Preprint, Institute of Mathematics, AS CR, Prague. 2010-9-29

Convergence of the Neumann series in BEM for the Neumann problem of the Stokes system

D. Medková

Institute of Mathematics, Academy of Sciences of the Czech Republic, Žitná 25, 115 67 Praha 1, CZECH REPUBLIC

Abstract

A weak solution of the Neumann problem for the Stokes system in Sobolev space is studied in a bounded Lipschitz domain with connected boundary. A solution is looked for in the form of a hydrodynamical single layer potential. It leads to an integral equation on the boundary of the domain. Necessary and sufficient conditions for the solvability are given. Moreover, it is shown that we can obtain a solution of this integral equation using the successive approximation method. Then the consequences for the direct boundary integral equation method are treated. A solution of the Neumann problem for the Stokes system is the sum of the hydrodynamical single layer potential corresponding to the boundary condition and the hydrodynamical double layer potential corresponding to the trace of the velocity part of the solution. The boundary behavior of potentials leads to an integral equation on the boundary of the domain where the trace of the velocity part of the solution is an unknown. It is shown that we can obtain a solution of this integral equation using the successive approximation method.

Keywords: Stokes system; Neumann problem; single layer potential; double layer potential; integral equation method; successive approximation

Mathematics Subject Classification: 76D10, 76D07, 65N38.

1 Introduction.

The use of a harmonic single layer potential for solving the Neumann problem for the Laplace equation has a long history. If we look for a solution of the Neumann problem in the form of a harmonic single layer potential with an unknown density, we obtain an integral equation on the boundary of the domain. (It is so called indirect boundary integral equation method.) It is a classical result that a solution of this integral equation can be constructed by the Neumann series for a bounded convex domain (see [19], [20], [21], [22], [12], [11], [7]). (For the history of the problem see [30].) In 2001 O. Steinbach and W. L. Wendland studied a weak solution in $H^1(G)$ of the Neumann problem for the Laplace equation on a bounded domain G with connected Lipschitz boundary in R^2 and R^3 by the indirect boundary integral equation method ([27]). They proved that a solution of the corresponding integral equation can be calculated by the Neumann series. In 2007 the same result was proved by different methods by M. Constanda (see

[3]) and D. Medková (see [15]). (A similar result for more general open sets can an interested reader find in the paper [16] published in 2009.) O. Steinbach and W. L. Wendland used their result in the study of the Neumann problem for the Laplace equation by the direct integral equation method (see [26] and [9]). This method utilizes the fact that a harmonic function $u \in H^1(G)$ is the sum of the harmonic double layer potential with density given by the trace of u and the harmonic single layer potential corresponding to the normal derivative of u. Using the boundary behavior of harmonic potentials we obtain an integral equation on ∂G, the boundary of G, where the trace of u is an unknown density. Since this equation is adjoint to the integral equation obtained for the indirect boundary integral equation method, they deduced that a solution of the integral equation can be calculated by the Neumann series.

In this paper we obtain a similar result for the Neumann problem of the Stokes system. We construct a solution $\mathbf{u} \in H^1(G)$, $p \in L^2(G)$ of the Neumann problem for the Stokes system

$$\Delta \mathbf{u} = \nabla p \quad \text{in} \quad G, \qquad \nabla \cdot \mathbf{u} = 0 \quad \text{in} \quad G, \tag{1}$$

$$T(\mathbf{u}, p)\mathbf{n}^G = \mathbf{g} \quad \text{on} \quad \partial G \tag{2}$$

using methods of hydrodynamical potential theory. Here G is a bounded domain with connected Lipschitz boundary in R^m, $\mathbf{n}^G$ is the outward unit normal vector of G, $\mathbf{u} = (u_1, \ldots, u_m)$ is a velocity field, p is a pressure and

$$T(\mathbf{u}, p) = 2\hat{\nabla}\mathbf{u} - pI \tag{3}$$

is the corresponding stress tensor. Here I denotes the identity matrix and

$$\hat{\nabla}\mathbf{u} = \frac{1}{2}[\nabla \mathbf{u} + (\nabla \mathbf{u})^T] \tag{4}$$

is the strain tensor, with $(\nabla \mathbf{u})^T$ as the matrix transposed to $\nabla \mathbf{u} = (\partial_j u_k)$, $(k, j = 1, \ldots, m)$. Remark that $\nabla \cdot \mathbf{u} = \partial_1 u_1 + \cdots + \partial_m u_m$ is the divergence of $\mathbf{u}$.

Many authors have studied the Neumann problem for the Stokes system. Y. Shibata, S. Shimizu studied in [25] the Neumann problem for the Stokes system on bounded domains with $C^{1,2}$ boundary and boundary conditions given by the trace of functions from $W^{1,q}(G)$. The Neumann problem for the Stokes system with boundary condition $\mathbf{g} \in L^2(\partial G)$ was studied in [4]. M. Kohr studied by the integral equation method classical solutions of the Neumann problem for the Stokes system on domains with smooth boundary (see [10]). The same problem was studied by the author in [17]. The authors of [10] and [17] found necessary and sufficient conditions for the existence of a classical solution of the problem. The solution of the problem was looked for in the form of a hydrodynamical single layer potential and it was shown that the original problem is equivalent to some boundary integral equation on the space of Hölder continuous functions on the boundary. It was shown in [17] that a solution of the integral equation can be calculated by the Neumann series and the successive approximation method converges.

In the present paper we develop necessary and sufficient conditions for the existence of a solution of the Neumann problem for the Stokes system in $H^1(G)$, where G is bounded domain with connected Lipschitz boundary in R^m. First we study the problem using the indirect integral equation method. We look for a solution in the form of a hydrodynamical potential with an unknown density from $H^{-1/2}(\partial G)$.We construct a solution of the integral equation in the form of a Neumann series. We also prove that a solution of the integral equation can be approximated using the successive approximation method. Then we turn to the direct integral equation method. We shall show again that a solution of the corresponding integral equation can be obtained by the successive approximation. The integral equations corresponding to the direct and indirect BEM are not uniquely solvable. It might be a problem. In a numerical application we substitute the boundary condition $\mathbf{g}$ by some function $\tilde{\mathbf{g}}$ which is close to $\mathbf{g}$. Now we want the solution of the integral equation $S\tilde{\mathbf{\Psi}} = \tilde{\mathbf{g}}$ to be close to the solution of the original equation $S\mathbf{\Psi} = \mathbf{g}$. But the new equation might not be solvable. So, we shall find a modified integral operator $\tilde{S}$ such that the integral equation $\tilde{S}\mathbf{\Psi} = \mathbf{g}$ is uniquely solvable and the corresponding solution $\mathbf{\Psi}$ is also a solution of the original integral equation $S\mathbf{\Psi} = \mathbf{g}$ provided only this equation is solvable. Then we shall construct a solution of the modified integral equation $\tilde{S}\mathbf{\Psi} = \mathbf{g}$ in the form of a Neumann series and prove that a solution of the integral equation $\tilde{S}\mathbf{\Psi} = \mathbf{g}$ can be approximated using the successive approximation method.

2 The surface potentials

The aim of this section is to assemble some basic facts on hydrodynamical potentials.

Let $G \subset R^m$ be an open set with compact Lipschitz boundary ∂G. Denote $G^e := R^m \setminus \mathrm{cl}\, G$ its complement with $\partial G^e = \partial G$. Here $\mathrm{cl}\, G$ denotes the closure of G and ∂G the boundary of G.

If $X(M)$ is a vector space of real functions (or distributions) on a set M denote by $X(M,C)$ its complexification, i.e. $X(M,C) = \{v_1 + iv_2; v_1 \in X(M,R) = X(M), v_2 \in X(M)\}$. If $K = R$ or $K = C$ and $k \in N$, we denote $X(M,K^k) = \{\mathbf{u} = (u_1, \ldots, u_k); u_j \in X(M,K) \text{ for } j = 1, \ldots, k\}$.

Denote by ω_m the surface of the unit sphere in R^m. For $\mathbf{x} \in R^m$ and $j, k = 1, \ldots, m$ define

$$E_{jk}(\mathbf{x}) = \begin{cases} \frac{1}{2\omega_m}\left[\delta_{jk}\frac{|\mathbf{x}|^{2-m}}{m-2} + \frac{x_j x_k}{|\mathbf{x}|^m}\right], & m > 2, \\ \frac{1}{4\pi}\left[\delta_{jk}\ \ln\frac{1}{|\mathbf{x}|} + \frac{x_j x_k}{|\mathbf{x}|^2}\right], & m = 2, \end{cases} \tag{5}$$

$$Q_k(\mathbf{x}) = \frac{x_k}{\omega_m |\mathbf{x}|^m}. \tag{6}$$

For $\mathbf{\Psi} = [\Psi_1, \dots, \Psi_m] \in L^1(\partial G, R^m)$ define the hydrodynamical single layer potential with density $\mathbf{\Psi}$ by

$$(E_G \mathbf{\Psi})(\mathbf{x}) = \int\limits_{\partial G} E(\mathbf{x} - \mathbf{y}) \mathbf{\Psi}(\mathbf{y}) \; d\mathbf{y} \tag{7}$$

whenever it makes sense and the corresponding pressure

$$(Q_G \mathbf{\Psi})(\mathbf{x}) = \int\limits_{\partial G} Q(\mathbf{x} - \mathbf{y}) \mathbf{\Psi}(\mathbf{y}) \; d\mathbf{y}, \qquad \mathbf{x} \in R^m \setminus \partial G. \tag{8}$$

Then $E_G \mathbf{\Psi} \in C^\infty(R^m \setminus \partial G, R^m)$, $Q_G \mathbf{\Psi} \in C^\infty(R^m \setminus \partial G, R^1)$, $\nabla Q_G \mathbf{\Psi} - \Delta E_G \mathbf{\Psi} = 0$, $\nabla \cdot E_G \mathbf{\Psi} = 0$ in $R^m \setminus \partial G$. We have the following decay behavior as $|\mathbf{x}| \to \infty$:

$$E_G \mathbf{\Psi}(\mathbf{x}) = O(|\mathbf{x}|^{2-m}), \quad m > 2,$$

$$E_G \mathbf{\Psi}(\mathbf{x}) = O(\ln |\mathbf{x}|), \quad m = 2,$$

$$Q_G \mathbf{\Psi}(\mathbf{x}), \; |(\nabla E_G \mathbf{\Psi})(\mathbf{x})| = O(|\mathbf{x}|^{1-m}).$$

If $m = 2$ and

$$\int\limits_{\partial G} \mathbf{\Psi}(\mathbf{y}) \; d\mathbf{y} = 0$$

then

$$E_G \mathbf{\Psi}(\mathbf{x}) = O(|\mathbf{x}|^{-1}), \quad |\nabla E_G \mathbf{\Psi}(\mathbf{x})| = O(|\mathbf{x}|^{-2}).$$

Now we define a hydrodynamical double layer potential. Fix $\mathbf{y} \in \partial G$ such that there is the unit outward normal $\mathbf{n}^G(\mathbf{y})$ of G at $\mathbf{y}$. For $\mathbf{x} \in R^m \setminus \{\mathbf{y}\}$, $j, k \in \{1, \dots, m\}$ put

$$K^G_{jk}(\mathbf{x}, \mathbf{y}) = \frac{m}{\omega_m} \frac{(y_j - x_j)(y_k - x_k)(\mathbf{y} - \mathbf{x}) \cdot \mathbf{n}^G(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|^{m+2}},$$

$$\Pi^G_k(\mathbf{x}, \mathbf{y}) = \frac{2}{\omega_m} \left\{ -m \frac{(y_k - x_k)(\mathbf{y} - \mathbf{x}) \cdot \mathbf{n}^G(\mathbf{y})}{|\mathbf{y} - \mathbf{x}|^{m+2}} + \frac{n^G_k(\mathbf{y})}{|\mathbf{y} - \mathbf{x}|^m} \right\}.$$

For $\mathbf{\Psi} = [\Psi_1, \dots, \Psi_m] \in L^1(\partial G, R^m)$ define the hydrodynamical double layer potential with density $\mathbf{\Psi}$ by

$$(D_G \mathbf{\Psi})(\mathbf{x}) = \int\limits_{\partial G} K^G(\mathbf{x}, \mathbf{y}) \mathbf{\Psi}(\mathbf{y}) \; d\mathbf{y} \tag{9}$$

and the corresponding pressure

$$(\Pi_G \mathbf{\Psi})(\mathbf{x}) = \int\limits_{\partial G} \Pi^G(\mathbf{x}, \mathbf{y}) \mathbf{\Psi}(\mathbf{y}) \; d\mathbf{y} \tag{10}$$

in $R^m \setminus \partial G$. Then $D_G \mathbf{\Psi} \in C^\infty(R^m \setminus \partial G, R^m)$, $\Pi_G \mathbf{\Psi} \in C^\infty(R^m \setminus \partial G, R^1)$ and $\nabla \Pi_G \mathbf{\Psi} - \Delta D_G \mathbf{\Psi} = 0$, $\nabla \cdot D_G \mathbf{\Psi} = 0$ in $R^m \setminus \partial G$. We have the following decay behavior as $|\mathbf{x}| \to \infty$:

$$(D_G \mathbf{\Psi})(\mathbf{x}) = O(|\mathbf{x}|^{1-m}),$$

$$|(\nabla D_G \mathbf{\Psi})(\mathbf{x})|, \ \Pi_G \mathbf{\Psi}(\mathbf{x}) = O(|\mathbf{x}|^{-m}).$$

Now we gather boundary properties of hydrodynamical potentials. For $\mathbf{\Psi} = [\Psi_1, \dots, \Psi_m] \in L^1(\partial G, R^m)$ and $\mathbf{x} \in \partial G$ define

$$K_G \mathbf{\Psi}(\mathbf{x}) = \lim_{\epsilon \searrow 0} \int_{\partial G \setminus B(\mathbf{x};\epsilon)} K^G(\mathbf{x}, \mathbf{y}) \mathbf{\Psi}(\mathbf{y}) \, d\mathbf{y},$$

$$K'_G \mathbf{\Psi}(\mathbf{x}) = \lim_{\epsilon \searrow 0} \int_{\partial G \setminus B(\mathbf{x};\epsilon)} K^G(\mathbf{y}, \mathbf{x}) \mathbf{\Psi}(\mathbf{y}) \, d\mathbf{y}$$

whenever these limits exist.

If $\mathbf{x} \in \partial G$, $a > 0$ denote the non-tangential approach regions of opening a at the point $\mathbf{x}$ by

$$\Gamma_a(\mathbf{x}) := \{\mathbf{y} \in G; |\mathbf{x} - \mathbf{y}| < (1 + a) \operatorname{dist}(\mathbf{y}, \partial G)\}.$$

Denote

$$\Gamma^e_a(\mathbf{x}) := \{\mathbf{y} \in G^e; |\mathbf{x} - \mathbf{y}| < (1 + a) \operatorname{dist}(\mathbf{y}, \partial G^e)\}$$

the non-tangential approach regions of opening a at the point $\mathbf{x}$ corresponding to $G^e = R^m \setminus \operatorname{cl} G$. We fix $a > 0$ large enough such that $\mathbf{x} \in \operatorname{cl} \Gamma_a(\mathbf{x}) \cap \operatorname{cl} \Gamma^e_a$ for every $\mathbf{x} \in \partial G$. We shall write $\Gamma(\mathbf{x}) = \Gamma_a(\mathbf{x})$, $\Gamma^e(\mathbf{x}) = \Gamma^e_a(\mathbf{x})$. If now $\mathbf{v}$ is a vector function defined in G and $\mathbf{x} \in \partial G$ then the non-tangential maximal function of $\mathbf{v}$ is defined by

$$\mathbf{v}^*(\mathbf{x}) = \sup_{\mathbf{y} \in \Gamma(\mathbf{x})} \mathbf{v}(\mathbf{y})$$

and

$$\mathbf{v}_+(\mathbf{x}) = \lim_{\substack{\mathbf{y} \to \mathbf{x} \\ \mathbf{y} \in \Gamma(\mathbf{x})}} \mathbf{v}(\mathbf{y})$$

is the non-tangential limit of $\mathbf{v}$ at $\mathbf{x}$ with respect to G. Similarly, if $\mathbf{v}$ is a vector function defined in G^e and $\mathbf{x} \in \partial G$ then

$$\mathbf{v}_-(\mathbf{x}) = \lim_{\substack{\mathbf{y} \to \mathbf{x} \\ \mathbf{y} \in \Gamma^e(\mathbf{x})}} \mathbf{v}(\mathbf{y})$$

is the non-tangential limit of $\mathbf{v}$ at $\mathbf{x}$ with respect to G^e.

If $\mathbf{\Psi} \in L^2(\partial G, R^m)$ then there is $K_G \mathbf{\Psi}(\mathbf{x})$ at almost all $\mathbf{x} \in \partial G$ and K_G is a bounded linear operator in $L^2(\partial G, R^m)$ (see [14], Proposition 3.2 and [14], Corollary 3.3; compare also [4], [6]). Moreover, if G is bounded then $[D_G\mathbf{\Psi}]^* \in L^2(\partial G, R^1)$ and

$$[D_G\mathbf{\Psi}]_+(\mathbf{x}) = \frac{1}{2}\mathbf{\Psi}(\mathbf{z}) + K_G\mathbf{\Psi}(\mathbf{z}), \quad [D_G\mathbf{\Psi}]_-(\mathbf{x}) = -\frac{1}{2}\mathbf{\Psi}(\mathbf{z}) + K_G\mathbf{\Psi}(\mathbf{z}) \tag{11}$$

for almost all $\mathbf{x} \in \partial G$ (see [14], Proposition 3.2).

We shall study the Neumann problem in the Sobolev space $H^1(G; R^m)$. We denote by $H^s(G)$ the Sobolev-Slobodetski space of order s. Remark that $H^0(G) = L^2(G)$ and $H^1(G) = \{f \in L^2(G); \nabla f \in L^2(G; R^m)\}$ is equipped with the norm

$$\|f\|_{H^1(G)} = \left\{ \int\limits_G \left[f^2 + |\nabla f|^2 \right] \, d\mathbf{x} \right\}^{1/2}.$$

If φ is a Lipschitz function on R^{m-1} and $S = \{[\mathbf{x}, \varphi(\mathbf{x})]; \mathbf{x} \in R^{m-1}\}$ we say that $f \in H^s(S)$ if $f(\mathbf{x}, \varphi(\mathbf{x})) \in H^s(R^{m-1})$. Since G has Lipschitz boundary there are bounded open sets $U_1, \ldots, U_k$ and Lipschitz functions $\varphi_1, \ldots, \varphi_k$ such that $\partial G \subset U_1 \cup \cdots \cup U_k$ and for each $j \in \{1, \ldots, k\}$ there is a coordinate system such that $U_j \cap \partial G = U_j \cap S_j$ with $S_j = \{[\mathbf{x}, \varphi_j(\mathbf{x})]; \mathbf{x} \in R^{m-1}\}$. Choose $\omega_j \in C^\infty(R^m)$ supported in U_j with $0 \le \omega_j \le 1$ for $j = 1, \ldots, k$ such that $\omega_1 + \omega_2 + \ldots \omega_k = 1$ on a neighborhood of ∂G. We say that $f \in H^s(\partial G)$ if $\omega_j f \in H^s(S_j)$ for $j = 1, \ldots, k$.

Recall that $H^{1/2}(\partial G)$ is the space of traces of $H^1(G)$ endowed with the norm

$$\|v\|_{H^{1/2}(\partial G)} = \inf\{\|u\|_{H^1(G)}; u \in H^1(G), v = u|\partial G\} \tag{12}$$

and $H^{-1/2}(\partial G)$ is the dual space of $H^{1/2}(\partial G)$.

If G is bounded and $\mathbf{\Psi} \in H^{1/2}(\partial G; R^m)$ then $D_G\mathbf{\Psi} \in H^1(G; R^m)$ and there is a constant C depending only on G such that

$$\|D_G\mathbf{\Psi}\|_{H^1(G;R^m)} \le C\|\mathbf{\Psi}\|_{H^{1/2}(\partial G;R^m)}$$

(see [14], Theorem 4.4). Since $H^{1/2}(\partial G; R^m) \subset L^2(\partial G; R^m)$, we see using the relation (11) that $\frac{1}{2}\mathbf{\Psi} + K_G\mathbf{\Psi}$ is the trace of $D_G\mathbf{\Psi}$. Moreover, K_G is a bounded linear operator on $H^{1/2}(\partial G; R^m)$ (compare [14], Proposition 4.5).

K'_G is a bounded linear operator in $L^2(\partial G, R^m)$ which is the adjoint operator of K_G. If $\mathbf{\Psi} \in L^2(\partial G, R^m)$ then there is $K'_G\mathbf{\Psi}(\mathbf{x})$ at almost all $\mathbf{x} \in \partial G$. If G is bounded then $[E_G\mathbf{\Psi}]^* \in L^2(\partial G, R^1)$, $[\nabla E_G\mathbf{\Psi}]^* \in L^2(\partial G, R^1)$, $[Q_G\mathbf{\Psi}]^* \in L^2(\partial G, R^1)$. For almost all $\mathbf{x} \in \partial G$ we have

$$[E_G\mathbf{\Psi}]_+(\mathbf{x}) = [E_G\mathbf{\Psi}]_-(\mathbf{x}) = E_G\mathbf{\Psi}(\mathbf{x}), \tag{13}$$

$$[T(E_G\mathbf{\Psi}, Q_G\mathbf{\Psi})]_+(\mathbf{x})\mathbf{n}^G(\mathbf{x}) = \frac{1}{2}\mathbf{\Psi}(\mathbf{x}) - K'_G\mathbf{\Psi}(\mathbf{x}), \tag{14}$$

$$[T(E_G\mathbf{\Psi}, Q_G\mathbf{\Psi})]_-(\mathbf{x})\mathbf{n}^G(\mathbf{x}) = -\frac{1}{2}\mathbf{\Psi}(\mathbf{x}) - K'_G\mathbf{\Psi}(\mathbf{x}). \tag{15}$$

(Compare [14], [6],[4]).

If G is bounded then $E_G : \mathbf{\Psi} \mapsto E_G\mathbf{\Psi}$ represents a bounded linear operator from $H^{-1/2}(\partial G, R^m)$ to $H^1(G, R^m)$ and $Q_G : \mathbf{\Psi} \mapsto E_Q\mathbf{\Psi}$ is a continuous linear operator from $H^{-1/2}(\partial G, R^m)$ to $L^2(G, R^1)$ (see [14], Theorem 4.4). If $\mathbf{\Psi} \in H^{-1/2}(\partial G, R^m)$ then $E_G\mathbf{\Psi}$ is the trace of $E_G\mathbf{\Psi}$ on ∂G. Moreover, $E_G : \mathbf{\Psi} \mapsto E_G\mathbf{\Psi}$ is a bounded linear operator from $H^{-1/2}(\partial G; R^m)$ to $H^{1/2}(\partial G; R^m)$ (see [14], Proposition 4.5). Since K_G is a bounded linear operator on $L^2(\partial G, R^m)$ and on $H^{1/2}(\partial G, R^m)$ the operator K_G' can be extended as a bounded linear operator on $H^{-1/2}(\partial G, R^m)$ which is the adjoint operator of K_G. If $\mathbf{\Psi} \in H^{-1/2}(\partial G, R^m)$ we can understand $\frac{1}{2}\mathbf{\Psi} - K_G'\mathbf{\Psi}$ as $T(E_G\mathbf{\Psi}, Q_G\mathbf{\Psi})\mathbf{n}^G$ on ∂G. Again $E_G\mathbf{\Psi} \in C^\infty(R^m \setminus \partial G, R^m)$, $Q_G\mathbf{\Psi} \in C^\infty(R^m \setminus \partial G, R^1)$, $\nabla Q_G\mathbf{\Psi} - \Delta E_G\mathbf{\Psi} = 0$, $\nabla \cdot E_G\mathbf{\Psi} = 0$ in $R^m \setminus \partial G$.

It is well-known that for G bounded, $\mathbf{u} \in C^2(\mathrm{cl}\, G, R^m)$ and $p \in C^1(\mathrm{cl}\, G, R^1)$ a classical solution of the Neumann problem for the Stokes system (1), (2) with boundary condition $\mathbf{g}$ in G we have

$$\mathbf{u}(\mathbf{x}) = E_G\mathbf{g}(\mathbf{x}) + D_G\mathbf{u}(\mathbf{x}), \tag{16}$$

$$p(\mathbf{x}) = Q_G\mathbf{g}(\mathbf{x}) + \Pi_G\mathbf{u}(\mathbf{x}) \tag{17}$$

(compare [13], Chapter 3, § 2). If $\mathbf{u} \equiv 0$, $p \equiv 1$ then $\mathbf{g} = -\mathbf{n}^G$ and (16), (17) give

$$E_G\mathbf{n}^G = 0, \quad Q_G\mathbf{n}^G = -1 \quad \text{in } G. \tag{18}$$

3 Weak solution of the problem

If $A, B \in R^{m\times m}$ are matrices with $A = (A_{ij})$, $B = (B_{ij})$ denote

$$A : B = \sum_{i,j=1}^{3} A_{ij}B_{ij}.$$

If G is a bounded open set with smooth boundary, $(\mathbf{u}, p)$ is a classical solution of the Neumann problem for the Stokes system (1), (2) and $\mathbf{v} \in C^2(R^m, R^m)$, then the Green formula yields

$$\int\limits_{\partial G} [T(\mathbf{u}, p)\mathbf{n}^G] \cdot \mathbf{v}\; d\mathbf{y} = 2\int\limits_G \hat{\nabla}\mathbf{u} : \hat{\nabla}\mathbf{v}\; d\mathbf{y} - \int\limits_G p(\nabla \cdot \mathbf{v})\; d\mathbf{y} \tag{19}$$

(see for example (2.15) in [2]). It is well known that if $\mathbf{u} \in H^1(G, R^m; \mathrm{div}) = \{\mathbf{v} \in H^1(G, R^m); \nabla \cdot \mathbf{v} = 0 \text{ in } G\}$, $p \in L^2(G, R^1)$ and

$$2\int\limits_G \hat{\nabla}\mathbf{u} : \hat{\nabla}\mathbf{v}\; d\mathbf{y} - \int\limits_G p(\nabla \cdot \mathbf{v})\; d\mathbf{y} = 0$$

for each $\mathbf{v} \in C^\infty(G; R^m)$ with compact support in G then $\mathbf{u}$, p is a solution of the Stokes system (1) in G. Using this fact we formulate a weak solution of the problem (1), (2) as follows:

Let $\mathbf{g} \in H^{-1/2}(\partial G, R^m)$ *and* $G \subset R^m$ *be a bounded open set with Lipschitz boundary. We say that* $\mathbf{u} \in H^1(G, R^m; \mathrm{div})$, $p \in L^2(G, R^1)$ *is a weak solution of the problem (1), (2) if*

$$2 \int_G \hat{\nabla} \mathbf{u} : \hat{\nabla} \mathbf{v} \; d\mathbf{y} - \int_G p(\nabla \cdot \mathbf{v}) \; d\mathbf{y} = \langle \mathbf{g}, \mathbf{v} \rangle \tag{20}$$

for each $\mathbf{v} \in H^1(G, R^m)$.

Remark that if $G \subset R^m$ is a bounded open set with Lipschitz boundary, $\mathbf{g} \in H^{-1/2}(\partial G, R^m)$ and $\mathbf{u} \in H^1(G, R^m; \mathrm{div})$, $p \in L^2(G, R^1)$ is a weak solution of the problem (1), (2) then

$$2 \int_G \hat{\nabla} \mathbf{u} : \hat{\nabla} \mathbf{v} \; d\mathbf{y} = \langle \mathbf{g}, \mathbf{v} \rangle \quad \forall \mathbf{v} \in H^1(G, R^m; \mathrm{div}). \tag{21}$$

Denote $\mathcal{R}_m = \{\mathbf{f}(\mathbf{x}) = A\mathbf{x} + \mathbf{b}; \mathbf{b} \in R^m, A$ is a skew symmetric matrix $(A^T = -A)\}$ the space of rigid body motions. Easy calculation yields that $\hat{\nabla} \mathbf{w} = 0$ and $\nabla \cdot \mathbf{w} = 0$ for each $\mathbf{w} \in \mathcal{R}_m$. Remark that $\dim \mathcal{R}_m = m(m+1)/2$.

Lemma 3.1. *Let* Ω *be a domain in* R^m, $\mathbf{u} \in L^1_{loc}(\Omega, R^m)$. *Then* $\hat{\nabla} \mathbf{u} \equiv 0$ *in* Ω *in the sense of distributions if and only if* $\mathbf{u} \in \mathcal{R}_m$.

Proof. Easy calculation yields that $\hat{\nabla} \mathbf{u} = 0$ in classical sense for each $\mathbf{u} \in \mathcal{R}_m$.

Let now $\hat{\nabla} \mathbf{u} \equiv 0$ in Ω. Since $\partial_r u_s + \partial_s u_r = 0$ for arbitrary $r, s \in \{1, \ldots, m\}$ and $\partial_j \partial_k u_l = \partial_k \partial_j u_l$ in the sense of distributions, we obtain for fixed $j, k, l \in \{1, \ldots, m\}$

$$\partial_j \partial_k u_l = \frac{1}{2}[\partial_j(\partial_k u_l) + \partial_k(\partial_j u_l)] = -\frac{1}{2}[\partial_j(\partial_l u_k) + \partial_k(\partial_l u_j)]$$

$$= -\frac{1}{2} \partial_l (\partial_j u_k + \partial_k u_j) = 0.$$

Since $\nabla \partial_k u_l = (0, \ldots, 0)$ in Ω, Lemma 6.4 in [28] gives that $\partial_k u_l$ is constant in Ω. This means that u_l is affine. Thus there is a matrix A of the type $m \times m$ and a vector $\mathbf{b} \in R^m$ such that $\mathbf{u}(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$. Since $\hat{\nabla} \mathbf{u} = 0$, the matrix A is a skew symmetric matrix.

Proposition 3.2. *Let* $G \subset R^m$ *be a bounded domain with Lipschitz boundary* ∂G, $\mathbf{g} \in H^{-1/2}(\partial G, R^m)$, $\mathbf{u} \in H^1(G, R^m; \mathrm{div})$, $p \in L^2(G, R^1)$. *If (21) holds true then* $\langle g, \mathbf{v} \rangle = 0$ *for each* $\mathbf{v} \in \mathcal{R}_m$. *If* $\mathbf{u}$, p *is a weak solution of the problem (1),* *(2) then $\{(\mathbf{u}+\mathbf{w},p);\mathbf{w}\in\mathcal{R}_m\}$ is the set of all weak solutions of the problem (1), (2).*

Proof. If $\mathbf{v}\in\mathcal{R}_m$ then $\hat{\nabla}\mathbf{v}=0$, $\nabla\cdot\mathbf{v}=0$ and (21) gives

$$\langle \mathbf{g},\mathbf{v}\rangle = 2\int_G \hat{\nabla}\mathbf{u}:\hat{\nabla}\mathbf{v}\;d\mathbf{y}-\int_G p(\nabla\cdot\mathbf{v})\;d\mathbf{y}=0.$$

Let $\mathbf{u}$, p be a weak solution of the problem (1), (2). If $\mathbf{w}\in\mathcal{R}_m$ and $\mathbf{v}\in H^1(G,R^m)$ then

$$2\int_G \hat{\nabla}(\mathbf{u}+\mathbf{w}):\hat{\nabla}\mathbf{v}\;d\mathbf{y}-\int_G p(\nabla\cdot\mathbf{v})\;d\mathbf{y}$$

$$=2\int_G \hat{\nabla}\mathbf{u}:\hat{\nabla}\mathbf{v}\;d\mathbf{y}-\int_G p(\nabla\cdot\mathbf{v})\;d\mathbf{y}=\langle \mathbf{g},\mathbf{v}\rangle$$

and $\mathbf{u}+\mathbf{w}$, p is a solution of the problem (1), (2). Let now $\mathbf{v}$, q be a solution of the problem (1), (2). Put $\mathbf{w}=\mathbf{v}-\mathbf{u}$. Since $\nabla\cdot\mathbf{w}=\nabla\cdot\mathbf{v}-\nabla\cdot\mathbf{u}=0$ we have

$$0=\left[2\int_G \hat{\nabla}\mathbf{v}:\hat{\nabla}\mathbf{w}\;d\mathbf{y}-\int_G q(\nabla\cdot\mathbf{w})\;d\mathbf{y}\right]-\left[2\int_G \hat{\nabla}\mathbf{u}:\hat{\nabla}\mathbf{w}\;d\mathbf{y}-\int_G p(\nabla\cdot\mathbf{w})\;d\mathbf{y}\right]$$

$$=2\int_G \hat{\nabla}\mathbf{w}:\hat{\nabla}\mathbf{w}\;d\mathbf{y}$$

Since $\hat{\nabla}\mathbf{w}=0$ almost everywhere in a domain G, Lemma 3.1 gives $\mathbf{w}\in\mathcal{R}_m$. Fix $\varphi\in C^\infty(G,R^1)$ and $j\in\{1,\dots,m\}$. Put $\mathbf{\Phi}=(\Phi_1,\dots,\Phi_m)$ with $\Phi_j=\varphi$, $\Phi_k=0$ for $k\neq j$. Then

$$0=\left[2\int_G \hat{\nabla}\mathbf{v}:\hat{\nabla}\mathbf{\Phi}\;d\mathbf{y}-\int_G q(\nabla\cdot\mathbf{\Phi})\;d\mathbf{y}\right]-\left[2\int_G \hat{\nabla}\mathbf{u}:\hat{\nabla}\mathbf{\Phi}\;d\mathbf{y}-\int_G p(\nabla\cdot\mathbf{\Phi})\;d\mathbf{y}\right]$$

$$=\int_G (p-q)(\nabla\cdot\mathbf{\Phi})\;d\mathbf{y}=\int_G (p-q)\partial_j\varphi\;d\mathbf{y}.$$

Since $\nabla(p-q)=0$ in G in the sense of distributions, $(p-q)$ is constant (see Lemma 6.4 in [28]). If φ and j is as above we have from the Green formula

$$\int_{\partial G} n_j(p-q)\varphi\;d\mathbf{y}=\int_G (p-q)\partial_j\varphi\;d\mathbf{y}=0.$$

Since φ was arbitrary we infer that $n_j(p-q)=0$ on ∂G. Since j was arbitrary we have $|p-q|=|(n_1(p-q),\dots,n_m(p-q))|=0$.

4 Indirect BEM

In this section we shall study the problem using the indirect boundary integral equation method. We shall look for a solution in the form of a hydrodynamical single layer potential $E_G\mathbf{\Psi}$ with a density $\mathbf{\Psi} \in H^{-1/2}(\partial G)$.

Lemma 4.1. *If $G \subset R^m$ is bounded open set with Lipschitz boundary then there is a sequence of C^∞ domains G_j with following properties:*

- $\mathrm{cl}\, G_j \subset G$.
- *There are $a > 0$ and homeomorphisms $\Lambda_j : \partial G \to \partial G_j$, such that $\Lambda_j(\mathbf{y}) \in \Gamma_a(\mathbf{y})$ for each j and each $\mathbf{y} \in \partial G$ and $\sup\{|\mathbf{y} - \Lambda_j(\mathbf{y})|; \mathbf{y} \in \partial G\} \to 0$ as $j \to \infty$.*
- *There are positive functions ω_j on ∂G bounded away from zero and infinity uniformly in j such that for any measurable set $E \subset \partial G$,*

$$\int_E \omega_j \, d\mathbf{y} = \int_{\Lambda_j(E)} 1 \, d\mathbf{y},$$

  and so that $\omega_j \to 1$ pointwise a.e. and in $L^2(\partial G, R^1)$.
- *The normal vectors to G_j, $\mathbf{n}(\Lambda_j(\mathbf{y}))$, converge pointwise a.e. and in $L^2(\partial G, R^m)$ to $\mathbf{n}(\mathbf{y})$.*

(For the proof see [29], Theorem 1.12.)

Proposition 4.2. *Let $G \subset R^m$ be a bounded open set with Lipschitz boundary, $m \geq 2$, $\mathbf{\Psi}, \mathbf{g} \in H^{-1/2}(\partial G, R^m)$. Then $\mathbf{u} = E_G\mathbf{\Psi}$, $p = Q_G\mathbf{\Psi}$ is a weak solution of the Neumann problem for the Stokes system with the boundary condition $\mathbf{g}$ (1), (2) if and only if $\frac{1}{2}\mathbf{\Psi} - K'_G\mathbf{\Psi} = \mathbf{g}$.*

Proof. Suppose first that $\mathbf{\Psi} \in L^2(\partial G, R^m)$. Then $\mathbf{h} = \frac{1}{2}\mathbf{\Psi} - K'_G\mathbf{\Psi} \in L^2(\partial G, R^m) \subset H^{-1/2}(\partial G, R^m)$. Fix $\mathbf{v} \in C^\infty(R^m, R^m)$. Let G_j be domains from Lemma 4.1. Since $E_G\mathbf{\Psi} \in H^1(G, R^m)$, $Q_G\mathbf{\Psi} \in L^2(G)$ we obtain using (19), Lemma 4.1, properties of hydrodynamical potentials and Lebesgue lemma

$$2\int_G \hat{\nabla}\mathbf{u} : \hat{\nabla}\mathbf{v} \, d\mathbf{y} - \int_G p(\nabla \cdot \mathbf{v}) \, d\mathbf{y} = \lim_{j\to\infty} \left[2\int_{G_j} \hat{\nabla}\mathbf{u} : \hat{\nabla}\mathbf{v} \, d\mathbf{y} - \int_{G_j} p(\nabla \cdot \mathbf{v}) \, d\mathbf{y} \right]$$

$$= \lim_{j\to\infty} \left\{ \int_{\partial G_j} [T(\mathbf{u}, p)\mathbf{n}] \cdot \mathbf{v} \, d\mathbf{y} \right\} = \int_{\partial G} [T(\mathbf{u}, p)\mathbf{n}] \cdot \mathbf{v} \, d\mathbf{y}$$

$$= \int_{\partial G} \left[\frac{1}{2}\mathbf{\Psi} - K'_G\mathbf{\Psi} \right] \cdot \mathbf{v} \, d\mathbf{y} = \int_{\partial G} \mathbf{h} \cdot \mathbf{v} \, d\mathbf{y} = \langle \mathbf{h}, \mathbf{v} \rangle.$$

Since $C^\infty(R^m, R^m)$ is a dense subset of $H^1(G, R^m)$ we have

$$2\int_G \hat{\nabla}\mathbf{u} : \hat{\nabla}\mathbf{v}\ d\mathbf{y} - \int_G p(\nabla\cdot\mathbf{v})\ d\mathbf{y} = \langle \mathbf{h}, \mathbf{v}\rangle \quad \forall\ \mathbf{v} \in H^1(G, R^m).$$

Let now $\mathbf{\Psi} \in H^{-1/2}(\partial G, R^m)$. Then there are $\mathbf{\Psi}_j \in L^2(\partial G, R^m)$ such that $\mathbf{\Psi}_j \to \mathbf{\Psi}$ in $H^{-1/2}(\partial G, R^m)$. Denote $\mathbf{h} = \frac{1}{2}\mathbf{\Psi} - K_G'\mathbf{\Psi}$, $\mathbf{h}_j = \frac{1}{2}\mathbf{\Psi} - K_G'\mathbf{\Psi}_j$. Since $E_G\Psi_j \to E_G\Psi$ in $H^1(G, R^m)$, $Q_G\Psi_j \to Q_G\Psi$ in $L^2(G, R^1)$ and $\mathbf{h}_j \to \mathbf{h}$ in $H^{-1/2}(\partial G, R^m)$ we have for $\mathbf{v} \in H^1(G, R^m)$

$$2\int_G \hat{\nabla}\mathbf{u} : \hat{\nabla}\mathbf{v}\ d\mathbf{y} - \int_G p(\nabla\cdot\mathbf{v})\ d\mathbf{y} = \lim_{j\to\infty}\Bigg[2\int_G (\hat{\nabla}E_G\mathbf{\Psi}_j) : \hat{\nabla}\mathbf{v}\ d\mathbf{y}$$

$$-\int_G (Q_G\mathbf{\Psi}_j)(\nabla\cdot\mathbf{v})\ d\mathbf{y}\Bigg] = \lim_{j\to\infty}\langle \mathbf{h_j}, \mathbf{v}\rangle = \langle \mathbf{h}, \mathbf{v}\rangle.$$

Proposition 4.3. *Let $G \subset R^m$ be an open set with compact Lipschitz boundary, $m \geq 2$. Let $\mathbf{\Psi}_1, \mathbf{\Psi}_2 \in H^{-1/2}(\partial G, R^m)$. If $m = 2$ and G is unbounded suppose moreover that $\langle \mathbf{\Psi}_1, \mathbf{c}\rangle = \langle \mathbf{\Psi}_2, \mathbf{c}\rangle = 0$ for each $\mathbf{c} \in R^m$. Then*

$$\left\langle \frac{1}{2}\mathbf{\Psi} - K_G'\mathbf{\Psi}_1, E_G\mathbf{\Psi}_2\right\rangle = 2\int_G (\hat{\nabla}E_G\mathbf{\Psi}_1) : (\hat{\nabla}E_G\mathbf{\Psi}_2)\ d\mathbf{y}. \tag{22}$$

Put $\mathbf{\Psi} = \mathbf{\Psi}_1 + i\mathbf{\Psi}_2$ where i is the imaginary unit. Denote $\overline{\mathbf{\Psi}} = \mathbf{\Psi}_1 - i\mathbf{\Psi}_2$ the conjugate of $\mathbf{\Psi}$. Then

$$\left\langle \frac{1}{2}\mathbf{\Psi} - K_G'\mathbf{\Psi}, E_G\overline{\mathbf{\Psi}}\right\rangle = 2\int_G (|\hat{\nabla}E_G\mathbf{\Psi}_1|^2 + |\hat{\nabla}E_G\mathbf{\Psi}_2|^2)\ d\mathbf{y} \geq 0. \tag{23}$$

Proof. We show (22). Suppose first that G is bounded. Proposition 4.2 gives that $\mathbf{u} = E_G\mathbf{\Psi}_1$, $p = Q_G\mathbf{\Psi}_1$ is a weak solution of the Neumann problem for the Stokes system (1), (2) with $\mathbf{g} = \frac{1}{2}\mathbf{\Psi}_1 - K_G'\mathbf{\Psi}_1$. Since $\mathbf{v} = E_G\mathbf{\Psi}_2 \in H^1(G; R^m)$ and $\nabla\cdot E_G\mathbf{\Psi}_2 = 0$ in G we have

$$\left\langle \frac{1}{2}\mathbf{\Psi}_1 - K_G'\mathbf{\Psi}_1, E_G\mathbf{\Psi}_2\right\rangle = 2\int_G (\hat{\nabla}E_G\mathbf{\Psi}_1) : (\hat{\nabla}E_G\mathbf{\Psi}_2)\ d\mathbf{y}$$

$$-\int_G Q_G\mathbf{\Psi}_1(\nabla\cdot E_G\mathbf{\Psi}_2)\ d\mathbf{y} = 2\int_G (\hat{\nabla}E_G\mathbf{\Psi}_1) : (\hat{\nabla}E_G\mathbf{\Psi}_2)\ d\mathbf{y}.$$

Let now G be unbounded. Fix $R > 0$ such that $\partial G \subset B(0; R)$ and denote $G(R) = G \cap B(0; R)$. Put $\mathbf{\Psi}_1 = 0 = \mathbf{\Psi}_2$ on $\partial B(0; R)$. Then

$$2\int_{G(R)} (\hat{\nabla}E_G\mathbf{\Psi}_1) : (\hat{\nabla}E_G\mathbf{\Psi}_2)\ d\mathbf{y} = \left\langle \frac{1}{2}\mathbf{\Psi}_1 - K_{G(R)}'\mathbf{\Psi}_1, E_G\mathbf{\Psi}_2\right\rangle$$

$$= \left\langle \frac{1}{2}\mathbf{\Psi}_1 - K'_G\mathbf{\Psi_1}, E_G\mathbf{\Psi}_2 \right\rangle + \int\limits_{\partial B(0;R)} [T(E_G\mathbf{\Psi}_1, Q_G\mathbf{\Psi}_1)\mathbf{n}^{B(0;R)}] \cdot E_G\mathbf{\Psi_2}\ d\mathbf{y}.$$

If $R \to \infty$ then the decay properties of hydrodynamical potentials give (22).

Using (22) we get

$$\left\langle \frac{1}{2}\mathbf{\Psi} - K'_G\mathbf{\Psi}, E_G\overline{\mathbf{\Psi}} \right\rangle = \left\langle \frac{1}{2}\mathbf{\Psi}_1 - K'_G\mathbf{\Psi}_1, E_G\mathbf{\Psi}_1 \right\rangle + \left\langle \frac{1}{2}\mathbf{\Psi}_2 - K'_G\mathbf{\Psi}_2, E_G\mathbf{\Psi}_2 \right\rangle$$

$$-i\left\langle \frac{1}{2}\mathbf{\Psi}_1 - K'_G\mathbf{\Psi}_1, E_G\mathbf{\Psi}_2 \right\rangle + i\left\langle \frac{1}{2}\mathbf{\Psi}_2 - K'_G\mathbf{\Psi}_2, E_G\mathbf{\Psi}_1 \right\rangle = 2\int\limits_G |\hat\nabla E_G\mathbf{\Psi}_1|^2\ d\mathbf{y}$$

$$+2\int\limits_G |\hat\nabla E_G\mathbf{\Psi_2}|^2\ d\mathbf{y} - 2i\int\limits_G (\hat\nabla E_G\mathbf{\Psi}_1) : (\hat\nabla E_G\mathbf{\Psi}_2)\ d\mathbf{y}$$

$$+2i\int\limits_G (\hat\nabla E_G\mathbf{\Psi}_1) : (\hat\nabla E_G\mathbf{\Psi}_2)\ d\mathbf{y} = 2\int\limits_G [|\hat\nabla E_G\mathbf{\Psi}_1|^2 + |\hat\nabla E_G\mathbf{\Psi}_2|^2]\ d\mathbf{y} \geq 0.$$

Corollary 4.4. *Let $G \subset R^m$ be an open set with compact Lipschitz boundary, $m \geq 2$. Let $\mathbf{\Psi} \in H^{-1/2}(\partial G, C^m)$. If $m = 2$ suppose moreover that $\langle \mathbf{\Psi}, \mathbf{c}\rangle = 0$ for each $\mathbf{c} \in R^m$. Then*

$$\langle \mathbf{\Psi}, E_G\overline{\mathbf{\Psi}}\rangle = 2\int\limits_{R^m\setminus\partial G} |\hat\nabla E_G\mathbf{\Psi}|^2\ d\mathbf{y} \geq 0. \tag{24}$$

If ∂G is connected and $\langle \mathbf{\Psi}, E_G\overline{\mathbf{\Psi}}\rangle = 0$ then $E_G\mathbf{\Psi} = 0$ in R^m and there is a constant c such that $\mathbf{\Psi} = c\mathbf{n}^G$.

Proof. Put $C = R^m \setminus \mathrm{cl}\, G$. Since $K'_G = -K'_C$ we get using Proposition 4.3

$$\langle \mathbf{\Psi}, E_G\overline{\mathbf{\Psi}}\rangle = \left\langle \frac{1}{2}\mathbf{\Psi} - K'_G\mathbf{\Psi}, E_G\overline{\mathbf{\Psi}} \right\rangle + \left\langle \frac{1}{2}\mathbf{\Psi} + K'_G\mathbf{\Psi}, E_G\overline{\mathbf{\Psi}} \right\rangle$$

$$= \left\langle \frac{1}{2}\mathbf{\Psi} - K'_G\mathbf{\Psi}, E_G\overline{\mathbf{\Psi}} \right\rangle + \left\langle \frac{1}{2}\mathbf{\Psi} - K'_C\mathbf{\Psi}, E_G\overline{\mathbf{\Psi}} \right\rangle = 2\int\limits_G |\hat\nabla E_G\mathbf{\Psi}|^2\ d\mathbf{y}$$

$$+2\int\limits_{R^m\setminus \mathrm{cl}\, G} |\hat\nabla E_G\mathbf{\Psi}|^2\ d\mathbf{y} = 2\int\limits_{R^m\setminus\partial G} [|\hat\nabla E_G\mathbf{\Psi}|^2\ d\mathbf{y} \geq 0.$$

Suppose now that ∂G is connected and $\langle \mathbf{\Psi}, E_G\overline{\mathbf{\Psi}}\rangle = 0$. Then $\hat\nabla E_G\mathbf{\Psi} = 0$ in $R^m \setminus \partial G$ by (24). Lemma 3.1 gives that there are $\mathbf{v}, \mathbf{w} \in \mathcal{R}_m$ such that $E_G\mathbf{\Psi} = \mathbf{v}$ on G, $E_G\mathbf{\Psi} = \mathbf{w}$ on C. For definiteness we can suppose that G is bounded and C be unbounded. Since $E_G\mathbf{\Psi}(\mathbf{x}) \to 0$ as $|\mathbf{x}| \to \infty$, we infer that $\mathbf{w} = 0$. The boundary behavior of a hydrodynamical single layer potential gives

$\mathbf{v} = E_G\mathbf{\Psi} = \mathbf{w} = 0$ on ∂G. Since $\mathbf{v} = (v_1, \dots, v_m) \in \mathcal{R}_m$, we see that v_j are linear function and $\Delta v_j = 0$. The maximum principle for harmonic functions gives that $v_j = 0$. Thus $E_G\mathbf{\Psi} = 0$ in R^m. Since $\mathbf{u} = E_G\mathbf{\Psi}$, $p = Q_G\mathbf{\Psi}$ is a solution of the Stokes system (1) in $R^m \setminus \partial G$, we have $\nabla Q_G\mathbf{\Psi} = \Delta E_G\mathbf{\Psi} = 0$ in $R^m \setminus \partial G$. So, there are constants c_1, c_2 such that $Q_G\mathbf{\Psi} = c_1$ on G, $Q_G\mathbf{\Psi} = 2$ on C. According to boundary behavior of a hydrodynamical potential

$$\mathbf{\Psi} = \left[\frac{1}{2}\mathbf{\Psi} - K'_G\mathbf{\Psi}\right] - \left[-\frac{1}{2}\mathbf{\Psi} - K'_G\mathbf{\Psi}\right] = [T(E_G\mathbf{\Psi}, Q_G\mathbf{\Psi})]_+\mathbf{n}^G$$

$$-[T(E_G\mathbf{\Psi}, Q_G\mathbf{\Psi})]_-\mathbf{n}^G = T(0, c_1)\mathbf{n}^G - T(0, c_2)\mathbf{n}^G = -c_1\mathbf{n}^G + c_2\mathbf{n}^G.$$

Definition 4.5. *Lex X be a Banach space. Denote by I the identity operator on X. If M is a subspace of X denote by $\dim M$ the dimension of M. If Y is a subspace of X such that $X = M \bigoplus Y$, i.e. X is the direct sum of M and Y, denote by $\operatorname{codim} Y = \dim M$ the codimension of Y. If T is a bounded linear operator in X, denote by $\operatorname{Ker} T = \{x \in X; Tx = 0\}$ the kernel of T, $\alpha(T) = \dim \operatorname{Ker} T$, $\beta(T) = \operatorname{codim} T(X)$. We say that T is upper semi-Fredholm if $T(X)$ is a closed and $\alpha(T) < \infty$. For an upper semi-Fredholm operator T denote $i(T) = \alpha(T) - \beta(T)$ the index of T. We say that T is Fredholm if T is upper semi-Fredholm and $\beta(T) < \infty$. If X is a complex Banach space denote by $\sigma(T)$ the spectrum of T and by $r(T) = \sup\{|\lambda|; \lambda \in \sigma(T)\}$ the spectral radius of T.*

Lemma 4.6. *Let $G \subset R^m$ be a bounded domain with connected Lipschitz boundary, $m \geq 2$. Then $(\frac{1}{2}I - K'_G)\mathbf{n}^G = \mathbf{n}^G$. Denote $\mathcal{R}^c_m = \{\mathbf{u} + i\mathbf{v}; \mathbf{u}, \mathbf{v} \in \mathcal{R}_m\}$, $H^{-1/2}(\partial G; C^m) \cap E_{-1}(\mathcal{R}^c_m) = \{\mathbf{\Psi} \in H^{-1/2}(\partial G; C^m); E_G\mathbf{\Psi} \in \mathcal{R}^c_m$ in $G\}$. Then $H^{-1/2}(\partial G; C^m) \cap E_{-1}(\mathcal{R}^c_m) = \operatorname{Ker}(\frac{1}{2}I - K'_G) \bigoplus \{c\mathbf{n}^G; c \in C\}$ and $\dim \operatorname{Ker}(\frac{1}{2}I - K'_G) \leq \dim \mathcal{R}^c_m = m(m+1)/2$.*

Proof. According to (18) we have $E_G\mathbf{n}^G = 0$ in G, $(\frac{1}{2}I - K'_G)\mathbf{n}^G = T(E_G\mathbf{n}^G, Q_G\mathbf{n}^G)\mathbf{n}^G = T(0, -1)\mathbf{n}^G = \mathbf{n}^G$.

If $\mathbf{\Psi} \in \operatorname{Ker}(\frac{1}{2}I - K'_G)$ then $\mathbf{u} = E_G\mathbf{\Psi}$, $p = Q_G\mathbf{\Psi}$ is a weak solution of the Neumann problem for the Stokes system (1), (2) with the boundary condition $\mathbf{g} = 0$ (see Proposition 4.2). Proposition 3.2 gives that $\mathbf{u} = E_G\mathbf{\Psi} \in \mathcal{R}^c_m$. Let now $\mathbf{\Phi} \in H^{-1/2}(\partial G; C^m) \cap E_{-1}(\mathcal{R}^c_m)$. Since $E_G\mathbf{\Phi} \in \mathcal{R}^c_m$ in G and $\mathbf{u} = E_G\mathbf{\Phi}$, $p = Q_G\mathbf{\Phi}$ is a solution of the Stokes system (1) in G, we obtain $\nabla Q_G\mathbf{\Phi} = \Delta E_G\mathbf{\Phi} = 0$ in G. Since G is connected there is a constant c such that $Q_G\mathbf{\Phi} = c$ in G. Put $\mathbf{\Psi} = \mathbf{\Phi} + c\mathbf{n}^G$. Since $Q_G\mathbf{n}^G = -1$, $E_G\mathbf{n}^G = 0$ in G by (18), we have $E_G\mathbf{\Psi} = E_G\mathbf{\Phi} \in \mathcal{R}^c_m$, $Q_G\mathbf{\Psi} = 0$ in G. Easy calculation yields that $E_G\mathbf{\Psi}$, $Q_G\mathbf{\Psi}$ solves the Neumann problem for the Stokes system in G with zero boundary condition. Proposition 4.2 gives that $\mathbf{\Psi} \in \operatorname{Ker}(\frac{1}{2}I - K'_G)$. Since $\mathbf{n}^G \notin \operatorname{Ker}(\frac{1}{2}I - K'_G)$, we infer that $\operatorname{Ker}(\frac{1}{2}I - K'_G) \bigoplus \{c\mathbf{n}^G; c \in C\} = H^{-1/2}(\partial G; C^m) \cap E_{-1}(\mathcal{R}^c_m)$.

Clearly, $\dim \operatorname{Ker}(\frac{1}{2}I - K'_G) \leq m + \dim\{\mathbf{\Psi} \in \operatorname{Ker}(\frac{1}{2}I - K'_G); \langle \mathbf{\Psi}, \mathbf{c}\rangle = 0 \forall \mathbf{c} \in R^m\}$. Let now $\mathbf{\Psi} \in \operatorname{Ker}(\frac{1}{2}I - K'_G)$, $\langle \mathbf{\Psi}, \mathbf{c}\rangle = 0$ for each $\mathbf{c} \in R^m$. If $E_G\mathbf{\Psi} =$

$\mathbf{b} \in C^m$ on G, the continuity properties of a hydrodynamical potential gives $E_G\mathbf{\Psi} = \mathbf{b}$ on ∂G and $0 = \langle \mathbf{\Psi}, \overline{\mathbf{b}} \rangle = \langle \mathbf{\Psi}, E_G\overline{\mathbf{\Psi}} \rangle$. Corollary 4.4 gives that $\mathbf{\Psi} = d\mathbf{n}^G$ for some $d \in C$. Since $\mathbf{n}^G \notin \operatorname{Ker}(\frac{1}{2}I - K'_G)$, we infer that $\mathbf{\Psi} = 0$. This gives $\dim\{\mathbf{\Psi} \in \operatorname{Ker}(\frac{1}{2}I - K'_G); \langle \mathbf{\Psi}, \mathbf{c} \rangle = 0 \forall \mathbf{c} \in R^m\} \le \dim \mathcal{R}^c_m - m$. Hence $\dim \operatorname{Ker}(\frac{1}{2}I - K'_G) \le \dim \mathcal{R}^c_m = m(m+1)/2$.

Proposition 4.7. *Let $G \subset R^m$ be a bounded domain with connected Lipschitz boundary. If $\lambda \in C$ is an eigenvalue of the operator $\frac{1}{2}I - K'_G$ in $H^{-1/2}(\partial G, C^m)$ then $0 \le \lambda \le 1$.*

Proof. Let $\mathbf{\Psi}$ be an eigenfunction corresponding to an eigenvalue λ. If $\mathbf{\Psi} = c\mathbf{n}^G$, $c \in C$, then $\lambda = 1$ by Lemma 4.6. Suppose now that $\mathbf{\Psi} \ne c\mathbf{n}^G$. Then

$$\langle \mathbf{\Psi}, E_G\overline{\mathbf{\Psi}} \rangle = 2 \int\limits_{R^m \setminus \partial G} |\hat\nabla E_G^{\mathbf{\Psi}}|^2 \, d\mathbf{x} > 0$$

by Corollary 4.4. According to Proposition 4.3 and Corollary 4.4

$$2 \int\limits_G |\hat\nabla E_G\mathbf{\Psi}(x)|^2 \, d\mathbf{x} = \left\langle \frac{1}{2}\mathbf{\Psi} - K'_G\mathbf{\Psi}, E_G\overline{\mathbf{\Psi}} \right\rangle$$

$$= \langle \lambda\mathbf{\Psi}, E_G\overline{\mathbf{\Psi}} \rangle = 2\lambda \int\limits_{R^m \setminus \partial G} |\hat\nabla E_G\mathbf{\Psi}|^2 \, d\mathbf{x}.$$

Since

$$\lambda = \frac{\int\limits_G |\hat\nabla E_G\mathbf{\Psi}|^2 \, d\mathbf{x}}{\int\limits_{R^m \setminus \partial G} |\hat\nabla E_G\mathbf{\Psi}|^2 \, d\mathbf{x}}$$

we infer $0 \le \lambda \le 1$.

Proposition 4.8. *Let $G \subset R^m$ be a bounded domain with Lipschitz boundary, T be a bounded linear operator from $H^1(G, R^m)$ to $\mathcal{R}_m$ such that $T\mathbf{w} = \mathbf{w}$ for every $\mathbf{w} \in \mathcal{R}_m$. Then there is a positive constant L such that*

$$\|\mathbf{w} - T\mathbf{w}\|_{H^1(G,R^m)} \le L\|\hat\nabla\mathbf{w}\|_{L^2(G,R^m)}$$

for each $\mathbf{w} \in H^1(G, R^m)$.

(For the proof see [1], Theorem 5.2.)

Corollary 4.9. *Let $G \subset R^m$ be a bounded domain with Lipschitz boundary, X be a closed subspace of $H^1(G, R^m)$ such that $H^1(G, R^m) = X \bigoplus \mathcal{R}_m$. Then $\|\hat\nabla\mathbf{w}\|_{L^2(G,R^m)}$ is a norm on X which is equivalent to the norm induced from $H^1(G, R^m)$.*

Proof. Let T be the projection of $H^1(G, R^m)$ onto $\mathcal{R}_m$ along X. According to Proposition 4.8 there is a constant L such that

$$\|\mathbf{w}\|_{H^1(G,R^m)} = \|\mathbf{w} - T\mathbf{w}\|_{H^1(G,R^m)} \le L\|\hat\nabla\mathbf{w}\|_{L^2(G,R^m)}$$

for each $\mathbf{w} \in X$. Since

$$\|\hat{\nabla}\mathbf{w}\|_{L^2(G,R^m)} \leq \|\mathbf{w}\|_{H^1(G,R^m)},$$

both norms are equivalent.

Proposition 4.10. *Let $G \subset R^m$ be a bounded domain with connected Lipschitz boundary, $m > 2$. Denote*

$$H^{1/2}(\partial G, R^m) \cap \mathbf{n}^\perp = \left\{ \mathbf{u} \in H^{1/2}(\partial G, R^m); \langle \mathbf{n}^G, \mathbf{u}\rangle \left(= \int\limits_{\partial G} \mathbf{u} \cdot \mathbf{n}^G \mathrm{d}\mathbf{y} \right) = 0 \right\}$$

and by $H^{-1/2}(\partial G, R^m) \cap \mathbf{n}^\perp$ the closure of $H^{1/2}(\partial G, R^m) \cap \mathbf{n}^\perp$ in the space $H^{-1/2}(\partial G, R^m)$. Then E_G is a continuously invertible linear operator from the space $H^{-1/2}(\partial G, R^m) \cap \mathbf{n}^\perp$ onto $H^{1/2}(\partial G, R^m) \cap \mathbf{n}^\perp$.

Proof. Recall that $H^0(\partial G, R^m) = L^2(\partial G, R^m)$, $H^1(\partial G, R^m) = H^1(\partial G, R^m)$ and $H^{-1}(\partial G, R^m)$ is the dual space of $H^1(\partial G, R^m)$. Denote

$$H^0(\partial G, R^m) \cap \mathbf{n}^\perp = \left\{ \mathbf{u} \in H^1(\partial G, R^m); \int\limits_{\partial G} \mathbf{u} \cdot \mathbf{n}^G \mathrm{d}\mathbf{y} = 0 \right\},$$

$H^1(\partial G, R^m) \cap \mathbf{n}^\perp = H^1(\partial G, R^m) \cap (H^0(\partial G, R^m) \cap \mathbf{n}^\perp)$ and by $H^{-1}(\partial G, R^m) \cap \mathbf{n}^\perp$ denote the closure of $H^0(\partial G, R^m) \cap \mathbf{n}^\perp$ in $H^{-1}(\partial G, R^m)$. The operator E_G is one-to-one linear operator $H^0(\partial G, R^m) \cap \mathbf{n}^\perp$ onto $H^1(\partial G, R^m) \cap \mathbf{n}^\perp$ by [6], p. 792. Since $H^1(\partial G, R^m)$, $H^{-1}(\partial G, R^m)$ are dual spaces and $H^{-1}(\partial G, R^m) = \{c\mathbf{n}^G; c \in R\} \bigoplus H^{-1}(\partial G, R^m) \cap \mathbf{n}^\perp$, there is a nonzero $\mathbf{u} \in H^1(\partial G, R^m)$ such that $\langle \mathbf{\Psi}, u\rangle = 0$ for each $\mathbf{\Psi} \in H^{-1}(\partial G, R^m) \cap \mathbf{n}^\perp$. For $\mathbf{\Psi} \in H^{-1}(\partial G, R^m) \cap \mathbf{n}^\perp$ and $a \in R$ put $S(\mathbf{\Psi} + a\mathbf{n}^G) = E_G \mathbf{\Psi} + a\mathbf{u}$. Then S is a bounded one-to-one linear operator $H^0(\partial G, R^m)$ onto $H^1(\partial G, R^m)$. Since S is a continuously invertible operator $H^0(\partial G, R^m)$ onto $H^1(\partial G, R^m)$(see [5], Theorem 1.42), its adjoint operator S' is a continuously invertible operator $H^{-1}(\partial G, R^m)$ onto $H^0(\partial G, R^m)$. If $\mathbf{\Psi}, \mathbf{\Phi} \in H^0(\partial G, R^m) \cap \mathbf{n}^\perp \subset H^{-1}(\partial G, R^m) \cap \mathbf{n}^\perp$ and $a, b \in R$ then Fubini's theorem gives

$$\langle a\mathbf{n}^G + \mathbf{\Psi}, S(b\mathbf{n}^G + \mathbf{\Phi})\rangle = ab\langle \mathbf{n}^G, \mathbf{u}\rangle + a\langle \mathbf{n}^G, E_G \mathbf{\Phi}\rangle + b\langle \mathbf{\Psi}, \mathbf{u}\rangle + \langle \mathbf{\Psi}, E_G \mathbf{\Phi}\rangle$$

$$= \int\limits_{\partial G} [ab\mathbf{n}^G \cdot \mathbf{u} + \mathbf{\Psi} \cdot E_G \mathbf{\Phi}] \, \mathrm{d}\mathbf{y} = \int\limits_{\partial G} [ab\mathbf{n}^G \cdot \mathbf{u} + \mathbf{\Phi} \cdot E_G \mathbf{\Psi}] \, \mathrm{d}\mathbf{y} = \langle S(a\mathbf{n}^G + \mathbf{\Psi}), b\mathbf{n}^G + \mathbf{\Phi}\rangle.$$

Since $H^0(\partial G, R^m)$ is a dense subset of $H^{-1}(\partial G, R^m)$, we deduce that $S' = S$. Since S is a continuously invertible operator $H^0(\partial G, R^m)$ onto $H^1(\partial G, R^m)$ and $H^{-1}(\partial G, R^m)$ onto $H^0(\partial G, R^m)$, the real interpolation gives that S is a continuously invertible operator $H^{-1/2}(\partial G, R^m)$ onto $H^{1/2}(\partial G, R^m)$. Thus $S(H^{-1/2}(\partial G, R^m) \cap \mathbf{n}^G)$ is a closed subspace of $H^{1/2}(\partial G, R^m)$ of codimension 1 and S is a continuously invertible operator $H^{-1/2}(\partial G, R^m) \cap \mathbf{n}^G$ onto

$S(H^{-1/2}(\partial G, R^m) \cap \mathbf{n}^G)$. Since $S = E_G$ on $H^{-1/2}(\partial G, R^m) \cap \mathbf{n}^G$ we see that E_G is a continuously invertible operator the space $H^{-1/2}(\partial G, R^m) \cap \mathbf{n}^G$ onto $S(H^{-1/2}(\partial G, R^m) \cap \mathbf{n}^G) = E_G(H^{-1/2}(\partial G, R^m) \cap \mathbf{n}^G) \subset H^{1/2}(\partial G, R^m) \cap \mathbf{n}^G$. Since the image $E_G(H^{-1/2}(\partial G, R^m) \cap \mathbf{n}^G)$ is a subset of $H^{1/2}(\partial G, R^m)$ of codimension 1, we infer $E_G(H^{-1/2}(\partial G, R^m) \cap \mathbf{n}^G) = H^{1/2}(\partial G, R^m) \cap \mathbf{n}^G$.

Proposition 4.11. *Let $G \subset R^m$ be a bounded domain with connected Lipschitz boundary, $m \geq 2$. Then there is a closed subspace Y of $H^{-1/2}(\partial G, R^m)$ with finite codimension such that $\sqrt{\langle [(1/2)I - K'_G]\mathbf{\Psi}, E_G\mathbf{\Psi}\rangle}$, $\sqrt{\langle \mathbf{\Psi}, E_G\mathbf{\Psi}\rangle}$, and $\|E_G\mathbf{\Psi}\|_{H^{1/2}(\partial G)}$ are three norms on Y which are equivalent to the norm induced from $H^{-1/2}(\partial G, R^m)$.*

Proof. First we show that there are a closed subspace X of $H^{-1/2}(\partial G, R^m)$ with finite codimension and a constant C_1 such that

$$\|\mathbf{\Psi}\|_{H^{-1/2}(\partial G)} \leq C_1 \|E_G\mathbf{\Psi}\|_{H^{1/2}(\partial G)} \quad \forall \mathbf{\Psi} \in X. \tag{25}$$

If $m > 2$ we can put $X = H^{-1/2}(\partial G, R^m) \cap \mathbf{n}^{\perp}$ and use Proposition 4.10. Let now $m = 2$. We can suppose that $0 \in G$. Denote by $\operatorname{diam} G$ the diameter of G. Fix $r > \operatorname{diam} G$. Put $U = \{\mathbf{x}/r; \mathbf{x} \in G\}$. Then $\operatorname{diam} U < 1$. According to [23], Lemma 2.2 there is a closed subspace Z of $H^{-1/2}(\partial U, R^2)$ with codimension 1 and positive constant L such that

$$\|\mathbf{\Psi}\|_{H^{-1/2}(\partial U)} \leq L\sqrt{\langle \mathbf{\Psi}, E_U\mathbf{\Psi}\rangle}, \quad \mathbf{\Psi} \in Z.$$

If $\mathbf{\Psi} \in Z \setminus \{0\}$ then

$$\|\mathbf{\Psi}\|_{H^{-1/2}(\partial U)} \leq L\langle \mathbf{\Psi}, E_U\mathbf{\Psi}\rangle / \|\mathbf{\Psi}\|_{H^{-1/2}(\partial U)} \leq L\|E_U\mathbf{\Psi}\|_{H^{1/2}(\partial U)}.$$

Denote $J_1\mathbf{u}(\mathbf{x}) = \mathbf{u}(r\mathbf{x})$. Then J_1 is a continuously invertible linear operator from $H^{1/2}(\partial G, R^2)$ onto $H^{1/2}(\partial U, R^2)$. Moreover, there is a continuously invertible operator J_2 from $H^{-1/2}(\partial G, R^2)$ onto $H^{-1/2}(\partial U, R^2)$ such that $J_2 = J_1$ on $H^{1/2}(\partial G, R^2)$. Put $X = \{\mathbf{\Psi} \in H^{-1/2}(\partial G, R^2); J_2\mathbf{\Psi} \in Z, \langle \mathbf{\Psi}, \mathbf{c}\rangle = 0\ \forall \mathbf{c} \in R^2\}$. Then X is a closed subspace of $H^{-1/2}(\partial G, R^2)$ of finite codimension. Easy calculation yields that there is a positive constant b such that $J_1 E_G\mathbf{\Psi} = bE_U J_2\mathbf{\Psi}$ for $\mathbf{\Psi} \in X$. If $\mathbf{\Psi} \in X$ then

$$\|\mathbf{\Psi}\|_{H^{-1/2}(\partial G)} \leq \|J_2^{-1}\|\|J_2\mathbf{\Psi}\|_{H^{-1/2}(\partial U)} \leq L\|J_2^{-1}\|\|E_U J_2\mathbf{\Psi}\|_{H^{1/2}(\partial U)}$$

$$= Lb^{-1}\|J_2^{-1}\|\|J_1 E_G\mathbf{\Psi}\|_{H^{1/2}(\partial U)} \leq Lb^{-1}\|J_2^{-1}\|\|J_1\|\|E_G\mathbf{\Psi}\|_{H^{1/2}(\partial G)}.$$

If $\mathbf{\Psi} \in X$ then Proposition 4.3 and Corollary 4.4 give

$$0 \leq \langle [(1/2)I - K'_G]\mathbf{\Psi}, E_G\mathbf{\Psi}\rangle \leq \langle \mathbf{\Psi}, E_G\mathbf{\Psi}\rangle \leq \|E_G\mathbf{\Psi}\|_{H^{1/2}(\partial G)}\|\mathbf{\Psi}\|_{H^{-1/2}(\partial G)}$$

$$\leq \|E_G\|_{H^{-1/2}(\partial G)\to H^{1/2}(\partial G)}\|\mathbf{\Psi}\|^2_{H^{-1/2}(\partial G)}.$$

So, it is enough to prove that there is a closed subspace Y of X with finite codimension and a constant C such that

$$\|\mathbf{\Psi}\|_{H^{-1/2}(\partial\Omega)} \leq C\langle[(1/2)I - K'_G]\mathbf{\Psi}, E_G\mathbf{\Psi}\rangle \quad \forall\mathbf{\Psi} \in Y.$$

Denote

$$V = \left\{\mathbf{v} \in H^1(G, R^m); \int\limits_{\partial G} \mathbf{w}\cdot\mathbf{v}\; d\mathbf{y} = 0\ \forall\mathbf{w} \in \mathcal{R}_m\right\}.$$

Since $H^1(G, R^m) = V \bigoplus \mathcal{R}_m$, Corollary 4.9 gives that there is a positive constant C_2 such that

$$\|\mathbf{v}\|_{H^1(G)} \leq C_2\|\hat{\nabla}\mathbf{v}\|_{L^2(G)} \quad \forall\mathbf{v} \in V. \tag{26}$$

Denote $Y = \{\mathbf{\Psi} \in X; E_G \in V\}$. Since E_G is a continuously invertible operator X onto $E_G(X) \subset H^1(G, R^m)$ (compare (25) and (12)) and V is a closed subspace of $H^1(G, R^m)$ with finite codimension, we infer that X is a closed subspace of $H^{-1,2}(G, R^m)$ with finite codimension. Fix $\mathbf{\Psi} \in X$. Since $E_G\mathbf{\Psi}$ is the trace of $E_G\mathbf{\Psi}$ on ∂G we obtain using (25), (12) and Proposition 4.3

$$\|\mathbf{\Psi}\|^2_{W^{-1/2}(\partial G, R^m)} \leq C_1\|E_G\mathbf{\Psi}\|^2_{W^{1/2}(\partial G, R^m)} \leq C_1\|E_G\mathbf{\Psi}\|^2_{H^1(G,R^m)}$$

$$\leq C_1C_2\int\limits_G |\hat{\nabla}E_G\mathbf{\Psi}|^2\; d\mathbf{y} = \frac{C_1C_2}{2}\langle[(1/2)I - K'_G]\mathbf{\Psi}, E_G\mathbf{\Psi}\rangle.$$

Theorem 4.12. *Let $G \subset R^m$ be a bounded domain with connected Lipschitz boundary, $m \geq 2$. Then $\sigma(\frac{1}{2}I - K'_G) \subset \langle 0, 1\rangle$ in $H^{-1/2}(\partial G, C^m)$ and $\frac{1}{2}I - K'_G$ is a Fredholm operator with index* 0.

Proof. According to Proposition 4.11 there is a closed subspace Y of the space $H^{-1/2}(\partial G, R^m)$ with finite codimension and a positive constant L such that $\|\mathbf{\Psi}\|^2_{H^{-1/2}(\partial G)} \leq L\langle(\mathbf{\Psi}, E_G\mathbf{\Psi}\rangle$, $\|\mathbf{\Psi}\|^2_{H^{-1/2}(\partial G)} \leq L\langle(\frac{1}{2}I - K'_G)\mathbf{\Psi}, E_G\mathbf{\Psi}\rangle$ for each $\mathbf{\Psi} \in Y$. Put $Z = \{\mathbf{\Psi} = \mathbf{\Psi}_1 + i\mathbf{\Psi}_2; \mathbf{\Psi}_1, \mathbf{\Psi}_2 \in Y$ and $\langle\mathbf{\Psi}, \mathbf{c}\rangle = 0\ \forall\mathbf{c} \in C^m\}$. Then Z is a closed subspace of $H^{-1/2}(\partial G, C^m)$ with finite codimension. Proposition 4.3 and Corollary 4.4 give

$$\|\mathbf{\Psi}\|^2_{H^{-1/2}(\partial G)} \leq L\langle(\mathbf{\Psi}, E_G\overline{\mathbf{\Psi}}\rangle, \quad \|\mathbf{\Psi}\|^2_{H^{-1/2}(\partial G)} \leq L\langle[\frac{1}{2}I - K'_G]\mathbf{\Psi}, E_G\overline{\mathbf{\Psi}}\rangle \tag{27}$$

for each $\mathbf{\Psi} \in Z$.

If $\lambda \in R$ then (27) gives

$$\langle[(1/2-\lambda)I - K'_G]\mathbf{\Psi}, E_G\overline{\mathbf{\Psi}}\rangle = \langle[(1/2)I - K'_G]\mathbf{\Psi}, E_G\overline{\mathbf{\Psi}}\rangle - \lambda\langle(\mathbf{\Psi}, E_G\overline{\mathbf{\Psi}}\rangle \in R. \tag{28}$$

If $\lambda < 0$ then (27) and (28) give

$$L\langle[(1/2-\lambda)I - K'_G]\mathbf{\Psi}, E_G\overline{\mathbf{\Psi}}\rangle \geq L\langle[(1/2)I - K'_G]\mathbf{\Psi}, E_G\overline{\mathbf{\Psi}}\rangle \geq \|\mathbf{\Psi}\|^2_{H^{-1/2}(\partial G)}.$$

If $\lambda > 1$ then Corollary 4.4, Proposition 4.3 and (27) give

$$L|\langle[(1/2-\lambda)I - K'_G]\mathbf{\Psi}, E_G\overline{\mathbf{\Psi}}\rangle| \geq L\{\lambda\langle\mathbf{\Psi}, E_G\overline{\mathbf{\Psi}}\rangle - \langle[(1/2)I - K'_G]\mathbf{\Psi}, E_G\overline{\mathbf{\Psi}}\rangle\}$$

$$= 2L\lambda \int\limits_{R^m\setminus\partial G} |\hat\nabla E_G\mathbf{\Psi}|^2 \, \mathrm{d}\mathbf{y} - 2L\int\limits_{G} |\hat\nabla E_G\mathbf{\Psi}|^2 \, \mathrm{d}\mathbf{y} \geq 2L(\lambda-1) \int\limits_{R^m\setminus\partial G} |\hat\nabla E_G\mathbf{\Psi}|^2 \, \mathrm{d}\mathbf{y}$$

$$= L(\lambda-1)\langle\mathbf{\Psi}, E_G\overline{\mathbf{\Psi}}\rangle \geq (\lambda-1)\|\mathbf{\Psi}\|^2_{H^{-1/2}(\partial G)}.$$

If $\lambda = \lambda_1 + i\lambda_2 \in C$, $\lambda_2 \neq 0$ and $\mathbf{\Psi} \in Z$ then (27) and (28) give

$$|\langle[(1/2-\lambda)I - K'_G]\mathbf{\Psi}, E_G\overline{\mathbf{\Psi}}\rangle| = |\langle[(1/2-\lambda_1)I - K'_G]\mathbf{\Psi}, E_G\overline{\mathbf{\Psi}}\rangle - i\lambda_2\langle\mathbf{\Psi}, E_G\overline{\mathbf{\Psi}}\rangle|$$

$$\geq |\lambda_2||\langle\mathbf{\Psi}, E_G\overline{\mathbf{\Psi}}\rangle| \geq |\lambda_2|L^{-1}\|\mathbf{\Psi}\|^2_{H^{-1/2}(\partial G)}.$$

Fix $\lambda \in C \setminus (0'1\rangle$. We have proved that there is a positive constant M such that

$$\|\mathbf{\Psi}\|^2_{H^{-1/2}(\partial G)} \leq M\langle[(1/2-\lambda)I - K'_G]\mathbf{\Psi}, E_G\overline{\mathbf{\Psi}}\rangle.$$

for each $\mathbf{\Psi} \in Z$. If $\mathbf{\Psi} \in Z \setminus \{0\}$ then

$$\|\mathbf{\Psi}\|_{H^{-1/2}(\partial G)} \leq M\langle[(1/2-\lambda)I - K'_G]\mathbf{\Psi}, E_G\overline{\mathbf{\Psi}}\rangle / \|\mathbf{\Psi}\|_{H^{-1/2}(\partial G)}$$

$$\leq M\|E_G\|_{H^{-1/2}(\partial G)\to H^{-1/2}(\partial G)}\|[(1/2-\lambda)I - K'_G]\mathbf{\Psi}\|_{H^{-1/2}(\partial G)}.$$

So, the operator $\frac{1}{2}I - K'_G - \lambda I$ is upper semi-Fredholm by [18], § 16, Theorem 8. Since the index $i(\frac{1}{2}I - K'_G - \mu I)$ is constant on $C\setminus(0,1\rangle$ (see [18],§ 18, Corollary 3) and $\frac{1}{2}I - K'_G - \mu I$ is invertible for $|\mu| > \|\frac{1}{2}I - K'_G\|$ (see [24], Lemma 6.5), we infer that $i(\frac{1}{2}I - K'_G - \lambda I) = 0$. Thus $\frac{1}{2}I - K'_G - \lambda I$ is a Fredholm operator with index 0. If $\lambda \neq 0$ then $\alpha(\frac{1}{2}I - K'_G - \lambda I) = 0$ by Proposition 4.7 and $i(\frac{1}{2}I - K'_G - \lambda I) = 0$ forces that the operator $\frac{1}{2}I - K'_G - \lambda I$ is onto. Therefore $\frac{1}{2}I - K'_G - \lambda I$ is a continuously invertible operator (see [5], Theorem 1.42).

Proposition 4.13. *Let $G \subset R^m$ be a bounded domain with connected Lipschitz boundary. Then $(\frac{1}{2}I - K'_G)(H^{-1/2}(\partial G, C^m)) = \{\mathbf{\Psi} \in H^{-1/2}(\partial G, C^m);$ $\langle\mathbf{\Psi}, \mathbf{w}\rangle = 0 \ \forall \mathbf{w} \in \mathcal{R}^c_m\}$ and $H^{-1/2}(\partial G, C^m)$ is the direct sum of $\mathrm{Ker}(\frac{1}{2}I - K'_G)$ and $(\frac{1}{2}I - K'_G)(H^{-1/2}(\partial G, C^m))$. If we denote by $\tilde K'_G$ the restriction of K'_G onto $(\frac{1}{2}I - K'_G)(H^{-1/2}(\partial G, C^m))$ then $\sigma(\frac{1}{2}I - \tilde K'_G) \subset (0,1\rangle$.*

Proof. If $\{\mathbf{\Psi} \in H^{-1/2}(\partial G, C^m)$ then $\mathbf{u} = E_G\mathbf{\Psi}$, $p = Q_G\mathbf{\Psi}$ is a solution of the Neumann problem for the Stokes system with the boundary condition $(\frac{1}{2}I - K'_G)\mathbf{\Psi}$ by Proposition 4.2. Proposition 3.2 gives that $\langle\mathbf{\Psi}, \mathbf{w}\rangle = 0$ for all $\mathbf{w} \in \mathcal{R}^c_m$. Thus $(\frac{1}{2}I - K'_G)(H^{-1/2}(\partial G, C^m)) \subset \{\mathbf{\Psi} \in H^{-1/2}(\partial G, C^m); \langle\mathbf{\Psi}, \mathbf{w}\rangle = 0 \ \forall \mathbf{w} \in \mathcal{R}^c_m\}$ and $\mathrm{codim}(\frac{1}{2}I - K'_G)(H^{-1/2}(\partial G, C^m)) \geq \dim \mathcal{R}^c_m = m(m+1)/2$. Since $\dim \mathrm{Ker}(\frac{1}{2}I - K'_G) \leq \dim \mathcal{R}^c_m = m(m+1)/2$ by Lemma 4.6, Theorem 4.12 gives $\mathrm{codim}(\frac{1}{2}I - K'_G)(H^{-1/2}(\partial G, C^m)) = \dim \mathrm{Ker}(\frac{1}{2}I - K'_G) = \dim \mathcal{R}^c_m =$

$m(m+1)/2$, and thus $(\frac{1}{2}I - K'_G)(H^{-1/2}(\partial G, C^m)) = \{\mathbf{\Psi} \in H^{-1/2}(\partial G, C^m);$ $\langle \mathbf{\Psi}, \mathbf{w}\rangle = 0\ \forall \mathbf{w} \in \mathcal{R}^c_m\}$.

Let now $\mathbf{\Psi} \in \operatorname{Ker}(\frac{1}{2}I - K'_G) \cap (\frac{1}{2}I - K'_G)(H^{-1/2}(\partial G, C^m))$. Then $\langle \mathbf{\Psi}, \mathbf{w}\rangle = 0\ \forall \mathbf{w} \in \mathcal{R}^c_m$. Since $E_G\mathbf{\Psi} \in \mathcal{R}^c_m$ by Lemma 4.6, we obtain $\langle \mathbf{\Psi}, E_G\overline{\mathbf{\Psi}}\rangle = 0$. Since $\langle \mathbf{\Psi}, \mathbf{c}\rangle = 0$ for each $\mathbf{c} \in C^m$, Corollary 4.4 gives that $\mathbf{\Psi} = b\mathbf{n}^G$ for some $b \in R^1$. Since $\mathbf{n}^G \not\in \operatorname{Ker}(\frac{1}{2}I - K'_G)$ by Lemma 4.6, we infer that $b = 0$. Since $\mathbf{\Psi} \in \operatorname{Ker}(\frac{1}{2}I - K'_G) \cap (\frac{1}{2}I - K'_G)(H^{-1/2}(\partial G, C^m)) = \{0\}$ and $\operatorname{codim}(\frac{1}{2}I - K'_G)(H^{-1/2}(\partial G, C^m)) = \dim \operatorname{Ker}(\frac{1}{2}I - K'_G)$, we deduce that $H^{-1/2}(\partial G, C^m) = \operatorname{Ker}(\frac{1}{2}I - K'_G) \bigoplus (\frac{1}{2}I - K'_G)(H^{-1/2}(\partial G, C^m))$.

Since $H^{-1/2}(\partial G, C^m) = \operatorname{Ker}(\frac{1}{2}I - K'_G) \bigoplus (\frac{1}{2}I - K'_G)(H^{-1/2}(\partial G, C^m))$, we have $\sigma(\frac{1}{2}I - \tilde{K}'_G) \subset \sigma(\frac{1}{2}I - K'_G) \subset \langle 0, 1\rangle$. Moreover, the operator $(\frac{1}{2}I - \tilde{K}'_G)$ is one-to-one and onto. Thus $0 \not\in \sigma(\frac{1}{2}I - \tilde{K}'_G)$ (see [5], Theorem 1.42.)

Proposition 4.14. *Let X be a Banach space, T be a bounded linear operator on X. Suppose that X is the direct sum of $\operatorname{Ker}(I-T)$ and $(I-T)(X)$. Denote by $\tilde{T}$ the restriction of T onto $(I-T)(X)$. Suppose that*

$$\lim_{j\to\infty} \|\tilde{T}^j\|^{1/j} < 1. \tag{29}$$

Fix now $y \in (I-T)(X)$, $x_0 \in X$. Put

$$x_{j+1} = Tx_j + y \tag{30}$$

for a nonnegative integer j. Then there exists

$$x = \lim_{j\to\infty} x_j$$

and

$$\|x - x_j\| \le Cq^j(\|y\| + \|x_0\|) \tag{31}$$

for arbitrary j, where $C > 0$, $0 < q < 1$ are constants depending only on T.

(For the proof see ([17]), Proposition 3.)

Theorem 4.15. *Let $G \subset R^m$ be a bounded domain with connected Lipschitz boundary, $m \ge 2$. Fix $\mathbf{g} \in H^{-1/2}(\partial G, R^m)$. Then there is a weak solution of the Neumann problem for the Stokes system (1), (2) with the boundary condition $\mathbf{g}$ if and only if*

$$\langle \mathbf{g}, \mathbf{w}\rangle = 0\ \forall \mathbf{w} \in \mathcal{R}_m. \tag{32}$$

Suppose now that $\mathbf{g}$ satisfies (32) and $\mathbf{\Psi}_0 \in H^{-1/2}(\partial G, R^m)$. For a nonnegative integer k put

$$\mathbf{\Psi}_{k+1} = [(1/2)I + K'_G]\mathbf{\Psi}_k + \mathbf{g}. \tag{33}$$

Then there is $\mathbf{\Psi} \in H^{-1/2}(\partial G, R^m)$ such that $\mathbf{\Psi}_k \to \mathbf{\Psi}$ in $H^{-1/2}(\partial G, R^m)$ as $k \to \infty$. Moreover, there are constants $0 < q < 1$, $C > 0$ depending only on G such that

$$\|\mathbf{\Psi}_k - \mathbf{\Psi}\|_{H^{-1/2}(\partial G, R^m)} \le Cq^k \biggl(\|\mathbf{g}\|_{H^{-1/2}(\partial G, R^m)} + \|\mathbf{\Psi}_0\|_{H^{-1/2}(\partial G, R^m)} \biggr). \tag{34}$$

If we put $\mathbf{u} = E_G\mathbf{\Psi}$, $p = Q_G\mathbf{\Psi}$ then $\mathbf{u}$, p is a weak solution of the problem (1), (2).

Proof. Suppose first that there is a weak solution of the Neumann problem for the Stokes system in G with the boundary condition $\mathbf{g}$. Proposition 3.2 gives that (32) holds.

Suppose now that that (32) holds. Put $T = (1/2)I + K'_G$, $\tilde{T}$ the restriction of T onto $[(1/2)I - K'_G](H^{-1/2}(\partial G, C^m))$. Proposition 4.13 gives that $H^{-1/2}(\partial G, R^m) = \operatorname{Ker}(I-T) \bigoplus (I-T)(H^{-1/2}(\partial G, R^m))$ and $\sigma(I-\tilde{T}) \subset (-1, 1)$. Since $r(\tilde{T}) < 1$, [31], Chapter VIII, §2 gives (29). According to Proposition 4.14 there exists $\mathbf{\Psi} \in H^{-1/2}(\partial G, R^m)$ such that $\mathbf{\Psi}_k \to \mathbf{\Psi}$ as $k \to \infty$ in $H^{-1/2}(\partial G, R^m)$ and (34) holds with constants $0 < q < 1$, $C > 0$ depending only on G.

Put $\mathbf{u} = E_G\mathbf{\Psi}$, $p = Q_G\mathbf{\Psi}$. Letting $k \to \infty$ in (33) we get $\mathbf{\Psi} = [(1/2)I + K'_G]\mathbf{\Psi} + \mathbf{g}$. Proposition 4.2 forces that $\mathbf{u}$, p is a weak solution of the problem (1), (2).

5 Direct BEM

Let now $G \subset R^m$ be a bounded domain with connected Lipschitz boundary, $m \geq 2$, $\mathbf{g} \in H^{-1/2}(\partial G, R^m)$ be such that $\langle \mathbf{g}, \mathbf{w} \rangle = 0$ for all $\mathbf{w} \in \mathcal{R}_m$. According to Theorem 4.15 there is a weak solution $\mathbf{u}$, p of the Neumann problem for the Stokes system (1), (2) with the boundary condition $\mathbf{g}$. Denote by $\tilde{\mathbf{u}}$ the trace of $\mathbf{u}$. Since

$$\mathbf{u}(\mathbf{x}) = E_G\mathbf{g}(\mathbf{x}) + D_G\tilde{\mathbf{u}}(\mathbf{x}), \tag{35}$$

$$p(\mathbf{x}) = Q_G\mathbf{g}(\mathbf{x}) + \Pi_G\tilde{\mathbf{u}}(\mathbf{x}) \tag{36}$$

in G it is enough to determine $\tilde{\mathbf{u}}$. Using boundary behavior of hydrodynamical potentials we get

$$\frac{1}{2}\tilde{\mathbf{u}} - K_G\tilde{\mathbf{u}} = E_G\mathbf{g} \quad \text{on } \partial G. \tag{37}$$

Proposition 5.1. *Let $G \subset R^m$ be a bounded domain with connected Lipschitz boundary, $m \geq 2$. Then $\frac{1}{2}I - K_G$ is a Fredholm operator with index 0 in $H^{1/2}(\partial G, C^m)$, $H^{1/2}(\partial G, C^m) = \operatorname{Ker}(\frac{1}{2}I - K_G) \bigoplus (\frac{1}{2}I - K_G)(H^{1/2}(\partial G, C^m))$ and $\operatorname{Ker}(\frac{1}{2}I - K_G) = \mathcal{R}_m^c$. If we denote by $\tilde{K}_G$ the restriction of K_G onto $(\frac{1}{2}I - K_G)(H^{1/2}(\partial G, C^m))$ then $\sigma(\frac{1}{2}I - \tilde{K}_G) \subset (0, 1\rangle$.*

Proof. Since $\frac{1}{2}I - K_G$ and $\frac{1}{2}I - K'_G$ are adjoint operators, $\frac{1}{2}I - K_G$ is a Fredholm operator with index 0 by Theorem 4.12 and [24], Theorem 5.15 and $\sigma(\frac{1}{2}I - K_G) \subset \langle 0, 1\rangle$ by Theorem 4.12 and [24], Theorem 6.24. According to Proposition 4.13 and [24], Chapter 3, §3.3, we have $(\frac{1}{2}I - K_G)(H^{1/2}(\partial G, C^m)) = \{\mathbf{w} \in H^{1/2}(\partial G, C^m); \langle \mathbf{\Psi}, \mathbf{w}\rangle = 0 \ \forall \mathbf{\Psi} \in \operatorname{Ker}(\frac{1}{2}I - K'_G)\}$ and $\operatorname{Ker}(\frac{1}{2}I - K_G) = \{\mathbf{w} \in H^{1/2}(\partial G, C^m); \langle \mathbf{\Psi}, \mathbf{w}\rangle = 0 \ \forall \mathbf{\Psi} \in (\frac{1}{2}I - K'_G)(H^{-1/2}(\partial G, C^m))\} = \mathcal{R}_m^c$.

Since $H^{-1/2}(\partial G, C^m) = \mathrm{Ker}(\frac{1}{2}I - K'_G) \bigoplus (\frac{1}{2}I - K'_G)(H^{-1/2}(\partial G, C^m))$ we deduce $H^{1/2}(\partial G, C^m) = \mathrm{Ker}(\frac{1}{2}I - K_G) \bigoplus (\frac{1}{2}I - K_G)(H^{1/2}(\partial G, C^m))$. This forces $\sigma(\frac{1}{2}I - \tilde{K}_G) \subset (0, 1\rangle$.

Theorem 5.2. *Let $G \subset R^m$ be a bounded domain with connected Lipschitz boundary, $m \geq 2$, $\mathbf{g} \in H^{-1/2}(\partial G, R^m)$ be such that $\langle \mathbf{g}, \mathbf{w} \rangle = 0$ for all $\mathbf{w} \in \mathcal{R}_m$. Fix $\tilde{\mathbf{u}}_0 \in H^{1/2}(\partial G, R^m)$. For a nonnegative integer k put*

$$\tilde{\mathbf{u}}_{k+1} = [(1/2)I + K_G]\tilde{\mathbf{u}}_k + E_G\mathbf{g}. \tag{38}$$

Then there is $\tilde{\mathbf{u}} \in H^{1/2}(\partial G, R^m)$ such that $\tilde{\mathbf{u}}_k \to \tilde{\mathbf{u}}$ as $k \to \infty$ in $H^{1/2}(\partial G, R^m)$. Moreover, there are constants $0 < q < 1$, $C > 0$ depending only on G such that

$$\|\tilde{\mathbf{u}}_k - \tilde{\mathbf{u}}\|_{H^{1/2}(\partial G, R^m)} \leq Cq^k \bigg(\|\mathbf{g}\|_{H^{-1/2}(\partial G, R^m)} + \|\tilde{\mathbf{u}}_0\|_{H^{1/2}(\partial G, R^m)} \bigg). \tag{39}$$

The function $\tilde{\mathbf{u}}$ is a solution of the equation (37). If $\mathbf{u}$, p are given by (35), (36) in G, then $\mathbf{u}$, p is a weak solution of the problem (1), (2) and $\tilde{\mathbf{u}}$ is the trace of $\mathbf{u}$ on ∂G.

Proof. Put $T = (1/2)I + K_G$ and denote by $\tilde{T}$ the restriction of T onto $[(1/2)I - K_G](H^{1/2}(\partial G, C^m))$. Proposition 5.1 gives that $H^{1/2}(\partial G, R^m) = \mathrm{Ker}(I - T) \bigoplus (I - T)(H^{1/2}(\partial G, R^m))$ and $\sigma(I - \tilde{T}) \subset (-1, 1)$. Since $r(\tilde{T}) < 1$, [31], Chapter VIII, §2 gives (29). According to Theorem 4.15 there is a weak solution $\mathbf{v}$, q of the problem (1), (2). By virtue of (35), (36) and (37) we receive that $E_G\mathbf{g} \in (I - T)(H^{1/2}(\partial G, R^m))$. Proposition 4.14 gives that there is $\tilde{\mathbf{u}} \in H^{1/2}(\partial G, R^m)$ such that $\tilde{\mathbf{u}}_k \to \tilde{\mathbf{u}}$ as $k \to \infty$ in $H^{1/2}(\partial G, R^m)$ and

$$\|\tilde{\mathbf{u}}_k - \tilde{\mathbf{u}}\|_{H^{1/2}(\partial G, R^m)} \leq \tilde{C}q^k \bigg(\|E_G\mathbf{g}\|_{H^{1/2}(\partial G, R^m)} + \|\tilde{\mathbf{u}}_0\|_{H^{1/2}(\partial G, R^m)} \bigg).$$

holds with constants $0 < q < 1$, $\tilde{C} > 0$ depending only on G. So, (39) holds with $C = \tilde{C}(1 + \|E_G\|)$.

Letting $k \to \infty$ we get that $\tilde{\mathbf{u}}$ is a solution of the equation (37). Since $\mathbf{v}$ is also a solution of the equation (37), Proposition 5.1 forces that $\mathbf{w} = \tilde{\mathbf{u}} - \mathbf{v} \in \mathcal{R}_m$. Since $\mathbf{v}$, q is a solution of the problem (1), (2), we have $\mathbf{v} = E_G\mathbf{g} + D_G\mathbf{v}$, $q = Q_G\mathbf{g} + \Pi_G\mathbf{v}$ in G. Since $\mathbf{v} + \mathbf{w}$, q is a solution of the problem (1), (2) (see Proposition 3.2), we have also $\mathbf{v} + \mathbf{w} = E_G\mathbf{g} + D_G(\mathbf{v} + \mathbf{w}) = \mathbf{u}$, $q = Q_G\mathbf{g} + \Pi_G(\mathbf{v} + \mathbf{w}) = p$ in G. Thus $\tilde{\mathbf{u}} = \mathbf{v} + \mathbf{w}$ is the trace of $\mathbf{u} = \mathbf{v} + \mathbf{w}$ on ∂G.

6 Invertible integral operator for the indirect method

Let $\mathbf{g} \in H^{-1/2}(\partial G, R^m)$. Suppose that the Neumann problem (1), (2) is solvable. We know that there is a solution in the form $\mathbf{u} = E_G\mathbf{\Psi}$, $p = Q_G\mathbf{\Psi}$, where

$\mathbf{\Psi}$ is a solution of the integral equation $\frac{1}{2}\mathbf{\Psi} - K'_G\mathbf{\Psi} = \mathbf{g}$. In the numerical practice we approximate $\mathbf{g}$, so we solve the equation $\frac{1}{2}\tilde{\mathbf{\Psi}} - K'_G\tilde{\mathbf{\Psi}} = \tilde{\mathbf{g}}$ where $\tilde{\mathbf{g}}$ is close to $\mathbf{g}$. But there exists a sequence $\mathbf{g}_k$ such that $\mathbf{g}_k \to \mathbf{g}$ and the equation $\frac{1}{2}\mathbf{\Psi} - K'_G\mathbf{\Psi} = \mathbf{g}_k$ is not solvable. We would like to find a modified integral operator M' such that the integral equation $\frac{1}{2}\mathbf{\Psi} - M'\mathbf{\Psi} = \mathbf{g}$ is uniquely solvable and if the Neumann problem for the Stokes system with boundary condition $\mathbf{g}$ is solvable and $\mathbf{\Psi}$ is a solution of the equation $\frac{1}{2}\mathbf{\Psi} - M'\mathbf{\Psi} = \mathbf{g}$ then $\frac{1}{2}\mathbf{\Psi} - K'_G\mathbf{\Psi} = \mathbf{g}$. By virtue of Proposition 4.2 the functions $\mathbf{u} = E_G\mathbf{\Psi}$, $p = Q_G\mathbf{\Psi}$ solve the problem (1), (2).

Let $\mathbf{f}_1, \dots \mathbf{f}_{m(m+1)/2}$ form a basis of the space of rigid body motions $\mathcal{R}_m$. Let $c_1, \dots, c_{m(m+1)/2}$ be nonzero constants. Put

$$M'\mathbf{\Psi} = K'_G\mathbf{\Psi} + \sum_{j=1}^{m(m+1)/2} c_j \mathbf{f}_j \langle \mathbf{\Psi}, \mathbf{f}_j \rangle.$$

We shall show in Proposition 6.2 that the integral equation $\frac{1}{2}\mathbf{\Psi} - M'\mathbf{\Psi} = \mathbf{g}$ is uniquely solvable and if $\mathbf{\Psi}$ is a solution of the equation $\frac{1}{2}\mathbf{\Psi} - M'\mathbf{\Psi} = \mathbf{g}$ for $\mathbf{g} \in H^{-1/2}(\partial G, R^m)$ such that $\langle \mathbf{g}, \mathbf{w} \rangle = 0$ for all $\mathbf{w} \in \mathcal{R}_m$, then $\frac{1}{2}\mathbf{\Psi} - K'_G\mathbf{\Psi} = \mathbf{g}$. So, we can solve the equation $\frac{1}{2}\mathbf{\Psi} - M'\mathbf{\Psi} = \mathbf{g}$ instead the original equation $\frac{1}{2}\mathbf{\Psi} - K'_G\mathbf{\Psi} = \mathbf{g}$. If $\tilde{\mathbf{g}} \in H^{-1/2}(\partial G, R^m)$, $\|\mathbf{g} - \tilde{\mathbf{g}}\| \le \epsilon$ and $\tilde{\mathbf{\Psi}}, \mathbf{\Psi} \in H^{-1/2}(\partial G, R^m)$ are solutions of $\frac{1}{2}\mathbf{\Psi} - M'\mathbf{\Psi} = \mathbf{g}$, $\frac{1}{2}\tilde{\mathbf{\Psi}} - M'\tilde{\mathbf{\Psi}} = \tilde{\mathbf{g}}$, then $\|\mathbf{\Psi} - \tilde{\mathbf{\Psi}}\| \le \|(\frac{1}{2}I - M')^{-1}\|\epsilon$. If we want to express M^{-1} in the form of a Neumann series or to use the successive approximation method for the evaluation of a solution $\mathbf{\Psi}$ of the equation $\frac{1}{2}\mathbf{\Psi} - M'\mathbf{\Psi} = \mathbf{g}$, we need the spectrum of M to be in the same half space. For this aim we need a particular choice of $\mathbf{f}_1, \dots \mathbf{f}_{m(m+1)/2}$ and $c_1, \dots, c_{m(m+1)/2}$ (see Proposition 6.3 bellow).

Lemma 6.1. *Let $G \subset R^m$ be a bounded domain with connected Lipschitz boundary, $m \ge 2$, $\mathbf{g}, \mathbf{\Psi} \in H^{-1/2}(\partial G, R^m)$, $\frac{1}{2}\mathbf{\Psi} - M'\mathbf{\Psi} = \mathbf{g}$. If $\langle \mathbf{g}, \mathbf{w} \rangle = 0$ for all $\mathbf{w} \in \mathcal{R}_m$, then $\langle \mathbf{\Psi}, \mathbf{w} \rangle = 0$ for all $\mathbf{w} \in \mathcal{R}_m$.*

Proof. For $j, k \in \{1, 2, \dots, m(m+1)/2\}$ put

$$a_{jk} = \int_{\partial G} \mathbf{f}_j(\mathbf{x}) \cdot \mathbf{f}_k(\mathbf{x}) \, d\mathbf{x}.$$

Since $\mathbf{f}_1, \dots \mathbf{f}_{m(m+1)/2}$ is a basis of a finite dimensional subspace of $L^2(\partial G, R^m)$, the matrix $\{a_{jk}\}$ is regular.

For $j \in \{1, 2, \dots, m(m+1)/2\}$ put

$$d_j = c_j \langle \mathbf{\Psi}, \mathbf{f}_j \rangle.$$

Fix $k \in \{1, 2, \dots, m(m+1)/2\}$. Since $\mathbf{f}_k \in \mathcal{R}_m$, Proposition 4.13 gives

$$0 = \langle \mathbf{g}, \mathbf{f}_k \rangle = \langle \frac{1}{2}\mathbf{\Psi} - K'_G\mathbf{\Psi}, \mathbf{f}_k \rangle - \sum_{j=1}^{m(m+1)/2} a_{kj} d_j = - \sum_{j=1}^{m(m+1)/2} a_{kj} d_j.$$

Since the matrix $\{a_{jk}\}$ is regular we infer that $d_j = 0$ for $j = 1, \ldots, m(m+1)/2$. Since $\langle \mathbf{\Psi}, \mathbf{f}_j \rangle = 0$ for $j = 1, \ldots, m(m+1)/2$ we deduce $\langle \mathbf{\Psi}, \mathbf{w} \rangle = 0$ for all $\mathbf{w} \in \mathcal{R}_m$.

Proposition 6.2. *Let $G \subset R^m$ be a bounded domain with connected Lipschitz boundary, $m \geq 2$. Then the operator $\frac{1}{2}I - M'$ is a continuously invertible in $H^{-1/2}(\partial G, R^m)$. If $\mathbf{\Psi}, \mathbf{g} \in H^{-1/2}(\partial G, R^m)$, $\frac{1}{2}\mathbf{\Psi} - M'\mathbf{\Psi} = \mathbf{g}$ and $\langle \mathbf{g}, \mathbf{w} \rangle = 0$ for all $\mathbf{w} \in \mathcal{R}_m$, then $\frac{1}{2}\mathbf{\Psi} - K'_G \mathbf{\Psi} = \mathbf{g}$.*

Proof. Suppose first that $\mathbf{\Psi}, \mathbf{g} \in H^{-1/2}(\partial G, R^m)$, $\frac{1}{2}\mathbf{\Psi} - M'\mathbf{\Psi} = \mathbf{g}$ and $\langle \mathbf{g}, \mathbf{w} \rangle = 0$ for all $\mathbf{w} \in \mathcal{R}_m$. Lemma 6.1 gives that $\langle \mathbf{\Psi}, \mathbf{w} \rangle = 0$ for all $\mathbf{w} \in \mathcal{R}_m$. Since $\mathbf{f}_1, \ldots \mathbf{f}_{m(m+1)/2}$ form a basis of $\mathcal{R}_m$ we deduce

$$\frac{1}{2}\mathbf{\Psi} - K'_G \mathbf{\Psi} = \frac{1}{2}\mathbf{\Psi} - M'\mathbf{\Psi} = \mathbf{g}.$$

Now we prove that $\frac{1}{2}I - M'$ is one-to-one. Suppose $\frac{1}{2}I - M'\mathbf{\Psi} = 0$. Then $\langle \mathbf{\Psi}, \mathbf{w} \rangle = 0$ for all $\mathbf{w} \in \mathcal{R}_m$ by Lemma 6.1 and $\frac{1}{2}\mathbf{\Psi} - K'_G\mathbf{\Psi} = \frac{1}{2}\mathbf{\Psi} - M'\mathbf{\Psi} = 0$. Since $\frac{1}{2}I - K'_G$ is injective on $\{\mathbf{f} \in H^{-1/2}(\partial G, R^m); \langle \mathbf{f}, \mathbf{w} \rangle = 0$ for all $\mathbf{w} \in \mathcal{R}_m\}$ by Proposition 4.13, we infer that $\mathbf{\Psi} = 0$.

The operator $M' - K'_G$ is a finite rank operator and therefore compact (see [24], p. 88). Since $\frac{1}{2}I - K'_G$ is a Fredholm operator with index 0 by Theorem 4.12, the operator $\frac{1}{2}I - M'$ is a Fredholm operator with index 0, too (see [18], § 16, Theorem 16). Since $\frac{1}{2}I - M'$ is one-to-one it is also onto and therefore continuously invertible (see [5], Theorem 1.42).

Proposition 6.3. *Let $G \subset R^m$ be a bounded domain with connected Lipschitz boundary, $m \geq 2$. Suppose that*

$$\int\limits_{\partial G} \mathbf{f}_j(\mathbf{y}) \cdot \mathbf{f}_k(\mathbf{y})\, d\mathbf{y} = \begin{cases} 1 & \text{for j = k,} \\ 0 & \text{for j} \neq \text{k} \end{cases}$$

and $c_j = -1$ for $j = 1, \ldots, m(m+1)/2$, i.e. $M'\mathbf{\Psi} = K'_G\mathbf{\Psi} - \sum \mathbf{f}_j \langle \mathbf{\Psi}, \mathbf{f}_j \rangle$. Then there is an equivalent norm on $H^{-1/2}(\partial G, C^m)$ such that $\|\frac{1}{2}I + M'\| \leq q < 1$. Let now $\mathbf{g} \in H^{-1/2}(\partial G, C^m)$ be such that $\langle \mathbf{g}, \mathbf{w} \rangle = 0$ for all $\mathbf{w} \in \mathcal{R}_m$. Fix $\mathbf{\Psi}_0 \in H^{-1/2}(\partial G, C^m)$. For a nonnegative integer k put

$$\mathbf{\Psi}_{k+1} = \left(\frac{1}{2}I + M'\right)\mathbf{\Psi}_k + \mathbf{g}.$$

Then $\mathbf{\Psi}_k \to \mathbf{\Psi}$ in $H^{-1/2}(\partial G, C^m)$, $\frac{1}{2}\mathbf{\Psi} - M'\mathbf{\Psi} = \mathbf{g}$ and $\|\mathbf{\Psi} - \mathbf{\Psi}_j\| \leq q^j[\|\mathbf{g}\| + \|\mathbf{\Psi}_0\|]$ for arbitrary j.

Proof. Let λ be an eigenvalue of $\frac{1}{2}I - M'$ and $\mathbf{\Psi}$ be a corresponding eigenvector. Then $\mathbf{\Psi} = \mathbf{f} + \mathbf{g}$, where $\mathbf{g} \in \mathcal{R}_m^c$ and $\langle \mathbf{f}, \mathbf{w} \rangle = 0$ for each $\mathbf{w} \in \mathcal{R}_m$. We obtain

$$\lambda \mathbf{f} + \lambda \mathbf{g} = \left(\frac{1}{2}I - M'\right)\mathbf{\Psi} = \left(\frac{1}{2}I - K'_G\right)\mathbf{\Psi} + \mathbf{g}.$$

Since $H^{-1/2}(\partial G, C^m)$ is the direct sum of $[(1/2)I - K'_G](H^{-1/2}(\partial G, C^m)) = \{\mathbf{\Phi} \in H^{-1/2}(\partial G, C^m); \langle \mathbf{\Phi}, \mathbf{w} \rangle = 0 \ \forall \mathbf{w} \in \mathcal{R}^c_m\}$ and $\mathcal{R}^c_m$ (compare Proposition 4.13), we infer that

$$[(1/2)I - K'_G]\mathbf{\Psi} = \lambda \mathbf{f}, \qquad \lambda \mathbf{g} = \mathbf{g}.$$

If $\mathbf{g} \neq 0$ then $\lambda = 1$. If $\mathbf{g} = 0$ then $\mathbf{\Psi} = \mathbf{f} \in [(1/2)I - K'_G](H^{-1/2}(\partial G, C^m))$. Since λ is an eigenvalue of $[(1/2)I - \tilde{K}'_G]$, Theorem 4.13 and Proposition 6.2 give that $0 < \lambda \leq 1$.

Fix $\lambda \in C \setminus (0, 1\rangle$. The operator $\frac{1}{2}I - K'_G - \lambda I$ is a Fredholm operator with index 0 by Theorem 4.12. Since $M' - K'_G$ is a finite rank operator and so compact (see [24], p. 88), the operator $\frac{1}{2}I - M' - \lambda I$ is a Fredholm operator with index 0 (see [18], §16, Theorem 16). If $\lambda \in \sigma(\frac{1}{2}I - M')$ then λ is an eigenvalue of $\frac{1}{2}I - M'$. We have proved that λ is not an eigenvalue of $\frac{1}{2}I - M'$. Thus $\sigma(\frac{1}{2}I - M') \subset (0, 1\rangle$. Since $\sigma(\frac{1}{2}I + M') \subset \langle 0, 1)$ we have $r(\frac{1}{2}I + M') < 1$. If we fix $r(\frac{1}{2}I + M') < q < 1$ then there exists an equivalent norm $\| \cdot \|$ on $H^{-1/2}(\partial G, C^m)$ such that $\|\frac{1}{2}I + M'\| \leq q$. (see [8]). The rest is a consequence of Proposition 4.14.

7 Invertible operator for the direct method

Let $\mathbf{g} \in H^{-1/2}(\partial G, R^m)$. Suppose that the Neumann problem (1), (2) is solvable. If $\mathbf{u}$, p solves the Neumann problem (1), (2), then $\frac{1}{2}\mathbf{u} - K_G \mathbf{u} = E_G \mathbf{g}$. If $\mathbf{f}$ is a solution of the equation

$$\frac{1}{2}\mathbf{f} - K_G \mathbf{f} = E_G \mathbf{g}, \tag{40}$$

then there is a rigid body motion $\mathbf{w}$ such that $\mathbf{f} = \mathbf{u} + \mathbf{w}$ (see Proposition 5.1). If we put $\mathbf{v} = \mathbf{u} + \mathbf{w}$, then $\mathbf{v}$, p solves the Neumann problem (1), (2). Moreover,

$$\mathbf{v} = E_G \mathbf{g} + D_G \mathbf{f}, \quad p = Q_G \mathbf{g} + \Pi_G \mathbf{f}.$$

So we would like to find any solution of the integral equation (40). If we approximate $E_G \mathbf{g}$ then the new equation $\frac{1}{2}\tilde{\mathbf{f}} - K_G \tilde{\mathbf{f}} = \mathbf{h}$ might not be solvable. We shall study instead of this equation a modified equation which is uniquely solvable and a solution of this new equation solves also the equation (40).

Let $\mathbf{f}_1, \ldots \mathbf{f}_{m(m+1)/2}$ form a basis of the space of rigid body motions $\mathcal{R}_m$. Let $c_1, \ldots, c_{m(m+1)/2}$ be nonzero constants. Put

$$M\mathbf{\Psi} = K_G \mathbf{\Psi} + \sum_{j=1}^{m(m+1)/2} c_j \mathbf{f}_j \langle \mathbf{\Psi}, \mathbf{f}_j \rangle.$$

Remark that M is the adjoint operator to the operator M' constructed in the previous section.

Proposition 7.1. *Let $G \subset R^m$ be a bounded domain with connected Lipschitz boundary, $m \geq 2$. The operator $\frac{1}{2}I - M$ is continuously invertible in $H^{1/2}(\partial G, C^m)$. If $\mathbf{f} \in H^{1/2}(\partial G, C^m)$, $\mathbf{h} \in (\frac{1}{2}I - K_G)(H^{1/2}(\partial G, C^m))$ and $\frac{1}{2}\mathbf{f} - M\mathbf{f} = \mathbf{h}$, then $\frac{1}{2}\mathbf{f} - K_G\mathbf{f} = \mathbf{h}$.*

Proof. Since $\frac{1}{2}I - M'$ is continuously invertible by Proposition 6.2, its adjoint operator $\frac{1}{2}I - M$ is also continuously invertible (see [24], Theorem 6.24). We have $H^{1/2}(\partial G, C^m) = \mathcal{R}^c_m \bigoplus (\frac{1}{2}I - K_G)(H^{1/2}(\partial G, C^m))$ by Proposition 5.1. Since $\frac{1}{2}\mathbf{f} - K_G\mathbf{f} \in (\frac{1}{2}I - K_G)(H^{1/2}(\partial G, C^m))$, $(K_G - M)\mathbf{f} \in \mathcal{R}^c_m$ and $\mathbf{h} = [\frac{1}{2}\mathbf{f} - K_G\mathbf{f}] + (K_G - M)\mathbf{f} \in (\frac{1}{2}I - K_G)(H^{1/2}(\partial G, C^m))$, we infer that $(K_G - M)\mathbf{f} = 0$.

Proposition 7.2. *Let $G \subset R^m$ be a bounded domain with connected Lipschitz boundary, $m \geq 2$. Suppose that*

$$\int_{\partial G} \mathbf{f}_j(\mathbf{y}) \cdot \mathbf{f}_k(\mathbf{y})\, d\mathbf{y} = \begin{cases} 1 & \text{for j = k,} \\ 0 & \text{for j} \neq \text{k} \end{cases}$$

and $c_j = -1$ for $j = 1, \ldots, m(m+1)/2$, i.e. $M\mathbf{\Psi} = K_G\mathbf{\Psi} - \sum \mathbf{f}_j \langle \mathbf{\Psi}, \mathbf{f}_j \rangle$. Then there is an equivalent norm on $H^{1/2}(\partial G, C^m)$ such that $\|\frac{1}{2}I + M\| \leq q < 1$. Let now $\mathbf{h} \in H^{1/2}(\partial G, C^m)$. Fix $\mathbf{f}_0 \in H^{1/2}(\partial G, C^m)$. For a nonnegative integer k put

$$\mathbf{f}_{k+1} = \left(\frac{1}{2}I + M\right)\mathbf{f}_k + \mathbf{h}.$$

Then $\mathbf{f}_k \to \mathbf{f}$ in $H^{1/2}(\partial G, C^m)$, $\frac{1}{2}\mathbf{f} - M\mathbf{f} = \mathbf{h}$ and $\|\mathbf{f} - \mathbf{f}_j\| \leq q^j[\|\mathbf{h}\| + \|\mathbf{f}_0\|]$ for arbitrary j.

Proof. Since there is an equivalent norm on $H^{-1/2}(\partial G, C^m)$ such that $\|\frac{1}{2}I + M'\| \leq q < 1$ (see Proposition 6.3), we have $\|\frac{1}{2}I + M\| = \|\frac{1}{2}I + M'\| \leq q < 1$ (see [24], Theorem 3.3). The rest is a consequence of Proposition 4.14.

8 Acknowledgement

The work was supported by the Academy of Sciences of the Czech Republic, Institutional Research Plan No. AVOZ10190503 and grant No. I AA 100190804 financed by the GA AVČR.

References

[1] Alessandrini,G., Morassi, A., Rosset, E.: The linear constraint in Poincaré and Korn type inequalities. Forum Math. 20, 557–569 (2008)

[2] Brown, R., Mitrea, I., Mitrea, M., Wright, M.: Mixed boundary value problems for the Stokes system. Trans. Amer. Math. Soc. 362, 1211–1230 (2010)

[3] Constanda, M.: Some historical remarks on the positivity of boundary integral operators, In: Schanz M., Steinbach O. (eds.) Boundary Element analysis, pp. 1–28, Springer, Berlin (2007)

[4] Dahlberg, B. E. J., Kenig, C., Verchota, G. C.: Boundary value problems for the systems of elastics in Lipschitz domains. Duke Math. J. 57, 795–818 (1988)

[5] Douglas, R. G.: Banach algebra techniques in operator theory. Academic Press, New York and London (1972)

[6] Fabes, E. B., Kenig, C. E., Verchota, G.C.: The Dirichlet problem for the Stokes system on Lipschitz domains. Duke Math. J. 57, 769–793 (1988)

[7] Fabes, E., Sand, M., Seo, J. K.: The spectral radius of the classical layer potentials on convex domains. IMA Vol. Math. Appl. 42, 129-137 (1992)

[8] Gohberg, I., Marcus, A.: Some remarks on topologically equivalent norms. Izvestija Mold. Fil. Akad. Nauk SSSR 10(76), 91–95 (1960) (in Russian)

[9] Hsiao, G. C., Wendland, W.L.: Boundary Integral Equations. Springer, Heidelberg (2008)

[10] Kohr, M.: Boundary value problems for a compressible Stokes system in bounded domains in R^n. J. Comput. Appl. Math. 201, 128–145 (2007)

[11] Král, J.: Integral Operators in Potential Theory. Springer, Berlin (1980)

[12] Král, J. , Netuka, I.: Contractivity of C. Neumann's operator in potential theory. J. Math. Anal. Appl. 61, 607–619 (1977)

[13] Ladyzenskaya, O. A.: The mathematical theory of viscous incompressible flow. Gordon and Breach, New York - London - Paris (1969)

[14] Maz'ya, V., Mitrea, M., Shaposhnikova, T.: The inhomogeneous Dirichlet problem for the Stokes system in Lipschitz domains with unital close to VMO. Funct. Anal. Appl. 43, 217–235 (2009)

[15] Medková, D.: The Neumann problem for the Laplace equation on general domains. Czech. Math. J. 57, 1107–1139 (2007)

[16] Medková, D.: The integral equation method and the Neumann problem for the Poisson equation on NTA domains. Integr. Equ. Oper. Theory 63, 227–247 (2009)

[17] Medková, D.: Integral representation of a solution of the Neumann problem for the Stokes system. Numer. Algorithms 54, 459–484 (2010), DOI 10.1007/s11075-9346-4

[18] Müller, V.: Spectral Theorey of Linear Operators and Spectral Systems in Banach Algebras. Operator Theory Advances and Applications. Birkhäuser, Basel (2007)

[19] Neumann, C.: Untersuchungen über das logarithmische und Newtonsche Potential. Teubner Verlag, Leipzig (1877)

[20] Neumann, C.: Zur Theorie des logarithmischen und des Newtonschen Potentials. Berichte über die Verhandlungen der Königlich Sachsischen Gesellschaft der Wissenschaften zu Leipzig 22, 49–56, 264–321 (1870)

[21] Neumann, C.: Über die Methode des arithmetischen Mittels. Hirzel, Leipzig (1887)

[22] Plemelj, J.: Potentialtheoretische Untersuchungen. B. G. Teubner, Leipzig (1911)

[23] Reidinger, B., Steinbach, O.: A symmetric boundary element method for the Stokes problem in multiple connected domains. Math. Meth. Appl. Sci. 26, 77–93 (2003)

[24] Schechter, M.: Principles of Functional Analysis. American Mathematical Society, Providence - Rhode Island (2002)

[25] Shibata, Y., Shimizu, S.: On the $L_p - L_q$ maximal regularity of the Neumann problem for the Stokes equations in a bounded domain. J. reine angew. Math. 615, 157–209 (2008)

[26] Steinbach, O.: Numerical Approximation Methods for Elliptic Boundary Value Problems. Finite and Boundary Elements. Springer, New York (2008)

[27] Steinbach, O., Wendland, W. L.: On C. Neumann's method for second-order elliptic systems in domains with non-smooth boundaries. J. Math. Anal. Appl. 262, 733–748 (2001)

[28] Tartar, L.: An Introduction to Sobolev Spaces and Interpolation Spaces. Springer, Berlin Heidelberg (2007)

[29] Verchota, G.: Layer potentials and regularity for the Dirichlet problem for Laplace's equation in Lipschitz domains. J. Funct. Anal. 59, 572–611 (1984)

[30] Wendland, W. L.: On the double layer potential. Adv. Appl. 193, 319–334 (2009)

[31] Yosida, K.: Functional Analysis. Springer, Berlin (1965)