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Infinite dimensional Fenchel duality and truncated moment problems in several real variables

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Abstract

We use an infinite dimensional version of the Fenchel duality technique to prove that, whenever a truncated problem of powers moments on a multidimensional dimensional Euclidian space has representing densities, it will have also a distinguished representing density of a concrete form, namely the unique density maximizing the entropy functional subject to the equations of moments.

Keywords: multidimensional, truncated moments problem, positive representing measure.

Mathematics Subject Classification: MSC 44A60, 49J99

1 Introduction

We consider the following truncated problem of moments in several variables. Fix $n, m \geq 1$ and let $\mathbb{R}^n$ be endowed with the Lebesgue measure. Let $A = \{\alpha \in \mathbb{Z}_+^n : |\alpha| \leq 2m\}$, where $|\alpha| = \alpha_1 + \cdots + \alpha_n$ for any multiindex α. Set $u_\alpha(t) = t^\alpha = t_1^{\alpha_1} \cdots t_n^{\alpha_n}$ for $\alpha \in A$ and $t \in \mathbb{R}^n$. Given a finite set $\gamma = (\gamma_\alpha)_\alpha$ of numbers γ_α $(\alpha \in A)$, the problem is to establish whether there exist measures $\mu \geq 0$ on $\mathbb{R}^n$ such that

$$\int_{\mathbb{R}^n} u_\alpha(t) d\mu(t) = \gamma_\alpha \quad (\alpha \in A). \tag{1}$$

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In particular, we look for absolutely continuous measures $\mu = f\,dt$ with $f \geq 0$, where the symbol dt stands as usual for the Lebesgue measure. If (1) holds, then μ (resp. f) is called a *representing* measure (resp. density) of the sequence γ. The problem is then to characterize those sets γ which have nonnegative representing measures, study the set of the solutions and find or approximate such measures μ.

In the case of densities supported on a given compact subset in $\mathbb{R}^n$ we gave such a characterization [] in terms of the existence of a certain distinguished representing density f having a very particular form $f = e^p$, namely the exponential of a polynomial p of degree $\leq 2m$,

$$\int u_\alpha(t) e^{p(t)} dt = \gamma_\alpha$$

for all $\alpha \in A$.

The aim of this work is **to prove similar results in the case of unbounded support** ($= \mathbb{R}^n$ for instance), that is well known to be more difficult than the compactly supported case, for various problems of this type. Moreover, the method we use here, namely optimization of the entropy functional $H(f) = -\int f \ln f\,dt$ by means of Fenchel duality, gives also a new, simpler proof in the compact case.

2 Preliminaries

We consider absolutely continuous representing measures $f\,dt$, with nonnegative density f from $L^1(\mathbb{R}^n)$ – the space of all (classes of) measurable functions that are Lebesgue integrable on T with respect to dt. Set $a := \operatorname{card} A$. We characterize the existence of such representing densities by the solvability of the following system

$$\int_{\mathbb{R}^n} u_\alpha(t)\, \mathrm{e}^{\sum_{\beta \in A} x_\beta\, u_\beta(t)}\, dt \;=\; \gamma_\alpha \qquad (\alpha \in A) \tag{2}$$

of a equations with a unknowns x_α ($\alpha \in A$). Therefore if our problem (1) has any absolutely continuous solution $\mu = f\,dt$, then it will necessarily have also a solution of the form from above.

When the system (2) (see (**??**) and (**??**), too) has a solution, this is unique and provides the (also unique) representing density f_* having maximal entropy, by the formula

$$f_*(t) = f_{*,x}(t) = \exp\,(\sum_{\alpha\in A} x_\alpha u_\alpha(t)) \quad (t \in T).$$

Note that the solution in the above example is valid if and only if $\gamma_0 > 0$ and $\gamma_0\gamma_2 - \gamma_1^2 > 0$. This agrees with the theory of the truncated problem of power moments existing for $n = 1$. Namely, a set $(\gamma_\alpha)_{\alpha=0}^{2k}$ ($\neq 0$) has nonnegative representing measures (resp. densities) on $\mathbb{R} \Leftrightarrow$ the quadratic form $(\gamma_{\alpha+\beta})_{\alpha,\beta=0}^{k}$ is nonnegative (resp. positive) definite [3].

Such characterizatins are based on the possibility to represent any nonnegative polynomial (resp. trigonometrical polynomial) on $\mathbb{R}$. as a sum of squares [**?**]. Namely the existence of μ can be characterized by the existence of a functional L_γ on polynomials such that $L_\gamma t^\alpha = \gamma_\alpha$ and $L_\gamma p \geq 0$ for any $p \geq 0$ on T [11]. Then by the Hahn–Banach theorem it suffices to ask $L_\gamma p \geq 0$ for those $p = \sum_{\alpha\in A} c_\alpha u_\alpha$ such that $p(t) \geq 0$, $t \in T$ [11], [15]. The problem is to describe these polynomials p. If they are (or can be expressed in terms of) sums of squares, then conditions like $L_\gamma(|q|^2) \geq 0$ for all q lead to characterizations as above. This method is not applicable for $n > 1$ when the set of nonnegative polynomials is more difficult to handle [6] (for instance, not all of them can be written as sums of squares). The same questions appear for trigonometric moment problems [**?**], [**?**].

Thus the moment problems for $n > 1$ and $A =$finite have received rather partial answers.

Our approach is based on a Shannon's idea. Set $F_\alpha(f) = \int_T u_\alpha(t)f(t)dm - \gamma_\alpha$ ($\alpha \in A$). Assume the existence of the representing densities $f \geq 0$ of γ with $\int_T f dm = 1$. Then among them there exists one probability density f_* having the maximum degree of randomness allowed by the conditions (1). Namely, this density maximizes the entropy functional

$$H(f) := -\int_T f \,\ln f \, dm$$

with the restrictions $F_\alpha(f) = 0$ ($\alpha \in A$). Since $f_* > 0$ on T, then it belongs in a certain sense to the interior of the domain of H. Hence we may apply the method of the Lagrange multipliers for the conditioned extremum. Then there are $x_\alpha \in \mathbb{R}$ ($\alpha \in A$) such that f_* be a critical point of the function $L := H + \sum_{\alpha\in A} x_\alpha F_\alpha$, namely $L'(f_*) = 0$. Thus $L'(f_*)g = 0$ for all g. Note that $L(f) = \int_T G(f)dm$ where $G(f) := -f\ln f + \sum_{\alpha\in A}(u_\alpha f - \gamma_\alpha)$.

By using the formula $L'(f)g = \lim_{s\to 0} s^{-1}(L(f+sg) - L(f))$, it follows that $\int_T G'(f_*) g dm = 0$ for all g. Hence $G'(f_*) = 0$. Now $G'(f) = -\ln f - 1 + \sum_{\alpha\in A} x_\alpha u_\alpha$. We obtain that ef_* is the exponential of a linear combination of the functions $u_\alpha = u_\alpha(t)$ $(\alpha \in A)$. Writing the conditions (1) for $\mu := f_* m$, we obtain the system (2) which must then have a solution $x = (x_\alpha)_{\alpha\in A}$. In the 2nd section we will rigorously state and prove these considerations, by means of elements of Fenchel duality [**?**], [].

The idea from above is a known natural approach to this type of problems, at least in the case $T =$finite. One way or another, it was also used or suggested in several problems in which maximum entropy distributions naturally arise [**?**], [**?**], [**?**], [12].

2.1 Notation and definitions

We remind that a subset M of a real linear space is said to be *affine* if $\lambda x + (1-\lambda)y \in M$ for all $\lambda \in \mathbb{R}$ and $x, y \in M$. Let $\operatorname{aff} S$ denote the *affine hull* of a subset S of an n-dimensional Euclidian space ($n \in \mathbb{N}$), that is, the smallest affine set M s.t. $S \subset M$. The *relative interior* of a convex set $C \subset \mathbb{R}^n$, denoted by $\operatorname{ri} C$, is the interior of C regarded as a subset of $\operatorname{aff} C$, namely [[?]]

$$\operatorname{ri} C = \{x \in \operatorname{aff} C : \exists \varepsilon > 0 \text{ s.t. } B(x,\varepsilon) \cap \operatorname{aff} C \subset C\}$$

where $B(x,\varepsilon)$ denotes the Euclidian ball of center x and radius ε. Note that $\operatorname{ri} C$ also is a convex set, $\operatorname{ri} C \neq \emptyset$ whenever $C \neq \emptyset$, and $\overline{\operatorname{ri} C} = \overline{C}$, see [Theorems II.6.2-3, [?]]. If C is convex and moreover $\lambda c \in C$ whenever $c \in C$ and $\lambda > 0$, we call C a convex *cone*.

We call a function $\phi : \mathbb{R} \to (-\infty, \infty]$ *proper* if $f \not\equiv \infty$ and *convex* if it is convex on its *effective domain* $\operatorname{dom} \phi = \{x : \phi(x) < \infty\}$, namely if $\phi(\lambda x + (1-\lambda)y) \leq \lambda\phi(x) + (1-\lambda)\phi(y)$ for every $\lambda \in (0,1)$ and $x, y \in \operatorname{dom} \phi$. The *convex conjugate* $\phi^* : \mathbb{R} \to [-\infty, \infty]$ of ϕ is defined by $\phi^*(y) = \sup\{\langle x, y\rangle - \phi(x) : x \in \operatorname{dom} \phi\}$, see [Section III.12, [?]].

Let ϕ be defined by: $\phi(x) = x \ln x$ for $x > 0$, $\phi(0) = 0$ and $\phi(x) = +\infty$ for $x < 0$. Then ϕ is proper, convex, lower semicontinuous, bounded from below, with effective domain $[0, \infty)$ and its convex conjugate is $\phi^*(y) = e^{y-1}$ for all $y \in \mathbb{R}$.

2.2 Fenchel duality

For the basic notions of Fenchel duality, we refer to Rockafeller's book [14] where the finite dimensional case studied. In the infinite dimensional case, unlike in $\mathbb{R}^n$, there is no unique theorem covering all situations, but there are various results in the same spirit. The necessary result that best corresponds to our context is the following Borwein and Lewis's theorem from below. Note that applying Theorem 1 to our framework will also require several topics from the finite dimensional case [14]. This minimax - type result turns here to be more suitable than the various versions of the Lagrange multipliers method on infinite dimensional cones (that are usually involved in optimization of entropy - like functionals subject to convex restrictions).

Theorem 1 [Corollary 2.6,[7]] *Let T be a space with finite measure $\mu \geq 0$, $1 \leq p \leq \infty$, and $a_i \in L^q(\mu)$, $b_i \in \mathbb{R}$ for $i = \overline{1,n}$ where $\frac{1}{p} + \frac{1}{q} = 1$. Let $\phi : \mathbb{R} \to (-\infty, \infty]$ be proper, convex and lower semicontinuous, with* $(0, \infty) \subset \operatorname{dom} \phi$. *Suppose there exists $x \in L^p(\mu)$ with $x(t) > 0$ almost everywhere s.t. $\phi \circ x \in L^1(\mu)$ and $\int_T x(t)a_i(t)d\mu(t) = b_i$ for $i = \overline{1,n}$. Then the values $P \in [-\infty, \infty)$ and $D \in [-\infty, \infty]$ defined respectively by*

$$P = \inf\{\int \phi(x(t))d\mu(t) : x \in L^p(\mu), x \geq 0 a.e., \phi \circ x \in L^1(\mu), \int x(t)a_i(t)d\mu(t) = b_i \forall i\}$$

and

$$D = \max\{\sum_{j=1}^{n} b_j\lambda_j - \int \phi^*(\sum_{i=1}^{n} \lambda_i a_i(t))d\mu(t) : \lambda_1, \dots, \lambda_n \in \mathbb{R}, \phi^* \circ \sum_{i=1}^{n} \lambda_i a_i \in L^1(\mu)\}$$

are equal, and the maximum in the definition of D is atteined.

Note for later use that whenever there exists a vector $(\lambda_i)_{i=1}^n$ s.t. $\phi^* \circ \sum_{i=1}^n \lambda_i a_i \in L^1(\mu)$, Theorem 1 gives $-\infty < P = D < \infty$ (otherwise the conclusion holds in the form $P = D = -\infty$ if we set as usual $\max_\emptyset = -\infty$). In particular, $-\infty < D < \infty$ whenever ϕ is bounded from below, since in this case $-\infty < \phi^*(0) < \infty$ and hence for $(\lambda_i)_i := 0$ we have the constant function $\phi^* \circ \sum_{i=1}^n \lambda_i a_i \equiv \phi^*(0) \in L^1(\mu)$ because $\mu(T) < \infty$.

For every finite measure $\mu \geq 0$ and $f \in L^\infty_+$, the function $f \ln f$ belongs to the space L^1_μ, as follows by letting $x := f(t)$ a.e. and $y := \|f\|_\infty + 1$ in the elementary inequalities $-e^{-1} \leq x \ln x \leq y \ln y$ that hold for any $0 \leq x \leq y$ with $y \geq 1$, and then integrating with respect to μ.

For any multiindex $i = (i_1, \dots, i_n) \in \mathbb{Z}^n_+$ we write as usual $i! = i_1! \cdots i_n!$, $|i| = i_1 + \cdots + i_n$ and $x^i = x_1^{i_1} \cdots x_n^{i_n}$ for a variable $x = (x_1, \dots, x_n)$. Also, $i \leq j$ means $i_1 \leq j_1$, …, $i_n \leq j_n$. Let $\deg p$ denote the degree of a polynomial p. Let p_h denote the homogeneous part of maximal degree of p.

Let $GL(n)$, resp. $O(n)$ denote as usual the group of all invertible, resp. orthogonal linear maps on $\mathbb{R}^n$.

Remind that a *positive definite form* in n variables is a polynomial $p = \sum_{i,j=1}^n a_{ij}X_iX_j$ s.t. the $n \times n$ matrix $[a_{ij}]_{i.j=1}^n$ is positive definite, namely $\sum_{i,j=1}^n a_{ij}x_ix_j > 0$ for every vector $(x_i)_{i=1}^n \neq 0$ in $\mathbb{R}^n$ or, equivalently, s.t. $p(x) \geq c\|x\|^2$ for some constant $c = c_p > 0$ ($\Leftrightarrow \lim_{\|x\| \to \infty} p(x) = +\infty$, too).

Definition We call an arbitrary polynomial $p \in \mathbb{R}[X]$ *positive definite* if there exist constants $c > 0$ and R s.t.

$$p(x) \geq c\|x\|^2$$

for all $x \in \mathbb{R}^n$ with $\|x\| \geq R$, or, equivalently, if there exist $c > 0$, c' s.t.

$$p(x) + c' \geq c\|x\|^2 \quad \forall\, x \in \mathbb{R}^n,$$

condition that easily proves also to be equivalent to

$$\lim_{\|x\|\to\infty} p(x) = +\infty.$$

Let $P = P_n = \{\, p \in \mathbb{R}[X_1, \ldots, X_n] \,:\, p \text{ is positive definite } \}$.

Remark 2 (a) If $p = \sum_{i,j=1}^{n} a_{ij} X_i X_j + \sum_{i=1}^{n} b_i X_i + c$, then $p \in P_n \Leftrightarrow$ the form $\sum_{i,j=1}^{n} a_{ij} X_i X_j$ is positive definite.

(b) P_n is a convex cone, stable under multiplication.

(c) If $p \in P_n$, then for every $T \in GL(n)$, $x_0 \in \mathbb{R}^n$ and $c \in \mathbb{R}$ the polynomial $p(TX + x_0) + c$ also is in P_n.

(d) If $X = (X^1, \ldots, X^k)$ is a partition of the set $X = (X_1, \ldots, X_n)$ of variables and $p_j \in \mathbb{R}[X^j] \subset \mathbb{R}[X]$ is a positive definite form in $\mathbb{R}[X^j]$ for each $j = \overline{1,k}$ then $p_1 + \cdots + p_k \in P_n$.

(e) P_n is the minimal set containing all polynomials $p_1 + \cdots + p_k$ with $1 \leq k \leq n$ from (e) and stable under the operations from (b) and (c).

(f) If $p \in P$, then $\deg p$ must be even ≥ 2.

(g) For p homegeneous, $p \in P \Leftrightarrow \inf_{\|x\|=1} p(x) > 0 \Leftrightarrow p(x) \geq c\|x\|^{deg\, p}\, \forall x$ for some $c > 0$.

(h) If the homogeneous part p_h of p is in P, then $p \in P$, but the converse is not true: for example, the polynomial $p = X_1^4 + X_2^2 \in \mathbb{R}[X_1, X_2]$ is in P_2 while $p_h = X_1^4 \notin P_2$.

We endow the space $\mathbb{R}^n$ with the usual inner product, that will be inherited by all its linear subspaces. For every linear subspace $Y \subset \mathbb{R}^n$, let $P_Y : \mathbb{R}^n \to \mathbb{R}^n$ be the orthogonal projection onto Y, and $Y^\perp$ denote the space orthogonal to Y. For any linear map $T : Y \to Y$, let $T^* : Y \to Y$ denote as usual the Hilbert space adjoint of T, and $\|T\|$ be the uniform operator norm of T.

Lemma 3 *Let Y, $Z \subset \mathbb{R}^n$ be linear subspaces. Let P, $Q : Y+Z \to Y+Z$ be the linear projections onto Y, Z respectively defined by $P(u+v+t) = u+t$ and $Q(u+v+t) = v+t$ for $u \in Y \cap (Y\cap Z)^{\perp}$, $v \in Z \cap (Y \cap Z)^{\perp}$ and $t \in Y \cap Z$. Then:*

(a) *There exists an invertible linear map $T : Y+Z \to Y+Z$ s.t. $P_Y T = P$ and $P_Z T = Q$.*

(b) *For any l, $m \in \mathbb{N}$ with $l \leq m$ there are constants A and $B > 0$ s.t. $\|u+v+t\|^l \leq A(\|u\|^m + \|t\|^m + \|v\|^l) + B$ for all u, v, t as above.*

Proof. (a) For every u, v, t as above and $w \in Y \cap Z$, we have the equalities

$$\langle P^* w, u+v+t\rangle = \langle w, P(u+v+t)\rangle = \langle w, u+t\rangle = \langle w, t\rangle$$

and

$$\langle Q^* w, u+v+t\rangle = \langle w, Q(u+v+t)\rangle = \langle w, v+t\rangle = \langle w, t\rangle.$$

This proves the implication

$$w \in Y \cap Z \;\Rightarrow\; P^* w = Q^* w. \tag{3}$$

Let $y \in Y$ and $z \in Z$ be arbitrary s.t. $P^* y = Q^* z$. Then for $u \in Y \cap (Y \cap Z)^{\perp}$,

$$\langle y, u\rangle = \langle y, Pu\rangle = \langle P^* y, u\rangle = \langle Q^* z, u\rangle = \langle z, Qu\rangle = 0.$$

It follows that $y \in Y \cap Z$. Symmetrically, we obtain also $z \in Y \cap Z$. Now for every $t \in Y \cap Z$, we have

$$\langle y, t\rangle = \langle y, Pt\rangle = \langle P^* y, t\rangle = \langle Q^* z, t\rangle = \langle z, Qt\rangle = \langle z, t\rangle.$$

Hence $y = z$. Therefore, we have also the implication

$$y \in Y,\; z \in Z, \quad P^* y = Q^* z \quad \Rightarrow \quad y = z. \tag{4}$$

We can define then a map $\sigma : Y+Z \to Y+Z$ by

$$\sigma(y+z) = P^* y + Q^* z \quad (y \in Y,\, z \in Z),$$

for if $y+z = y'+z'$ for other vectors $y' \in Y$, $z' \in Z$, then $w := y - y' = z' - z$ is in $Y \cap Z$ and by (3) we obtain $P^* y + Q^* z = P^* y' + Q^* z'$. Moreover σ is injective (and so, invertible) since $\sigma(y+z) = 0 \Rightarrow P^* y = Q^*(-z)$, which by (4) gives $y+z = 0$. Then the equalities

$$\sigma P_Y = P^*, \;\; \sigma P_Z = Q^*$$

hold on the space $Y+Z$, as follows: if $x \in Y$, $\sigma P_Y x = P^* x$ from the definition of σ, while if $x \in Y+Z$ is in $Y^\perp$, $x \in (\text{range of } P)^\perp = \ker P^*$ and so both $\sigma P_Y x$, $P^* x = 0$; thus $\sigma P_Y = P^*$, and similarly one checks the second equality. We let then $T = \sigma^*$.

(b) Use twice the inequality $(\frac{a+b}{2})^p \leq \frac{a^p+b^p}{2}$ $(a\,, b \geq 0,\, p \geq 1)$ for $p = l,\, m$ as follows:

$$\|u+v+t\|^l \leq (\|u+t\| + \|v\|)^l \leq 2^{l-1}(\|u+t\|^l + \|v\|^l);$$

if $\|u+t\| \geq 1$, we estimate the right hand side term from above using

$$\|u+t\|^l + \|v\|^l \leq \|u+t\|^m + \|v\|^l \leq 2^{m-1}(\|u\|^m + \|t\|^m) + \|v\|^l,$$

while if $\|u+t\| < 1$ we can estimate it as

$$2^{l-1}(\|u+t\|^l + \|v\|^l) \leq 2^{l-1}(1 + \|v\|^l) \leq 2^{l-1}(\|u\|^m + \|t\|^m + \|v\|^l) + 2^{l-1};$$

hence the desired inequality holds with $A = 2^{l+m-2}$ and $B = 2^{l-1}$.

Lemma 4 *Let p, $q \in \mathbb{R}[X]$ with q homogeneous s.t. $1 \leq deg\, q < deg\, p$. Let Y, $Z \subset \mathbb{R}^n$ be linear subspaces s.t. $p = p \circ P_Y$ and $q = q \circ P_Z$. Moreover, we assume that the spaces $Y \cap (Y \cap Z)^\perp$ and $Z \cap (Y \cap Z)^\perp$ are orthogonal. Suppose there are constants a and $c > 0$ s.t. $p(y) \leq -c\|y\|^m + a$ for all $y \in Y$. If $\delta := \sup\{d(z, Y) : z \in Z, \|z\| = 1, q(x) \geq 0\} < 1$, then there are constants a' and $c' > 0$ s.t. $(p+q)(x) \leq -c'\|x\|^l + a'$ for all $x \in Y+Z$.*

Proof. Let $l = \deg q$ and $m = \deg p$. Thus $1 \leq l < m$. Fix a constant $C > 0$ s.t. for all $s \in \mathbb{R}^n$, $q(s) \leq C\|s\|^l$. Since the spaces $Y \cap (Y \cap Z)^\perp$ and $Z \cap (Y \cap Z)^\perp$ are orthogonal, we have $d(v, Y) = \|v\|$ for every $v \in Z \cap (Y \cap Z)^\perp$. Let $K = K(\delta, C, l, m, ...) >> C$ (s.t. sa obtin ulterior o contradictie cu $\delta < 1$). Use the notation in Lemma 3, namely let P, $Q : Y+Z \to Y+Z$ be the linear projections onto Y, Z defined by $P(u+v+t) = u+t$ and $Q(u+v+t) = v+t$ for $u \in Y \cap (Y \cap Z)^\perp$, $v \in Z \cap (Y \cap Z)^\perp$ and $t \in Y \cap Z$. We claim that there exists a constant $c_1 > 0$ s.t.

$$q(t+v) \leq -c_1\|v\|^l + K\|t\|^l \quad (t \in Y \cap Z,\, v \in Z \cap (Y \cap Z)^\perp). \tag{5}$$

Indeed, supposing the opposite, we could find sequences $t_k \in Y \cap Z$ and $v_k \in Z \cap (Y \cap Z)^\perp$ $(k \geq 1)$ s.t. $q(t_k + v_k) > -\frac{1}{k}\|v_k\|^l + K\|t_k\|^l$ for all $k \geq 1$. Note that $d(v_k, Y) > 0$ for all $k \geq 1$, for otherwise some $v_k = 0$, which by the strict inequalities $K\|t_k\|^l < q(t_k) \leq C\|t_k\|^l$ and $K > C$ would lead to $t_k = 0$, that is impossible since $\|t_k + v_k\|^2 = \|t_k\|^2 + \|v_k\|^2 \neq 0$ (note also that $l \geq 1 \Rightarrow q(0) = 0$). Then for each k we can divide the last estimate by $d(v_k, Y)$, use that q is homogeneous of degree l, note $t'_k = t_k/d(v_k, Y)$ and $v'_k = v_k/d(v_k, Y)$ and use the compactness of the unit sphere to obtain some accumulation points $t_0 \in Y \cap Z$ and $v_0 \in Z \cap (Y \cap Z)^\perp$ of the sequences $(t'_k)_k$ and $(v'_k)_k$ respectively; to this aim, we note that $(v'_k)_k$ is bounded due to $\|v'_k\| = d(v'_k, Y) = 1$ and so $(t'_k)_k$ also is bounded by means of the estimates

$$K\|t'_k\|^l \leq q(t'_k + v'_k) + 1 \leq C\|t'_k + v'_k\|^l + 1$$

and

$$\|t'_k + v'_k\|^l \leq (\|t'_k\| + \|v'_k\|)^l \leq 2^{l-1}(\|t'_k\|^l + \|v'_k\|^l) = 2^{l-1}\|t'_k\| + 2^{l-1}.$$

Moreover, from the two estimates from above it follows also, due to $K >> C$, that $\|t'_k\|$ (and hence, $\|t_0\|$) is negligible... (comparat cu diferenta $1 - \delta$) a.i. $\|t_0 + v_0\| \approx \|v_0\| = 1$ and $d(v_0 + t_0, Y) = d(v_0, Y) = 1$ which is impossible (due to the hypothesys, we should have $d(t_0 + v_0, Y) < 1$ because $q(t_0 + v_0) \geq 0$).

Therefore, the existence of a constant $c_1 > 0$ satisfying (5) is proved. Moreover we may diminish c_1 so that $c'_1(:= c_1\epsilon^l ?\, sau\, c_1?) < c/2$.

Also, since $l < m$, there exists a constant a_1 s.t.

$$-c\|t\|^m + K\|t\|^l \leq -\frac{c}{2}\|t\|^m + a_1 \quad (t \in Y \cap Z). \tag{6}$$

Indeed, the estimate $-c\|t\|^m + K\|t\|^l \leq -\frac{c}{2}\|t\|^m$ holds for large t, with $\|t\| > R := (\frac{2K}{c})^{\frac{1}{m-l}}$, and since the inequality fails only on a (compact) closed unit ball, the difference of the two terms in (6) can be compensated on $\|t\| \leq R$ by the addition of a large, suitable constant $a_1 > 0$.

Let $T : Y + Z \to Y + Z$ and A, B be the mapping and constants provided by Lemma 3. Set $c' = c_1 A^{-1}\|T\|^{-l}$ and $a' = a + a_1 + c_1 A^{-1} B$.

Let $x \in Y + Z$ be arbitrary. Set $w = T^{-1}x$. Then $\|x\| = \|Tw\| \leq \|T\|\,\|w\|$. Write $w = u + v + t$ as in Lemma 3. Since $u + t \in Y$, our hypotheses implies that

$$p(P(w)) = p(P(u + v + t)) = p(u + t) \leq -c\|u + t\|^m + a.$$

Also, by (5), we have

$$q(Q(w)) = q(Q(u + v + t)) = q(v + t) \leq -c_1\|v\|^l + K\|t\|^l.$$

The last two estimates from above, together with the equalities

$$p \circ T = (p \circ P_Y) \circ T = p \circ P, \quad q \circ T = (q \circ P_Z) \circ T = q \circ Q,$$

and the inequality $\|u + t\|^m = (\|u\|^2 + \|t\|^2)^{m/2} \geq \|u\|^m + \|t\|^m$ (due to $m \geq 2$), imply that

$$(p \circ T + q \circ T)(w) = p(P(w)) + q(Q(w)) \leq -c\|u + t\|^m + a + K\|t\|^l - c_1\|v\|^l$$
$$\leq -c\|u\|^m - c\|t\|^m + K\|t\|^l - c_1\|v\|^l + a,$$

that together with (6), using also $c_1 < c/2$, the estimate (b), and the equality $u + v + t = w$, gives

$$(p + q)(x) = (p \circ T + q \circ T)(w) \leq -c\|u\|^m - \frac{c}{2}\|t\|^m + a_1 - c_1\|v\|^l + a \leq$$
$$-c_1(\|u\|^m + \|t\|^m + \|v\|^l) + a_1 + a \leq -c_1 A^{-1}\|u + v + t\|^l + c_1 A^{-1} B + a_1 + a$$
$$\leq -c_1 A^{-1}\|T\|^{-l}\|x\|^l + a' = -c'\|x\|^l + a'.$$

We note $X' = (X_1, \ldots, X_{n-1})$ so that $X = (X', X_n)$.

Lemma 5 *For any $p \in \mathbb{R}[X]$ there exists a unique minimal linear subspace $Y \subset \mathbb{R}^n$ s.t. $p = p \circ P_Y$.*

Proof. Let $L = \{Y \subset \mathbb{R}^n : Y \text{ linear subspace}, p = p \circ P_Y\}$. Then $L \neq \emptyset$ since $\mathbb{R}^n \in L$. Any totally ordered chain $\mathcal{L}$ in L decreasing with respect to the order induced by inclusion must be finite and become stationary at some element $Y_{\mathcal{L}}$ s.t. $Y_{\mathcal{L}} \subset Y$ for all $Y \in \mathcal{L}$, since we have the dimension function with valus in $\mathbb{N}$. By Zorn's lemma, there exist minimal elements Y in L. Now any two such minimal elements must coincide, since whenever Y, $Z \in L$ one can easily derive from the equalities $p = p \circ P_Y$ and $p = p \circ P_Z$ that we have $p = p \circ Q$ for every linear map $Q : \mathbb{R}^n \to \mathbb{R}^n$ from the sequence P, QP, PQP, $QPQP$, ..., whose limit is easily shown to be the orthogonal projection $P_{Y \cap Z}$ onto the intersection $Y \cap Z$; whence $p = p \circ P_{Y \cap Z}$ and due to the minimality me obtain Y, $Z \subset Y \cap Z$, that is, $Y = Z$.

Let $\operatorname{supp} p$ denote the unique minimal linear subspace provided by Lemma 5. We call $\operatorname{supp} p$ the *support* of the polynomial p.

Lemma 6 *Let $\tilde{\pi}$, $\tilde{q}$, $\tilde{r}$ be polynomials with $deg\,\tilde{r} < deg\,\tilde{q} (< deg\,\tilde{\pi}?)$ and $\tilde{q}$ homogeneous of degree k. Write $\tilde{q} = \sum_{j=0}^{k} P_j X_n^j$ with $P_j \in \mathbb{R}[X']$ homogeneogous of degree $k - j$. Suppose there is an index $j \in \{1, \ldots, k - 1\}$ s.t. $P_j \not\equiv 0$. Suppose also that $\tilde{\pi} \in \mathbb{R}[X']$. Then $e^{\tilde{\pi} + \tilde{q} + \tilde{r}} \notin L^1$.*

Proof. Vezi foi 1,2. Let $J \geq 1$ be the maximal index $j \geq 1$ s.t. $P_j \not\equiv 0$. Thus $\tilde{q} = \sum_{j=1}^{J} P_j X_n^j$ and $P_J \not\equiv 0$. Then $\int ... + \infty$.

Lemma 7 *Let π, q, $r \in \mathbb{R}[X]$ s.t. $deg\, r < deg\, q (< deg\, \pi?)$ and q is homogeneous. Let $Y \subset \mathbb{R}^n$ be a linear subspace s.t. $\pi = \pi \circ P_Y$. Suppose that $\sup\{d(z, Y) : z \in supp\, q \, \|z\| = 1, \, q(z) \geq 0\} = 1$. Then $e^{\pi+q+r} \notin L^1$.*

Proof. We denote $\operatorname{supp} q$ by Z. Thus Z is the minimal linear subspace of $\mathbb{R}^n$ s.t. $q = q \circ P_Z$. Set also $k = \deg q$. Thus $1 \leq k < \deg \pi$.

Since $\{z \in Z : q(z) \geq 0\} \not\subset Y$, the set $\{z \in Z : q(z) \geq 0, z \notin Y\}$ is nonempty and $\neq \{0\}$. Then it contains at least one vector v of norm 1, because q is homogeneous of degree ≥ 1. Since $v \notin Y$, there is a subspace $X \subset \mathbb{R}^n$ of codimension 1 s.t. $Y \subset X$ and $v \not/ X$. Therefore, $v \in Z \setminus X$ $(\neq \emptyset)$ and $q(v) \geq 0$. By composing the equality $\pi = \pi \circ P_Y$ with P_X we derive, using $P_Y \circ P_X = P_Y$ and $\pi = \pi \circ P_Y$, that $\pi \circ P_X = \pi$. There exists a change of coordinates $T \in GL(n)$ (de fapt e ortogonala)s.t. $T(X) = \mathbb{R}^{n-1} \times \{0\} \subset \mathbb{R}^n$ and $Tv = e_n := (0, \ldots, 0, 1)$. Then $\mathcal{P} := TP_X T^{-1}$ is a projection (nonorthogonal, generally) onto the linear subspace $\mathbb{R}^{n-1} \times \{0\}$ of $\mathbb{R}^n$. Set $\tilde{\pi} = \pi \circ T^{-1}$. From the equality $\pi \circ P_X = \pi$ we obtain $\tilde{\pi} = \tilde{\pi} \circ \mathcal{P}$. Let also $\tilde{q} = q \circ T^{-1}$ and $\tilde{r} = r \circ T^{-1}$. Then $(\pi + q + r) \circ T^{-1} = \tilde{\pi} + \tilde{q} + \tilde{r}$, $\deg \tilde{r} < \deg \tilde{q} < \deg \tilde{\pi}$ and $\tilde{q}$ is homogeneous. We write $\tilde{q} = \sum_{j=0}^{k} P_j X_n^j$ where $P_j \in \mathbb{R}[X']$ is homogeneogous of degree $k - j$. Since $\deg \tilde{q} \geq 1$, $\tilde{q} \not\equiv 0$ and so there is an index $j \in \{0, \ldots, k\}$ s.t. $P_j \not\equiv 0$. Let J be the maximal j with this property. Thus $\tilde{q} = \sum_{j=0}^{J} P_j X_n^j$ and $P_j \equiv 0$ for all $j = \overline{J+1, k}$. We claim that $J \geq 1$. Indeed, supposing $J = 0$ would imply that $\tilde{q} = P_0 \in \mathbb{R}[X']$, namely that $\tilde{q}$ depends only on the first $n-1$ variables $X_1, \ldots, X_n$. Then for every vector $x \in X$ and real number λ, writing $Tx = (x'_1, \ldots, x'_{n-1}, 0)$ for the appropriate linear functionals $x'_i = x'_i(x)$ with $i = \overline{1, n-1}$, we obtain

$$q(x + \lambda v) = \tilde{q}(Tx + \lambda Tv) = \tilde{q}(x'_1, \ldots, x'_{n-1}, \lambda) = \tilde{q}(x'_1, \ldots, x'_{n-1}, 0) = q(x).$$

This means $q \circ P = q$ where P is the linear projection defined by $P(x + \lambda v) = x$ onto the hyperplane X. Note that $Pv = 0$. By Lemma **??**, it follows that $q = q \circ P_{\ker(I-P^*)}$. Then $\tilde{X} := \ker(I - P^*)$ is another hyperplane and from $q = q \circ P_{\tilde{X}}$ and $q = q \circ P_Z$ we derive $q = q \circ P_{\tilde{X} \cap Z}$. Moreover $v \notin \tilde{X}$ since $v = P^* v$ would imply $\langle v, v \rangle = \langle P^* v, v \rangle = \langle v, Pv \rangle = \langle v, 0 \rangle = 0$ that is impossible since $\|v\| = 1$.

Now since $v \in Z \setminus \tilde{X}$ and $\tilde{X}$ has codimension 1, it follows that $\tilde{X} + Z = \mathbb{R}^n$. Hence $n = \dim(\tilde{X} + Z) = \dim \tilde{X} + \dim Z - \dim \tilde{X} \cap Z = n - 1 + \dim Z -$

$\dim \tilde{X} \cap Z$ and so we have obtained a linear subspace $\tilde{X} \cap Z$ s.t. $q = q \circ P_{\tilde{X} \cap Z}$ and $\dim \tilde{X} \cap Z < \dim Z$, that is impossible due to the minimality of the support Z of q.

Therefore $J \geq 1$, doua cazuri: $J < K$: which by Lemma 6 leads to $e^{\tilde{\pi}+\tilde{q}+\tilde{r}} \notin L^1$ ($\Longrightarrow e^{\pi+\cdots} \notin L^1$); iar pentru cazul $J = K$, vezi foaia 4 .

Proposition 8 *Let $p \in \mathbb{R}[X_1, \ldots, X_n]$ be arbitrary. Set $f(t) = e^{p(t)}$ for $t \in \mathbb{R}^n$. The following statements are equivalent:*

(a) *The function $f = e^p$ is Lebesgue integrable on $\mathbb{R}^n$.*

(b) *The polynomial $-p$ is positive definite in $\mathbb{R}[X_1, \ldots, X_n]$.*

Proof. (a) $\Rightarrow$ (b) Foaia 3 lateral, inductie etc. Suppose that $e^p \in L^1$. Let $k \in \mathbb{Z}_+$ be the degree of p. Obviously $k \geq 1$, since constant positive functions $e^c \notin L^1$. Set $p_h := \sum_{|i|=k} c_i X^i$ ($\not\equiv 0$). We have to show that p_h is negative definite. Due to the compactness of the unit sphere in $\mathbb{R}^n$, it is sufficient to prove that $p_h(v) < 0$ for every vector v with $\|v\| = 1$, and hence the conclusion $\sup_{\|t\|=1} p_h(t) < 0$ will follow. Suppose there is some unit vector $v \in \mathbb{R}^n$ s.t. $p(v) \geq 0$. We can assume $v = (0, \ldots, 0, 1)$ by composition with a rotation, since conditions (a) and (b) are invariant under orthogonal transforms. Set $X' = (X_1, \ldots, X_{n-1})$. Write $p_h = \sum_{j=0}^k p_j X_n^j$ where $p_j \in \mathbb{R}[X']$ is homegeneous of degree $k - j$. Since $p_h \not\equiv 0$, there exists a minimal $k' \geq 0$, $k' \leq k$ s.t. $p_{k'} \not\equiv 0$. Thus $p_h = p_0(X')X_n^0 + p_1(X')X_n^1 + \cdots + p_{k'}(X')X_n^{k'}$. Since $p_h(0, 1) \geq 0$, the conclusion follows by induction.

Combining the partial results from above, we can finally obtain the following theorem.

Main result:

Theorem *Let $g = (g_i)_{\in \mathbb{Z}_+^n, |i| \leq 2m}$ be a set of powers moments of a measure $\mu = f dt + \nu \geq 0$, with $f \in L^1(\mathbb{R}^n, dt) \setminus \{0\}$ and ν singular with respect to dt. Namely,*

$$\int_{\mathbb{R}^n} t^i d\mu(t) = g_i \ (|i| \leq 2m).$$

Then there exist $x_i \in \mathbb{R}$ ($|i| \leq 2m$), uniquely determined by g, such that the polynomial

$$p(t) = p_x(t) := \sum_{|j| \leq 2m} x_j t^j$$

satisfies $p(t) \leq -c\|t\|^2 + c'$ and

$$\int_{\mathbb{R}^n} t^i \; exp\,(\sum_{|j| \leq 2m} x_j t^j) dt = g_i \quad (|i| \leq 2m)$$

(and conversely, any vector $(x_i)_i$ as above provides a nonsingular representing density $f(t) = e^{p_x(t)}$ for g).

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