

# Selection of Corners for the BDDC Method

Pavel Burda, Marta Čertíková, Alexandr Damašek,  
Jaroslav Novotný, Jakub Šístek \*

## Abstract

The Balancing Domain Decomposition by Constraints (BDDC) method has evolved quite fast since its introduction in 2003, as the primal counterpart to the earlier FETI-DP method. Recent results have shown close connection of these methods and theoretically supported equivalent rate of convergence. In both methods, a fundamental role is played by the coarse space. Optimal choice of constraints on continuity of the coarse space is still not a satisfactorily solved problem. The usual basic choice is a ‘minimal’ set of coarse nodes (sometimes called corners), that assures invertibility of local subdomain problems and also of the global coarse problem. However, this set alone does not suffice for optimal preconditioning in 3D. For this reason, continuity of some generalized degrees of freedom, such as average values on edges or faces of subdomains, have to be added. While theoretically correct, this approach does not easily offer a flexible size of desired coarse problem. In our contribution, we compare this approach with adding more coarse nodes into the coarse problem, which is technically simpler and allows flexible setting of desired approximation.

## 1 Introduction

The Balancing Domain Decomposition based on Constraints (BDDC) is a numerically scalable, nonoverlapping (substructuring), primary domain decomposition method introduced in 2003 by Dohrmann [4]. Its algebraic theory developed by Mandel, Dohrmann and Tezaur in [14] demonstrates close relation to FETI-DP introduced by Farhat, Lesoinne and Pierson [5]: the eigenvalues of the preconditioned problem in BDDC and FETI-DP are nearly the same (see also [2], [13] and [16] for simplified proofs). Thus the performance of BDDC and FETI-DP is in principle the same, and some theoretical results obtained for one method apply readily to the other.

The *coarse space*, defined by continuity constraints across the interface (or coarse degrees of freedom), is essential for performance of both methods.

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\*Corresponding author at: Institute of Mathematics, Academy of Sciences of the Czech Republic, Žitná 25, CZ-115 67 Praha 1, Czech Republic. Tel.: +420 222 090 710; fax: +420 222 211 638, sistek@math.cas.cz.

A historical overview of an evolution of the concept of the coarse space is presented by Widlund in [21]. The easiest choice of coarse degrees of freedom are node constraints and the usual basic choice is a set of coarse nodes (sometimes called corners), ‘minimal’ in the sense that it assures invertibility of local subdomain problems and also the global coarse problem. For 2D problems this choice ensures good convergence properties. However, an efficient BDDC method for 3D elliptic problems requires also constraints on some generalized degrees of freedom, such as average values on edges or faces of subdomains. This fact was first discovered for FETI-DP: experimentally observed in Farhat, Lesoinne and Pierson [5] and theoretically supported by Klawonn, Widlund and Dryja in [11]. These observations apply also to BDDC because of the equivalence between the methods.

Optimal choice of constraints on continuity of the coarse space is still not a satisfactorily solved problem. Related work on choice of the coarse degrees of freedom has focused on selecting a small and effective coarse space. An algorithm for selecting the smallest set of coarse nodes to avoid coarse mechanism is described by Lesoinne in [12]. Another algorithm based already on pairs of subdomains was given by Dohrmann in [4]. This task has been recently further discussed by Brož and Kruis in [3] for 2D case. Klawonn and Widlund in [9] and [10] minimize a set of more general coarse degrees of freedom (like weighted averages over edges and faces) to achieve optimal convergence estimates, introducing concept of an acceptable path. Adaptive selection of coarse degrees of freedom based on local estimates using eigenvectors associated with faces is described by Mandel and Sousedík in [15].

In this paper, we explore the potential of adding more coarse nodes into the coarse problem, which is technically simpler, easily parallelizable and allows flexible setting of desired approximation. The main contribution is proposing a new algorithm for selection of corners. It is based on the idea of loosening the requirement of ‘minimal’ selection and motivated by the observation that although the earlier selection algorithms lead to invertible subdomain problems and the coarse problem, the performance of the BDDC preconditioner may be cheaply but considerably improved by selecting more corners, as presented by Šístek et al. in [19]. Numerical experiments on industrial 3D elasticity problems show that this approach is able to considerably speed up the computation.

## 2 BDDC method

After a discretization of a linearized partial differential equation of elliptic type in a given domain  $\Omega$  by means of finite element method (FEM), a system of linear algebraic equations

$$\mathbf{Ax} = \mathbf{f} \tag{1}$$

with a symmetric positive definite matrix  $\mathbf{A}$  and a right-hand side  $\mathbf{f}$  is solved for the unknown vector  $\mathbf{x}$ . Components of  $\mathbf{x}$  represent function values at mesh nodes and they are often called *degrees of freedom*. In 3D linear elasticity, there

are 3 unknown values of displacement (3 degrees of freedom) at every mesh node.

The first step in BDDC method is the reduction of the problem to the interface. This is quite standard and described in the literature, e.g., Toselli and Widlund [20]: the underlying discretized (meshed) domain  $\Omega$  is split into  $N$  nonoverlapping subdomains (also called *substructures*)  $\Omega_i$ ,  $i = 1, \dots, N$  with common interface  $\Gamma$  and problem (1) is reduced to the Schur complement problem with respect to interface

$$\mathbf{S}\mathbf{u} = \mathbf{g} \quad (2)$$

with a symmetric positive definite matrix  $\mathbf{S}$ . The vector  $\mathbf{u}$  now represents the subset of degrees of freedom in  $\mathbf{x}$  that are on the interface  $\Gamma$ . Solution  $\mathbf{u}$  of the problem (2) can be also represented as the minimum of the functional

$$\frac{1}{2}\mathbf{u}^T\mathbf{S}\mathbf{u} - \mathbf{u}^T\mathbf{g} \rightarrow \min, \quad \mathbf{u} \in \widehat{W} \quad (3)$$

on the space  $\widehat{W}$  of unknowns on the interface  $\Gamma$ . The space  $\widehat{W}$  can be identified with the space of *discrete harmonic functions*, that are fully determined by their values of unknowns on the interface  $\Gamma$  and have minimal energy on every subdomain.

The problem (2) is then solved by the preconditioned conjugate gradient (PCG) method, for which BDDC acts as the preconditioner. The main idea of the BDDC method is shortly described below. More details, together with connection to FETI-DP, can be found in Mandel, Dohrmann and Tezaur [14] or Mandel and Sousedík [16].

A preconditioner  $\mathbf{M}$  of a system (2) should be some approximation of  $\mathbf{S}^{-1}$  such that obtaining a preconditioned residual  $\mathbf{p} = \mathbf{M}\mathbf{r}$  can be considerably easier than solving the original problem (2). Construction of the BDDC preconditioner is based on the idea that instead of minimizing (3) on the space  $\widehat{W}$ , which represents solving the system (2), the minimization is performed on some larger space  $\widetilde{W}$  such that  $\widehat{W} \subset \widetilde{W}$ :

$$\frac{1}{2}\widetilde{\mathbf{u}}^T\widetilde{\mathbf{S}}\widetilde{\mathbf{u}} - \widetilde{\mathbf{u}}^T\widetilde{\mathbf{g}} \rightarrow \min, \quad \widetilde{\mathbf{u}} \in \widetilde{W}, \quad (4)$$

where  $\widetilde{\mathbf{S}}$  is a symmetric positive definite extension of  $\mathbf{S}$  to  $\widetilde{W}$  and  $\widetilde{\mathbf{g}}$  is an extension of  $\mathbf{g}$ . The space  $\widetilde{W}$  has to be chosen so that the symmetric positive definite extension  $\widetilde{\mathbf{S}}$  on  $\widetilde{W}$  exists, that solving the problem (4) is considerably easier than solving the original problem (3) and at the same time the solution of (4) is still a good approximation of the solution of the problem (3). The BDDC preconditioner is then defined as

$$\mathbf{M} = \mathbf{E}\widetilde{\mathbf{S}}^{-1}\mathbf{E}^T, \quad (5)$$

where  $\mathbf{E}$  represents a projection from  $\widetilde{W}$  onto  $\widehat{W}$  realized by a kind of averaging.

### 3 Coarse degrees of freedom

In BDDC, the space  $\widetilde{W}$  is specified by relaxing the requirement of the continuity of discrete harmonic functions across the interface. The functions of  $\widetilde{W}$  are forced to be continuous at selected degrees of freedom only, called *coarse degrees of freedom*. In this paper, we focus on the simplest choice of coarse degrees of freedom, which is a function value at a selected node on the interface. Such node is then called *coarse node* or *corner*. More general coarse degrees of freedom are commented at the end of this section and are considered in computations.

In terms of mechanics, the transition from  $\widehat{W}$  to  $\widetilde{W}$  can be interpreted as making incisions into the continuous function along the interface, leaving the function continuous across the interface only at the corners. A schematic illustration of the continuity constraints is depicted in Figure 1: functions from  $\widehat{W}$  are continuous across the interface, functions from  $\widetilde{W}$  are continuous only at selected coarse nodes.

The space  $\widetilde{W}$  can be decomposed as  $\widetilde{\mathbf{S}}$ -orthogonal direct sum

$$\widetilde{W} = \widetilde{W}_1 \oplus \cdots \oplus \widetilde{W}_N \oplus \widetilde{W}_C, \quad (6)$$

where  $\widetilde{W}_i$  are local, subdomain spaces and  $\widetilde{W}_C$  is the global *coarse space*, defined as the  $\widetilde{\mathbf{S}}$ -orthogonal complement of all spaces  $\widetilde{W}_i$ , i.e.

$$\mathbf{w}_C \in \widetilde{W}_C \Leftrightarrow \mathbf{w}_C^T \widetilde{\mathbf{S}} \mathbf{w} = 0 \quad \forall \mathbf{w} \in \widetilde{W}_i, \quad i = 1, \dots, N.$$

Functions from  $\widetilde{W}_i$  can have nonzero values only in  $\Omega_i$  except for coarse degrees of freedom. They have zero values at coarse degrees of freedom and they are fully determined by degrees of freedom on  $\Gamma$  and a discrete harmonic condition in interiors of subdomains. Similarly, functions from  $\widetilde{W}_C$  are fully determined by their values at coarse degrees of freedom (where they are continuous) and by a discrete harmonic condition in interiors of subdomains and on the rest of the interface (i.e. everywhere apart from the coarse nodes). Functions from the spaces  $\widetilde{W}_C$  and  $\widetilde{W}_i$  are generally discontinuous across  $\Gamma$  outside the corners.

According to decomposition (6), solution of the problem (4) can now be split into solution of  $N$  local subdomain problems on the spaces  $\widetilde{W}_i$  and one global coarse problem on the coarse space  $\widetilde{W}_C$ . All these problems are mutually independent and so can be naturally parallelized.

Coarse degrees of freedom have to be selected so that stable invertibility of both the coarse problem and the local problems is assured. Important role of the coarse space is to assure scalability by global error propagation over the whole domain. It was shown that while for 2D elasticity problems the BDDC (or FETI-DP) preconditioner is scalable for coarse space defined by coarse nodes (corners) only, in 3D elasticity problems more general coarse degrees of freedom, such as (weighted) average values over edges and faces, need to be used in order to achieve the scalability, see e.g. Toselli and Widlund [20].

Choice of the coarse degrees of freedom has a great impact on the performance of the preconditioner  $\mathbf{M}$ . The more coarse degrees of freedom are chosen, the more difficult is to obtain the solution of (4), which, on the other

hand, is then closer to the solution of the original problem (3). In the extreme case of selecting all interface nodes as coarse,  $\widetilde{W}_C \equiv \widetilde{W} \equiv \widehat{W}$ , coarse problem becomes the original problem (3) and  $\mathbf{M} \equiv \mathbf{S}^{-1}$ . In the opposite extreme, if no coarse degrees of freedom are selected,  $\widetilde{W}_C$  is empty and solution of (4) splits to  $N$  local problems only, some of which might not be invertible. Thus, the optimal choice of the coarse space lies somewhere in-between.

## 4 Geometry and selection of the coarse space in 3D

The interface  $\Gamma$  in 3 dimensions can be specified as a set of nodes belonging to at least two subdomains (subdomains are considered as closed sets). It consists of subdomain faces, edges and vertices. While there is an intuitive geometric notion what these three entities mean in a simple case of a cubic domain divided into cubic subdomains, there is no unique exact classification in more general case of domain with complicated geometry and subdomains obtained by a graph partitioning tool. We use a classification presented by Klawonn and Rheinbach in [8] in a little simplified form, which does not assume knowledge of boundary of the domain and is easy to implement:

### Definition 1

- *a face contains all nodes shared by the same two subdomains,*
- *an edge contains nodes shared by the same more than two subdomains,*
- *a vertex is a degenerated edge with only one node.*

Then every node of the interface belongs to just one of the entities defined above. Note that all faces and edges are open sets in the sense that a face does not contain its boundary and an edge does not contain its endpoints. Two subdomains are called *adjacent* if they share a face.

However, this classification does not reproduce our intuition in the case of cubic subdomains, as can be seen in Figure 2: for instance the interface of a domain consisting of two cubic subdomains has neither vertex nor edge, just one face (the leftmost case in Figure 2). Different definitions of faces and edges are discussed by Klawonn and Rheinbach in [7, Section 2].

In practice, there are often not enough vertices, edges or faces for satisfactory number of constraints. We have found it useful to introduce one additional entity:

### Definition 2

- *a corner is any interface node selected as coarse.*

In implementations of the BDDC method, it is often customary to distinguish between the following two kinds of constraints on continuity across interface.

### **Node constraints - corners**

The most obvious choice of coarse degrees of freedom are node constraints (at corners). The basic choice is a set of corners, that assures invertibility of local subdomain problems and also the global coarse problem. This is often put as a requirement on their selection (e.g. in [4], [19]).

Although vertices provide a good initial set of corners, they often do not suffice for assuring invertibility of subdomain problems and/or of the coarse problem (cf. Figure 2), and other constraints need to be added. When other nodes are selected as corners, they have to be excluded from corresponding faces or edges, so that every interface node is either a corner, or belongs to a face or an edge.

Corner constraints are not as efficient as constraints on averages on edges or faces, nevertheless they can be used as a substitute for these constraints, if enough corners are employed. Figure 6 left illustrates the typical dependence of the condition number of the preconditioned problem on number of corners randomly selected from the interface, starting from some basic set. For small numbers of corners, we can observe poor performance of the preconditioner even though all system matrices are invertible. Then, after a typical sudden drop, the condition number improves only slightly with adding more corners. Number of iterations reproduce this dependence, see Figure 6 centre.

Improving convergence by adding more corners leads to a larger coarse problem than adding averages on faces or edges, on the other hand, its implementation is straightforward and its scaling is easy to maintain.

For 2D problems, the basic set of corner constraints already ensures good convergence properties. Although an efficient BDDC method for 3D elliptic problems requires also constraints on some generalized degrees of freedom, such as average values on edges or faces of subdomains described below, for many industrial problems this simple approach also leads to good results.

### **Constraints on averages over edges and faces**

General coarse degrees of freedom beside corners can be constructed as any linear combinations of function values at nodes belonging to one face or one edge. This type of constraints is required for both BDDC and FETI-DP methods in three dimensions, if one expects the optimal polylogarithmic bound on condition number  $\kappa$  of the preconditioned operator

$$\kappa(\mathbf{MS}) \leq \text{const.} \left( 1 + \log \frac{H}{h} \right)^2, \quad (7)$$

where  $H$  is the subdomain size and  $h$  is the finite element size (see [11]).

One of the standard choices is an arithmetic average over unknowns separately for each component of displacement leading to three constraints in 3D elasticity. We have tested this standard choice applied to all edges, to all faces, or both. More sophisticated methods of weighted averaging were developed by Mandel and Sousedík [15] or Klawonn and Widlund [10].

## 5 Selection of the basic set of corners

In this section, we concentrate on the selection of the basic set of corners that leads to positive definiteness of matrix  $\tilde{\mathbf{S}}$  in (4). This task is equivalent to assuring invertibility of both local subdomain problems and the global coarse problem only by corner constraints, which is often required by implementations (cf. [4], [19]). Therefore, we investigate selection of corners independently of enforcing constraints on general coarse degrees of freedom.

From the mechanical point of view, the question of assuring invertibility of local subdomain problems corresponds to enforcing enough boundary conditions on a body to fix rigid body modes, with subdomain playing the role of the body. This goal is easily attained by selecting three nodes (not in a line) of the interface of a subdomain as corners.

It turns out, that the more difficult task is assuring invertibility of the coarse matrix since selection with respect to subdomain problems only may still lead to *mechanisms* in the coarse problem (see [12]). To see this, one can simply think of a domain divided into subdomains in a linear fashion. Figure 3 illustrates this on a 2D case, where two corners for each subdomain are sufficient for invertibility of subdomain stiffness matrices.

An algorithm attempting to select the smallest set of coarse nodes to avoid coarse mechanisms was given by Lesoinne in [12]. Minimization of the number of corners is obtained mainly by favouring already selected corners. Thus, it is serial in its nature.

Another algorithm for selecting corners was described already by Dohrmann in [4]. It is based on the investigation of all possible neighbourhoods between substructures and selecting three corners from each such set, that maximize the area of a triangle with corners at its vertices. However, this algorithm is based on an incomplete classification of interface into vertices, edges and faces, which does not differentiate between the last two groups. Also this algorithm favours already selected corners by selecting vertices in the interface as the initial vertices of the triangle to be maximized. Nevertheless, it has provided a good starting point for the new proposed algorithm.

The third algorithm, which is based on selection of corners along edges, was described in [18]. This idea is inspired by the definition of corners as end-points of edges by Klawonn and Widlund [9]. Although it was successfully used in a number of practical computations, it may fail to produce a mechanism-free coarse problem in the case of divisions where no edges are present (cf. the leftmost case in Figure 2).

The pressure on low number of corners inherent to all these algorithms is motivated by the fact, that low number of corners results in turn in a small size of the matrix of the coarse problem and its cheap factorization. However, it has been observed on a number of experiments (e.g. [19], also Section 7 in this paper) that this motivation may be misleading, and in fact, larger sets are preferable for the performance of the preconditioner often resulting in much lower number of PCG iterations. It has been also shown, that using more corners may lead to a considerable reduction of the computational time in spite of the longer time

spent in factorization of the larger matrix of the coarse problem, even in the case of considering averages on edges and faces.

Based on these observations and experience with the algorithms, we see several ideas that the new proposed algorithm should reflect:

1. selection with respect to faces (by Definition 1) as these are the basic building blocks of interface in 3D structures (Figure 2),
2. provide larger set of corners than the previous algorithms as this usually leads to much better preconditioning,
3. independence of selection subdomain by subdomain and of order of going through subdomains (better parallelization).

Points 2. and 3. are attained simply by not favouring already selected corners and selecting optimal distribution of at least three corners between each pair of substructures sharing a face, i.e. adjacent substructures, independently.

Let us now formalize this algorithm. For this, denote the set of faces of subdomain  $\Omega_i$  as  $\mathcal{F}(\Omega_i)$  and recall that  $N$  denotes the number of subdomains. A face  $\mathcal{F}_{ij}$  between subdomains  $\Omega_i$  and  $\Omega_j$  is present in both sets  $\mathcal{F}(\Omega_i)$  and  $\mathcal{F}(\Omega_j)$ .

**Algorithm 1 (Selection of corners for 3D elasticity problems)**

1. *Classify interface according to Definition 1 and use all vertices as corners.*
2. **For** subdomain  $\Omega_i$ ,  $i = 1, \dots, N$ ,
  - For** face  $\mathcal{F}_{ij} \in \mathcal{F}(\Omega_i)$ ,  $j = 1, \text{size}(\mathcal{F}(\Omega_i))$ ,
    - *find the set of all nodes shared with the adjacent subdomain (generally larger set than the face under consideration, as it may contain also edges and/or vertices),*
    - *select (in 3D) three corners within each such set as:*
      - (a) *pick an arbitrary node of the set,*
      - (b) *find the first corner as the most remote node from the arbitrary node,*
      - (c) *find the second corner as the most remote node from the first corner,*
      - (d) *find the third corner as the node maximizing the area of the triangle,*
- end,**
- end.**
3. *Select corners as the union of vertices and face-based selection above.*
4. *Remove selected corners from edges and faces.*

The algorithm assures that at least three corners are selected in an optimal way with respect to each face. This situation is often not obtained by favouring already selected corners, since corners optimally distributed for one face may be far from optimal distribution with respect to another face. Presented algorithm has also better potential for parallelization than algorithms favouring already selected corners, as step 2 is inherently parallel. It typically provides more corners than algorithms mentioned above, which we consider as an advantage rather than a drawback.

**Remark 1** *A modification of Algorithm 1 favouring already selected corners is simply possible by entering the face-based selection in any point (a), (b), (c), or (d), depending on how many corners are already selected between adjacent substructures. This modification leads to selection that is very similar to the algorithm by Dohrmann in [4]. In our experience, this modification, referred to as ‘minimal’, leads to lower number of corners, but also usually to worse results. Thus, we recommend using the (full) version as stated in Algorithm 1.*

**Remark 2** *A modification of Algorithm 1 for 2D problems (where no edges are present) is simply possible by finishing the face-based selection with point (c).*

## 6 Implementation

The BDDC method has been implemented on top of common components of existing finite element codes, namely the frontal solver and the element stiffness matrix generation. Such implementation requires only a minimal amount of additional code. In our case, most of the program is written in Fortran 77, with some parts in Fortran 90. The MPI library is used for parallelization.

The implementation relies on the separation of node constraints and enforcing the rest by Lagrange multipliers, as suggested already in Dohrmann [4]. One new aspect of the implementation is the use of reactions, which come naturally from the frontal solver, to avoid custom coding. An external parallel multifrontal solver MUMPS [1] is used for the solution of the coarse problem, instead of the serial frontal solver, as dimension of the coarse space could become a bottleneck.

Detailed description of the implementation can be found in [19], and some more experiments were presented in [18].

## 7 Numerical results

Presented numerical results were computed on SGI Altix 4700 computer with 1.5 GHz Intel Itanium 2 processors (OS Linux) in Czech Technical University Supercomputing Centre, Prague. For decompositions, we use the METIS graph partitioner [6].

Three different industrial problems have been tested. The first one is a problem of elasticity analysis of a turbine nozzle, through which the steam

enters the turbine blades (Figure 4). The geometry is discretized using 2696 quadratic elements, which leads to 40254 unknowns. The second one is a problem of elasticity analysis of a hip joint replacement which is loaded by pressure from human body weight. This mesh consists of 33186 quadratic elements resulting in 544734 unknowns. Both meshes are divided into 36 subdomains by METIS. The turbine nozzle problem was computed using 12 processors, for hip joint replacement 36 processors were used. The third problem is stress analysis of a mine reel loaded by its own weight and the weight of the steel rope (Figure 5). The mesh consists of 140816 quadratic elements and 1739211 unknowns. It was divided into 1024 subdomains by METIS. Problem was computed using 32 processors. Decomposition characteristics of the three industrial problems are summarized in Table 1.

Three algorithms for selecting the basic set of corners are tested: Algorithm 1 from Section 5 referred to as *full*, modified Algorithm 1 described in Remark 1 in Section 5 referred to as *minimal*, and the edge-based algorithm mentioned in Section 5, inspired by [9] and described in [18], referred to as *edge*. The number of PCG iterations was chosen as a measure of quality of the BDDC preconditioning. Numbers of the basic sets of corners obtained by the three algorithms for the three problems are recorded in Table 2 and corresponding number of PCG iterations are summarized in Table 3. For the two smaller problems (turbine nozzle and hip joint replacement), either constraints on corners only (referred to as  $C$ ), or constraints on corners and all averages (over all edges and faces) referred to as  $C+E+F$  are tested. For the larger problem of mine reel, corner constraints alone turned out to be too weak to achieve a reasonable convergence and the results are marked as ‘n/a’. The edge-based algorithm did not work properly for hip joint replacement problem in the case of the basic set of corners only, so the results are missing too.

As the basic sets of corners selected by different algorithms have different numbers of corners, for a fair comparison of the algorithms we added more corners selected randomly from the interface to the smaller sets in order to achieve the same number of corners. Comparison of the algorithms using the same number of corner constraints is summarized in Table 4.

Interesting results are obtained by adding more randomly selected corners to the basic set in order to improve convergence (see Figures 7 – 9 left): it seems that the initial choice of the basic set influences the convergence properties even when many more randomly selected corners are added. Graphs on the right side of these figures show that the best computational time is achieved for higher numbers of corners than the basic sets for all problems tested and all algorithms for selecting the basic set used.

It can be observed especially at the most difficult problem of mine reel (Fig. 9), that the basic set of corners provided by the new algorithm in its full version is much more efficient than the basic sets provided by the earlier approaches and considerably reduces the computational time.

## 8 Conclusion

It has been observed on a number of practical computations by the BDDC method, that the effort to find the minimal set of corners might be misleading and selecting more corners often considerably improves the performance of the preconditioner and reduces the computational time.

This has been the main motivation for presenting a new approach to selecting the basic set of corners, which is proposed in Section 5. It attempts to combine advantages of previous algorithms and it is based on selection of corners independently for each face, so it can be naturally parallelized. It does not aspire to minimizing the number of selected corners that assure the invertibility of all problems in BDDC and typically produces larger initial set of coarse nodes than the other algorithms. We have seen this to be beneficial for all performed computations.

Numerical experiments on three industrial problems show that for basic sets of corners, this approach gives better results than the other two algorithms used for comparison in all three tested problems. When more corners are added, better results are obtained in two of the problems (turbine nozzle and mine reel) and comparable results in the third case (hip joint replacement).

We are aware that for very large problems the solution of the coarse problem might eventually dominate the computation and another approach than a (parallel) direct solver could be necessary. In such cases, multilevel extension of the BDDC method (e.g. [17]) seems to be a promising way. However, we could have observed even for the largest test problem of the mine reel, that we have not reached this computational bottleneck when adding more corners into the coarse problem, and the curve of computational time with respect to the number of corners is still decreasing. This bottleneck is also pushed farther by the everlasting advances in parallel direct solvers.

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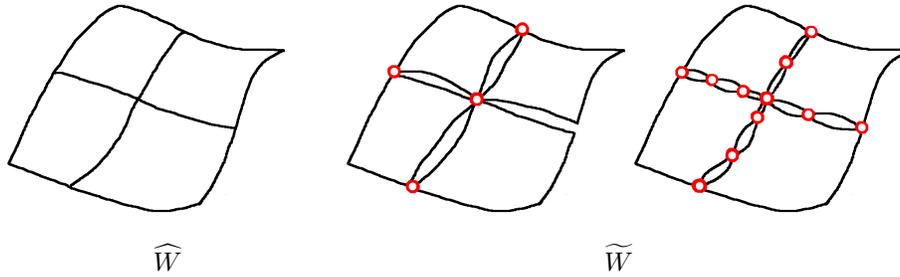


Figure 1: A schematic illustration of the continuity constraints: functions from  $\widehat{W}$  are continuous across the interface (left), functions from  $\widetilde{W}$  are continuous only at corners, marked by circles (centre and right, for two different choices of  $\widetilde{W}$ ).

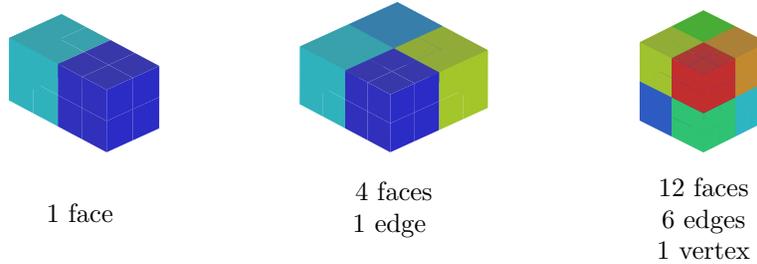


Figure 2: Examples of classification of the interface nodes as faces, edges and vertices according to *Definition 1*.

problem	subd.	vertices	edges	faces	intf. nodes	all nodes
Turbine nozzle	36	6	60	101	2 714	13 418
Hip replacement	36	1	19	78	9 222	181 578
Mine reel	1 024	2 451	1 209	4 164	117 113	579 737

Table 1: Decomposition characteristics of the tested problems.

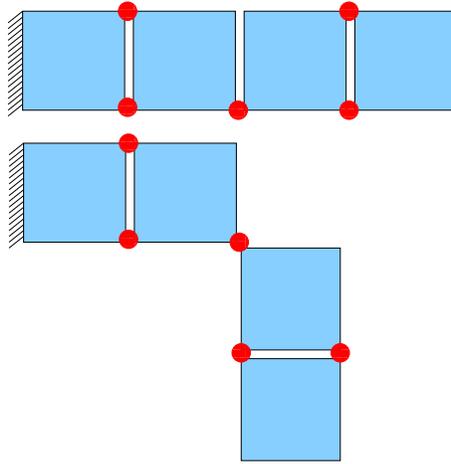


Figure 3: A 2D example of mechanism in the coarse problem for serial division into four subdomains, red dots denote corners.

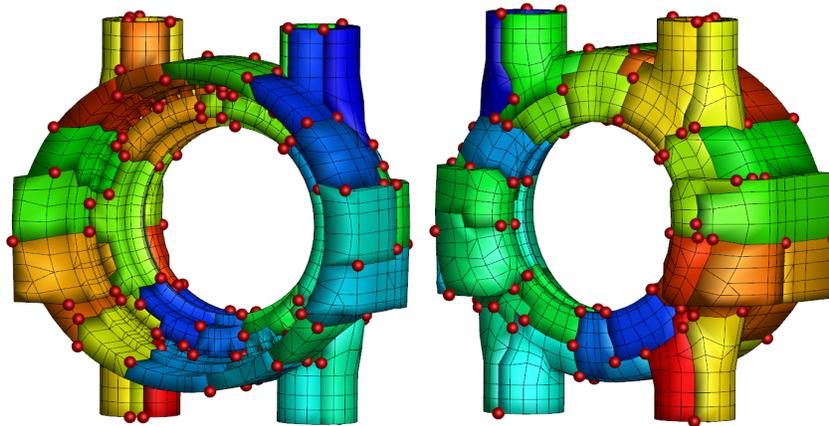


Figure 4: Turbine nozzle problem, 36 subdomains, initial set of 218 corners selected by the full version of Algorithm 1 marked by balls.

problem	full	min	edge
Turbine nozzle	218	145	115
Hip replacement	227	189	66
Mine reel	7 864	6 183	4 152

Table 2: Number of corners in the basic set selected by different algorithms.

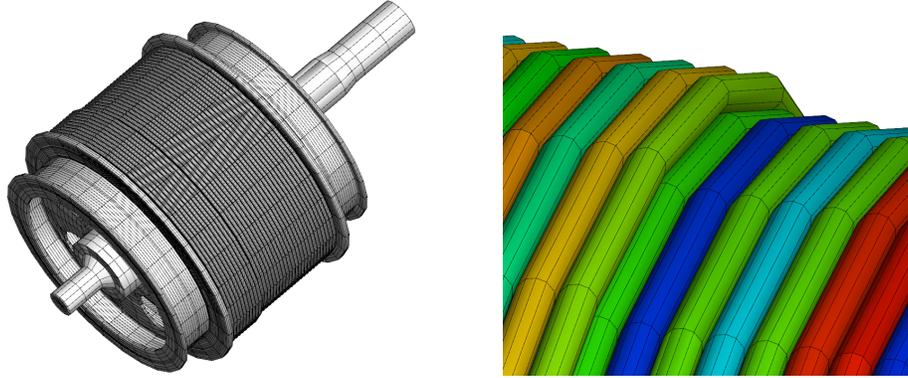


Figure 5: Mine reel problem, finite element mesh (left) and a detail of the steel rope with division into subdomains (right).

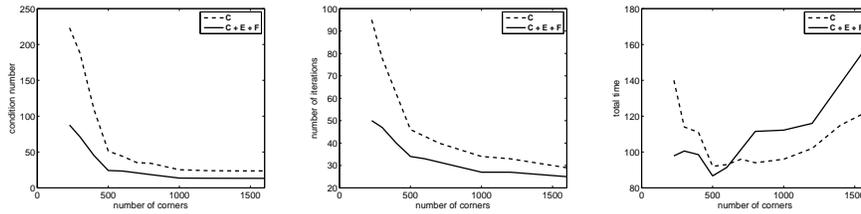


Figure 6: Typical dependence of the condition number (left), the number of iterations of the PCG (centre) and the total computational time (right) on the number of corner constraints. Dashed line - corner constraints only, full line - corner constraints and all face and edge averages. Hip joint replacement, 33 186 quadratic elements, 36 subdomains.

	C			C+E+F		
	full	min	edge	full	min	edge
Turbine nozzle	38	49	73	24	27	29
Hip replacement	95	99	n/a	50	52	n/a
Mine reel	n/a	n/a	n/a	935	1 841	4 637

Table 3: Number of PCG iterations needed for convergence for different algorithms of selecting the basic set of corners and different constraint type.

	C			C+E+F		
	full	min	edge	full	min	edge
Turbine nozzle	38	41	42	24	25	26
Hip replacement	95	91	> 138	50	50	61
Mine reel	n/a	n/a	n/a	935	1 674	≈1 800

Table 4: Number of PCG iterations needed for convergence for different algorithms of selecting the basic set of corners and different constraint type. For every problem, different basic sets were completed to the same number of corners by adding randomly selected corners.

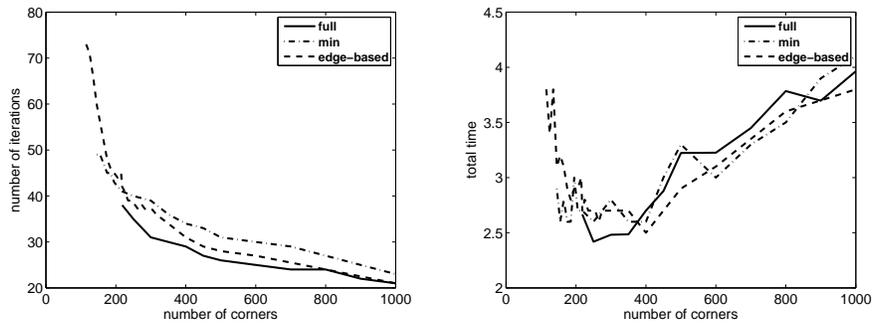


Figure 7: Turbine nozzle problem, 36 subdomains, corner constraints only. Dependence of the number of iterations (left) and the total computational time (right) on the number of corner constraints. Full line - full version of the Algorithm 1, dash-dotted line - minimalistic version, dashed line - the edge based algorithm.

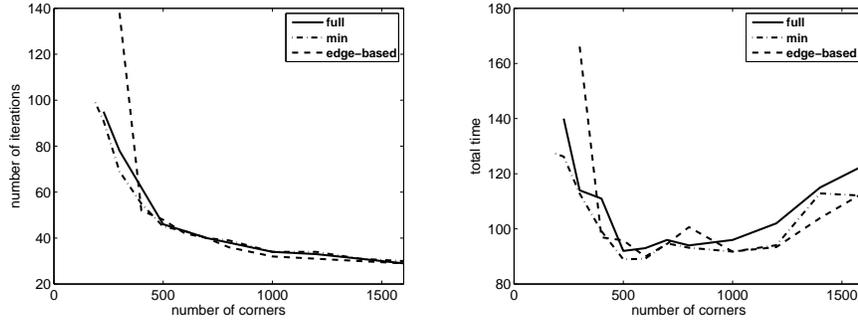


Figure 8: Hip joint replacement problem, 36 subdomains, corner constraints only. Dependence of the number of iterations (left) and the total computational time (right) on the number of corner constraints. Full line - full version of the Algorithm 1, dash-dotted line - minimalistic version, dashed line - the edge based algorithm.

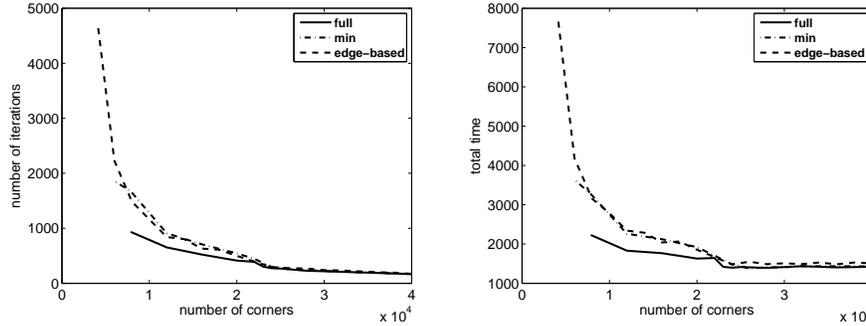


Figure 9: Mine reel problem, 1024 subdomains, corner and all edge and face constraints. A dependence of the number of iterations (left) and the total computational time (right) on the number of corner constraints. Full line - full version of the Algorithm 1, dash-dotted line - minimalistic version, dashed line - the edge based algorithm.