for any $n \in \mathbb{N}$ and $\alpha, \beta \in \Lambda$, $\alpha \ne \beta$.*
- *for each $x \in P$ there is an open ball $U_x \in P$ with centre x and a number $n_x \in \mathbb{N}$ such that*
  - *(i) if $i > n_x$, then $U_x \cap W_{i\alpha} = \emptyset$ for any $\alpha \in \Lambda$,*
  - *(ii) if $i \le n_x$, then $U_x \cap W_{i\alpha} \ne \emptyset$ for at most one $\alpha \in \Lambda$,*

Next, we need a result about Lipschitz partitions of unity. Let X be a normed linear space, $k \in \mathbb{N}\cup\{\infty\}$. We denote $C^k_L(X) = \{f \in C^k(X), f \text{ is Lipschitz}\}$ and $\mathcal{U}^k_L = \big\{f^{-1}\big((0,+\infty)\big);\ 0 \le f \le 1, f \in C^k_L(X)\big\}$.

Lemma 12. *Let X be a normed linear space and $k \in \mathbb{N}\cup\{\infty\}$. Then the following are equivalent:*

(i) The space X admits C^k_L partitions of unity.

(ii) If $A \subset W \subset X$, A is closed and W is open in X, then there is $U \in \mathcal{U}^k_L$ such that $A \subset U \subset W$.
(iii) If $A \subset W \subset X$, W is bounded and $\operatorname{dist}(A, X \setminus W) > 0$, then there is $U \in \mathcal{U}^k_L$ such that $A \subset U \subset W$.
(iv) The family $\mathcal{U}^k_L$ contains a σ-discrete basis of the topology of X.

Proof. The equivalence of (i) and (iv) follows from [JTZ, Lemma 1]. Note only, that to satisfy the condition (i) in [JTZ, Lemma 1] it suffices to multiply each $f \in S_0$ by a suitable constant so that all $f \in S_0$ have the same Lipschitz constant.

To see (i)⇒(ii), consider the partition of unity subordinated to the covering $\{W, X \setminus A\}$ of X. By (i) there are functions $\psi_1, \psi_2 \in C^k_L$ such that $\operatorname{supp} \psi_1 \subset W$, $\operatorname{supp} \psi_2 \subset X \setminus A$, $0 \le \psi_1 \le 1$, $0 \le \psi_2 \le 1$ and $\psi_1(x) + \psi_2(x) = 1$ for all $x \in X$. Hence $\psi_1 = 1$ on A and we may put $U = \psi_1^{-1}\big((0, +\infty)\big)$.

(ii) implies (iii) is trivial.

Finally, suppose (iii) holds. Let $\mathcal{D}_m = \{D^m_\alpha\}_{\alpha\in\Lambda}$ be a covering of X by open balls with radius $\frac{1}{m}$. By Lemma 11 there are open refinements $\{V^m_{n\alpha}\}_{n\in\mathbb{N},\alpha\in\Lambda}$, $\{W^m_{n\alpha}\}_{n\in\mathbb{N},\alpha\in\Lambda}$ of $\mathcal{D}_m$ such that $V^m_{n\alpha} \subset W^m_{n\alpha} \subset D^m_\alpha$, $\operatorname{dist}(V^m_{n\alpha}, X \setminus W^m_{n\alpha}) \ge 2^{-n}$ and the family $\{W^m_{n\alpha}\}_{\alpha\in\Lambda}$ is discrete for all $n \in \mathbb{N}$. Thus, by (iii), there are $U^m_{n\alpha} \in \mathcal{U}^k_L$ such that $V^m_{n\alpha} \subset U^m_{n\alpha} \subset W^m_{n\alpha}$. The family $\{U^m_{n\alpha}\}_{m,n\in\mathbb{N},\alpha\in\Lambda}$ is therefore a σ-discrete basis of the topology of X. □

Finally, we prove the main result of this section.

Theorem 13. *Let X, Y be normed linear spaces and $k \in \mathbb{N} \cup \{\infty\}$. Consider the following statements:*

(i) There is $C \in \mathbb{R}$ such that for any L-Lipschitz mapping $f\colon U_X \to Y$ and any $\varepsilon > 0$ there is a CL-Lipschitz mapping $g \in C^k(U_X, Y)$, such that $\sup_{x\in U_X} \|f(x) - g(x)\| \le \varepsilon$.
(ii) For any open $\Omega \subset X$, any mapping $f \in C^1(\Omega, Y)$ and any continuous function $\varepsilon\colon \Omega \to (0, +\infty)$ there is $g \in C^k(\Omega, Y)$, such that $\|f(x) - g(x)\| < \varepsilon(x)$ and $\|f'(x) - g'(x)\| < \varepsilon(x)$ for all $x \in \Omega$.
(iii) For any open $\Omega \subset X$, any L-Lipschitz mapping $f \in C^1(\Omega, Y)$, any continuous function $\varepsilon\colon \Omega \to (0, +\infty)$ and any $\eta > 1$ there is an ηL-Lipschitz mapping $g \in C^k(\Omega, Y)$, such that $\|f(x) - g(x)\| < \varepsilon(x)$ for all $x \in \Omega$.

Then (i)⇒(ii)⇒(iii)

Proof. (ii)⇒(iii) is obvious.

Suppose (i) holds. First notice that by translating and scaling we immediately obtain approximations on any open ball in X.

Second, (i) gives us also approximations of functions. Indeed, if $f \in C^1(U_X)$ is L-Lipschitz, then choose some $y \in S_Y$ and consider the mapping $\tilde{f} \in C^1(U_X, Y)$, $\tilde{f}(x) = f(x) \cdot y$. Let $\tilde{g} \in C^k(U_X, Y)$ be an approximation provided by (i) and $F \in Y^*$ be a Hahn-Banach extension of the norm-one functional $ty \mapsto t$ defined on $\operatorname{span}\{y\}$. Then $g = F \circ \tilde{g}$ is the desired approximation of the function f.

To prove (ii), let $\{U_\alpha = U(x_\alpha, r_\alpha)\}_{\alpha\in\Lambda}$ be a covering of Ω by open balls such that

$$\|f(x_\alpha) - f(x)\| < \frac{\varepsilon(x_\alpha)}{3} < \varepsilon(x) \quad \text{for each } x \in U_\alpha, \tag{4}$$

$$\|f'(x_\alpha) - f'(x)\| < \frac{\varepsilon(x_\alpha)}{9C} \quad \text{for each } x \in U_\alpha, \tag{5}$$

for each $\alpha \in \Lambda$. Let $\{W_{n\alpha}\}_{n\in\mathbb{N},\alpha\in\Lambda}$ be a refinement of $\{U_\alpha\}$ from Lemma 11.

The statement (iii) in Lemma 12 is satisfied using the approximations of functions discussed above. Hence there is a C^k_L partition of unity $\{\psi_{n\alpha}\}_{n\in\mathbb{N},\alpha\in\Lambda}$ subordinated to $\{W_{n\alpha}\}$. Let $L_{n\alpha}$ be the Lipschitz constant of $\psi_{n\alpha}$, and without loss of generality assume $L_{n\alpha} \ge 1$ and also $C \ge 1$.

For each $\alpha \in \Lambda$ let us define the mapping $g_\alpha\colon U_\alpha \to Y$ by $g_\alpha = f(x) - f'(x_\alpha)x$. Then, by (5) and the second inequality in (4),

$$\|g'_\alpha(x)\| < \frac{\varepsilon(x_\alpha)}{9C} < \frac{\varepsilon(x)}{3C} \le \frac{\varepsilon(x)}{3} \quad \text{for each } x \in U_\alpha. \tag{6}$$

For any $n \in \mathbb{N}$ and $\alpha \in \Lambda$, using (i) we approximate g_α by $h_{n\alpha} \in C^k(U_\alpha, Y)$ such that

$$\|h'_{n\alpha}(x)\| \le \frac{\varepsilon(x_\alpha)}{9} < \frac{\varepsilon(x)}{3} \quad \text{for each } x \in U_\alpha, \tag{7}$$

$$\|g_\alpha(x) - h_{n\alpha}(x)\| \le \frac{\varepsilon(x_\alpha)}{9 \cdot 2^n L_{n\alpha}} < \frac{\varepsilon(x)}{3 \cdot 2^n L_{n\alpha}} < \varepsilon(x) \quad \text{for each } x \in U_\alpha. \tag{8}$$

(The second inequalities follow from the second inequality in (4).)

Finally, we define the mapping $g\colon \Omega \to Y$ by

$$g(x) = \sum_{n\in\mathbb{N},\alpha\in\Lambda} \big(h_{n\alpha}(x) + f'(x_\alpha)x\big)\psi_{n\alpha}(x).$$

Since $\operatorname{supp} \psi_{n\alpha} \subset U_\alpha$ and the sum is locally finite, the mapping is well defined and moreover $g \in C^k(\Omega, Y)$.

Choose $x \in \Omega$ and let us compute how far $g(x)$ is from $f(x)$:

$$\|f(x) - g(x)\| = \left\| \sum_{n\in\mathbb{N},\alpha\in\Lambda} \big(f(x) - h_{n\alpha}(x) - f'(x_\alpha)x\big)\psi_{n\alpha}(x) \right\| = \left\| \sum_{n\in\mathbb{N},\alpha\in\Lambda} \big(g_\alpha(x) - h_{n\alpha}(x)\big)\psi_{n\alpha}(x) \right\|$$
$$\leq \sum_{\substack{n\in\mathbb{N}\\ \alpha\in\Lambda\colon x\in U_\alpha}} \|g_\alpha(x) - h_{n\alpha}(x)\| \, \psi_{n\alpha}(x) < \varepsilon(x) \sum_{\substack{n\in\mathbb{N}\\ \alpha\in\Lambda\colon x\in U_\alpha}} \psi_{n\alpha}(x) = \varepsilon(x),$$

where the last inequality follows from (8).

To estimate the distance between the derivatives at $x \in \Omega$, notice that by Lemma 11 for each $n \in \mathbb{N}$ there is at most one $\alpha \in \Lambda$ such that $x \in W_{n\alpha}$. Since $\overline{\operatorname{supp}}\, \psi_{n\alpha} \subset W_{n\alpha}$, there is a function $\beta\colon \mathbb{N} \to \Lambda$ such that for each $n \in \mathbb{N}$, $x \notin \overline{\operatorname{supp}}\, \psi_{n\alpha}$ whenever $\alpha \neq \beta(n)$ and moreover $x \in U_{\beta(n)}$. (For a fixed $n \in \mathbb{N}$, either there is exactly one $\alpha \in \Lambda$ such that $x \in W_{n\alpha}$ and in that case we put $\beta(n) = \alpha$, or $x \notin W_{n\alpha}$ for all $\alpha \in \Lambda$ and we can choose an arbitrary $\beta(n)$ for which $x \in U_{\beta(n)}$.) Hence,

$$\|f'(x) - g'(x)\| = \left\| \big(f(x) - g(x)\big)' \right\| = \left\| \sum_{n\in\mathbb{N},\alpha\in\Lambda} \big(g_\alpha(x) - h_{n\alpha}(x)\big)'\psi_{n\alpha}(x) + \sum_{n\in\mathbb{N},\alpha\in\Lambda} \big(g_\alpha(x) - h_{n\alpha}(x)\big)\psi'_{n\alpha}(x) \right\|$$
$$\leq \sum_{\substack{n\in\mathbb{N}\\ \alpha\in\Lambda\colon x\in U_\alpha}} \|g'_\alpha(x) - h'_{n\alpha}(x)\| \, \psi_{n\alpha}(x) + \sum_{n=1}^{\infty} \left\|g_{\beta(n)}(x) - h_{n\beta(n)}(x)\right\| \left\|\psi'_{n\beta(n)}(x)\right\|$$
$$\leq \sum_{\substack{n\in\mathbb{N}\\ \alpha\in\Lambda\colon x\in U_\alpha}} \Big(\|g'_\alpha(x)\| + \|h'_{n\alpha}(x)\|\Big)\psi_{n\alpha}(x) + \sum_{n=1}^{\infty} \left\|g_{\beta(n)}(x) - h_{n\beta(n)}(x)\right\| L_{n\beta(n)}$$
$$< \left(\frac{\varepsilon(x)}{3} + \frac{\varepsilon(x)}{3}\right) \sum_{\substack{n\in\mathbb{N}\\ \alpha\in\Lambda\colon x\in U_\alpha}} \psi_{n\alpha}(x) + \sum_{n=1}^{\infty} \frac{\varepsilon(x)}{3\cdot 2^n L_{n\beta(n)}} L_{n\beta(n)} = \varepsilon(x),$$

where the last inequality follows from (6), (7) and (8).

□

Notice that if we can construct Lipschitz partitions of unity on X without the aid of Lemma 12 (as is the case when X is separable), then in Theorem 13(i) it suffices to require only the approximation of Lipschitz mappings that are moreover C^1-smooth.

Corollary 14. *Let X be a separable normed linear space that admits a C^k-smooth Lipschitz bump function, $k \in \mathbb{N} \cup \{\infty\}$. Let Y be a Banach space. If at least one of the spaces X or Y is equal to either $B_0(V)$ for some topological space V, or $C_u(P)$ for some metric space P, then for any open $\Omega \subset X$, any mapping $f \in C^1(\Omega, Y)$ and any continuous function $\varepsilon\colon \Omega \to (0, +\infty)$ there is $g \in C^k(\Omega, Y)$, such that $\|f(x) - g(x)\| < \varepsilon(x)$ and $\|f'(x) - g'(x)\| < \varepsilon(x)$ for all $x \in \Omega$.*

Proof. By Corollary 8, the requirements of Theorem 13(i) are satisfied. Notice only that first we need to extend the mappings in Theorem 13(i) from U_X to the whole of X, which is easy, since first we extend it to B_X by the Lipschitzness and then to the whole of X, as B_X is a 2-Lipschitz retract of X.

□

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