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ON SOME DUALITY RELATIONS IN THE THEORY OF TENSOR PRODUCTS

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ABSTRACT. We study several classical duality results in the theory of tensor products, due mostly to Grothendieck, providing new proofs as well as new results. In particular, we show that the canonical mapping $Y^* \otimes_\pi X \to (\mathcal{L}(X,Y),\tau)^*$ is not always injective, answering a problem of Defant and Floret. We use the machinery of vector measures to give new proofs of the dualitites $(X \otimes_\varepsilon Y)^* = \mathcal{N}(X,Y^*)$, whenever Y^* has the RNP, and (a slight improvement of) the result of Rosenthal $(X \otimes_\varepsilon Y)^* \subset \overline{\mathcal{F}}(X,Y^*)$, whenever $\ell_1 \not\hookrightarrow Y$.

1. Inroduction and preliminaries

The goal of the present note is to study several classical duality results in the theory of tensor products, due mostly to Grothendieck, providing new proofs as well as new results.

An important result in the topological theory of tensor products is the theorem of Grothendieck that gives a description the linear topological dual of the space of bounded linear operators $\mathcal{L}(X,Y)$ equipped with the τ-topology of uniform convergence on compact sets. According to this result the continuous linear functionals on $(\mathcal{L}(X,Y),\tau)$ consist of all

$$\phi(T) = \sum_{i=1}^{\infty} \langle y_i^*, Tx_i \rangle, x_i \in X, y_i^* \in Y^*, \sum_{i=1}^{\infty} \|x_i\| \|y_i^*\| < \infty \tag{1}$$

This formulation of Grothendieck's theorem is taken from [LT77] (Prop. 1.e.3.). Its advantage is that it uses only elementary functional analytic language. However, it is more natural to rephrase this result using the language of tensor products: The canonical mapping (which is described by the formula (1)) $Y^* \otimes_\pi X \to (\mathcal{L}(X,Y),\tau)^*$ is surjective. This is the formulation to be found in [DF93] (Prop. 5.5). A natural question posed e.g. by Defant and Floret [DF93], p.65, is whether the canonical mapping above is also injective? In Theorem 2.5 we give a negative solution to this problem. We proceed by giving a new proof of the classical duality result $(X \otimes_\varepsilon Y)^* = \mathcal{N}(X,Y^*)$, whenever Y^* has the RNP. Our proof avoids the machinery of integral operators, and uses instead the theory of vector measures (Bochner integral). Using the same approach, we extend the recent result of Rosenthal which claims that $(X \otimes_\varepsilon Y)^* = \mathcal{I}(X,Y^*) \subset \mathcal{K}(X,Y^*)$ iff Y does not contain a copy of ℓ_1. We show that in fact the integral operators are approximable. In the final part we give a new proof of another celebrated result of Grothendieck which claims that X^* has a metric approximation property whenever it is a dual RNP space with the approximation property (AP for short).

We begin by collecting some basic definitions and results whose proofs may be found in [DU77], [DF93], [LT77], [Tal84] and the forthcoming monograph [F2]. Our

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notation is standard. By $\mathcal{F}(X,Y)$ we denote the space of all finite rank operators from $\mathcal{L}(X,Y)$. By $\mathcal{F}_{w^*}(X^*,Y)$ we denote the w^*-w continuous operators from $\mathcal{F}(X^*,Y)$. We denote $\mathcal{I}(X,Y)$ the space of all integral operators. We recall that a couple $\langle \mathcal{L}(X,Y^*), X\otimes Y\rangle$ forms a duality pair defined as follows. For $T\in\mathcal{L}(X,Y^*)$, $z=\sum_{i=1}^n x_i\otimes y_i\in X\otimes Y$ put

$$\langle T,z\rangle=\sum_{i=1}^n\langle T(x_i),y_i\rangle. \tag{2}$$

The pairing enables us to introduce the projective norm π on $X\otimes Y$ as follows.

$$\pi(z)=\sup\{\langle T,z\rangle,\|T\|\le 1, T\in\mathcal{L}(X,Y^*)\}. \tag{3}$$

The projective tensor product $X\otimes_\pi Y$ is the completion of $(X\otimes Y,\pi)$. Every element $z\in X\otimes_\pi Y$ admits a representation $z=\sum_{i=1}^\infty x_i\otimes y_i$ such that $\sum_{i=1}^\infty\|x_i\|\|y_i\|<\infty$ (WLOG $(\|x_i\|)\in c_0$ and $(\|y_i\|)\in\ell_1$) and

$$\pi(z)=\inf\{\sum_{i=1}^\infty\|x_i\|\|y_i\| : z=\sum_{i=1}^\infty x_i\otimes y_i\} \tag{4}$$

Moreover, we have

Proposition 1.1. *(*[DF93]*, Chap. 3) Let X,Y be Banach spaces. Then the canonical dual pairing gives the linear topological duality*

$$(X\otimes_\pi Y)^*=\mathcal{L}(X,Y^*). \tag{5}$$

Closely connected to the projective tensor product $X^*\otimes_\pi Y$ is the notion of nuclear operator. An operator $T:X\to Y$ is called nuclear if there exists a pair of sequences $\{x_i^*\}_{i=1}^\infty$ in X^* and $\{y_i\}_{i=1}^\infty$ in Y such that $\sum_{i=1}^\infty\|x_i^*\|\|y_i\|<\infty$ and $Tx=\sum_{i=1}^\infty\langle x_i^*,x\rangle y_i$. The nuclear norm is defined by

$$N(T)=\inf\{\sum_{i=1}^\infty\|x_i^*\|\|y_i\| : Tx=\sum_{i=1}^\infty\langle x_i^*,x\rangle y_i\}. \tag{6}$$

The Banach space of nuclear operators is denoted by $\mathcal{N}(X,Y)$. Let $J:\sum_{i=1}^\infty x_i^*\otimes y_i\to\sum_{i=1}^\infty x_i^*\otimes y_i$ be the formal identity mapping defined for all pairs of sequences $\{x_i^*,\}_{i=1}^\infty\in X^*$, $\{y_i\}_{i=1}^\infty\in Y$ such that $\sum_{i=1}^\infty\|x_i^*\|\|y_i\|<\infty$. The formal identity J is a well-defined quotient mapping $J:X^*\otimes_\pi Y\to\mathcal{N}(X,Y)$. Let X,Y be Banach spaces. Let τ be the locally convex topology on $\mathcal{L}(X,Y)$ of uniform convergence on compact sets in X, generated by seminorms $\|T\|_K$, $K\subset X$ norm compact set.

Theorem 1.2. *(Grothendieck)*
Let X be a Banach space. The following conditions are equivalent:
1. X has the approximation property.
2. For every Banach space Y, $\overline{\mathcal{F}}^\tau(X,Y)=\mathcal{L}(X,Y)$.
3. For every Banach space Y, $\overline{\mathcal{F}}^\tau(Y,X)=\mathcal{L}(Y,X)$.
4. $J:X^*\otimes_\pi X\to\mathcal{N}(X)$ is injective, or equivalently it is an isometry.
5. For every Banach space Y, $J:Y^*\otimes_\pi X\to\mathcal{N}(Y,X)$ is injective, or equivalently it is an isometry.

The next theorem is almost certainly known to specialists. As we have not found an explicit reference, we include its proof for the convenience of the reader.

Theorem 1.3. *Let X be a Banach space. The following conditions are equivalent:*
1. X^* has AP.
2. $J:X^*\otimes_\pi X^{**}\to\mathcal{N}(X,X^{**})$ is an isometry.
3. For every Banach space Y, $J:X^*\otimes_\pi Y\to\mathcal{N}(X,Y)$ is an isometry.

Proof. (2) $\Rightarrow$ (1) It is well-known ([Jar81], p.326) that the formal transposition mapping $t : E \otimes_\pi F \to F \otimes_\pi E$, $t(\sum_{i=1}^\infty e_i \otimes f_i) = \sum_{i=1}^\infty f_i \otimes e_i$ is an isometric linear isomorphism. Next, we note that $\mathcal{N}(X, X^{**})$ and $\mathcal{N}(X^*, X^*)$ are canonically isometric, via the transposition of their elements $z = \sum_{i=1}^\infty x_i^* \otimes x_i^{**} \leftrightarrow z' = \sum_{i=1}^\infty x_i^{**} \otimes x_i^*$. Indeed, $\mathcal{N}(X, X^{**})$ is a quotient (via J) of $X^* \otimes_\pi X^{**}$, while $\mathcal{N}(X^*, X^*)$ is a quotient (via J') of the isometric transpose $t(X^* \otimes_\pi X^{**}) = X^{**} \otimes_\pi X^*$. The kernels are described as follows.

$$Ker(J) = \{z = \sum_{i=1}^\infty x_i^* \otimes x_i^{**} : \sum_{i=1}^\infty x_i^*(x) x_i^{**} = 0 \text{ for all } x \in X\}. \tag{7}$$

$$Ker(J') = \{z' = \sum_{i=1}^\infty x_i^{**} \otimes x_i^* : \sum_{i=1}^\infty x_i^{**}(x^*) x_i^* = 0 \text{ for all } x^* \in X^*\}. \tag{8}$$

Both of these conditions are indeed equivalent to the single condition $z \in Ker(J) \Leftrightarrow t(z) \in Ker(J')$, which is to say $\sum_{i=1}^\infty x_i^{**}(x^*) x_i^*(x) = 0$ for all $x \in X, x^* \in X^*$. Using the transposition we may transform (2) of Theorem 1.3 into the equivalent statement that $J' : X^{**} \otimes_\pi X^* \to \mathcal{N}(X^*, X^*)$ is an isometry. By condition (4) of Theorem 1.2 we conclude that X^* has the AP. (3) $\Rightarrow$ (2) is immediate. It remains to show (1) $\Rightarrow$ (3). Let $0 \neq z = \sum_{i=1}^\infty x_i^* \otimes y_i \in X^* \otimes_\pi Y$; our goal is to show that $J(z) \neq 0$. WLOG we may assume that $\sum_{i=1}^\infty \|y_i\| < \infty$ and $\lim_{i\to\infty} \|x_i^*\| = 0$. We proceed by contradiction, assuming that $J(z)(y) = \sum_{i=1}^\infty x_i^*(x) y_i = 0$ for all $x \in X$. Given $\varepsilon > 0$, by condition (3) in Theorem 1.2 there is a

$$F = \sum_{k=1}^n u_k^{**} \otimes u_k^* \in \mathcal{F}(X^*), \text{ such that } \sup_i \|F(x_i^*) - x_i^*\| < \varepsilon. \tag{9}$$

We let $z' = \sum_{i=1}^\infty F(x_i^*) \otimes y_i \in X^* \otimes_\pi Y$. Note the important fact that $z' \in X^* \otimes Y$ is actually a finite tensor. Indeed,

$$z' = \sum_{i=1}^\infty (\sum_{k=1}^n u_k^{**}(x_i^*) u_k^*) \otimes y_i = \sum_{k=1}^n u_k^* \otimes (\sum_{i=1}^\infty u_k^{**}(x_i^*) y_i). \tag{10}$$

Next, $J(z')$ satisfies the following:

$$J(z')(x) = \sum_{i=1}^\infty \langle F(x_i^*), x\rangle y_i = \sum_{i=1}^\infty \langle x_i^*, F^*(x)\rangle y_i = 0, \text{ for every } x \in X. \tag{11}$$

Hence $J(z') = 0$, as an element of $\mathcal{K}(X, Y)$, and since z' is also a finite tensor we conclude that $z' = 0$ as an element of $X^* \otimes_\pi Y$. Hence we have an estimate

$$\pi(z) = \pi(z - z') = \pi(\sum_{i=1}^\infty x_i^* \otimes y_i - \sum_{i=1}^\infty F(x_i^*) \otimes y_i) \leq \varepsilon \sum_{i=1}^\infty \|y_i\|. \tag{12}$$

Since ε was arbitrarily small, we conclude that $\pi(z) = 0$ as desired. It is clear by the Banach open mapping theorem that J is an isometry. $\square$

2. Dual of $(\mathcal{L}(X, Y), \tau)$

Denote by $i : \mathcal{L}(X, Y) \to \mathcal{L}(X, Y^{**})$ the formal identity embedding. By Proposition 1.1 (and the transposition isometry $Y^* \otimes_\pi X = t(X \otimes_\pi Y^*)$) we have

$$(Y^* \otimes_\pi X)^* = \mathcal{L}(X, Y^{**}) = \mathcal{L}(Y^*, X^*). \tag{13}$$

We consider the w^*-topology on $\mathcal{L}(X, Y^{**})$, resp. $\mathcal{L}(Y^*, X^*)$, originating from this duality. Then we have the following.

Lemma 2.1. *The mapping*

$$i : (\mathcal{L}(X,Y),\tau) \to (\mathcal{L}(X,Y^{**}), w^*) \tag{14}$$

is continuous. In particular, the dual mapping

$$i^* : Y^* \otimes_\pi X \to (\mathcal{L}(X,Y),\tau)^* \tag{15}$$

is $w - w^*$ continuous (the topologies come from the duality pairs described above).

Proof. Every $z \in Y^* \otimes_\pi X$ admits a representation $z = \sum_{i=1}^\infty y_i^* \otimes x_i$, such that $(\|x_i\|) \in c_0$ and $(\|y_i^*\|) \in \ell_1$. Let $K = \overline{\mathrm{conv}}\{x_i\}_{i=1}^\infty$ be a compact and convex set in X. Let U be a τ-open set in $\mathcal{L}(X,Y)$ defined as $U = \{T : \sup_{x\in K} \|T(x)\| < 1\}$. Clearly, $T \in U$ implies $|y^*(T(x))| < \|y^*\|$ for all $y^* \in Y^*$, $x \in K$. Thus $|\langle T, \sum_{i=1}^\infty y_i^* \otimes x_i\rangle| \le \sum_{i=1}^\infty \|y_i^*\| < \infty$ for all $T \in U$, which finishes the proof. The second result follows by duality. □

Let us denote $\mathcal{T} : \mathcal{L}(X,Y) \to \mathcal{L}(Y^*,X^*)$, $\mathcal{T}(T) = T^*$ the conjugation operator. Of course, $\mathcal{T}$ is an isometric embedding whose target space is a dual space, which is canonically isometric with $\mathcal{L}(X,Y^{**})$.

Proposition 2.2. *Let X,Y be Banach spaces. The canonical image $\mathcal{L}(X,Y) \hookrightarrow \mathcal{L}(X,Y^{**})$, resp. $\mathcal{T}(\mathcal{L}(X,Y)) \hookrightarrow \mathcal{L}(Y^*,X^*)$, is w^*-dense if and only if i^* is injective.*

Proof. $\mathcal{L}(X,Y)$ is w^*-dense in $\mathcal{L}(X,Y^{**})$ iff for every $z \in Y^* \otimes_\pi X$ such that $z \in \mathcal{L}(X,Y)_\perp$ it holds that $z = 0$. Alternatively, $i^*(z) = 0$ implies $z = 0$ which is clearly equivalent to the injectivity of i^*. The respective case follows by standard transposition. □

The following is a more complete formulation of the Grothendieck duality result.

Theorem 2.3. *(Grothendieck,* [DF93]*, Prop. 5.5)*
The mapping $i^* : Y^* \otimes_\pi X \to (\mathcal{L}(X,Y),\tau)^*$ from (15) *is surjective. In other words, the continuous linear functionals on $(\mathcal{L}(X,Y),\tau)$ consist of all*

$$\phi(T) = \sum_{i=1}^\infty \langle y_i^*, Tx_i\rangle, x_i \in X, y_i^* \in Y^*, \sum_{i=1}^\infty \|x_i\|\|y_i^*\| < \infty \tag{16}$$

In some cases, the mapping i^* is injective. For example:

Theorem 2.4. *(*[DF93]*, p. 65) Let X,Y be Banach spaces. Suppose that either X or Y^* has the AP, or that Y is reflexive. Then the mapping $i^* : Y^* \otimes_\pi X \to (\mathcal{L}(X,Y),\tau)^*$ from* (15) *is injective. In particular, we may write $(\mathcal{L}(X,Y),\tau)^* = Y^* \otimes_\pi X$. The pairing is canonical,*

$$\langle z, T\rangle = \sum_{i=1}^\infty \langle y_i^*, Tx_i\rangle, T \in \mathcal{L}(X,Y), z = \sum_{i=1}^\infty y_i^* \otimes x_i \in Y^* \otimes_\pi X. \tag{17}$$

Our first main result is contained in the next characterization.

Theorem 2.5. *Let Y be a Banach space with the AP. Then the mapping $i^* : Y^* \otimes_\pi X \to (\mathcal{L}(X,Y),\tau)^*$ from* (15) *is injective for every Banach space X if and only if Y^* has the AP.*

Proof. We first assume the injectivity of i^* for every Banach space X. In fact, the case $X = Y^{**}$ is sufficient in order to prove the direct implication of our theorem. Our goal is to establish that Y^* has the AP. By using Theorem 1.3, it suffices to show that $J : Y^* \otimes_\pi X \to \mathcal{N}(Y,X)$ is an isometry. Recall that

$$Ker(i^*) = \{z = \sum_{i=1}^\infty y_i^* \otimes x_i : \langle z, S\rangle = \sum_{i=1}^\infty \langle y_i^*, S(x_i)\rangle = 0, \text{ for all } S \in \mathcal{L}(X,Y)\} \tag{18}$$

As Y is assumed to have the AP, we have by condition (3) in Theorem 1.2 that for every X, $\overline{\mathcal{F}}^{\tau}(X,Y)=\mathcal{L}(X,Y)$. Thus by the bipolar and Hahn-Banach theorem (18) is equivalent to the next condition.

$$Ker(i^*)=\{z=\sum_{i=1}^{\infty} y_i^*\otimes x_i : \langle z,S\rangle=\sum_{i=1}^{\infty}\langle y_i^*,S(x_i)\rangle=0,\ \text{for all } S\in\mathcal{F}(X,Y)\} \tag{19}$$

Next, compare this condition with the condition describing the kernel of J:

$$Ker(J)=\{z=\sum_{i=1}^{\infty} y_i^*\otimes x_i : \langle T,z\rangle=\sum_{i=1}^{\infty}\langle T(y_i^*),x_i\rangle=0,\ \text{for all } T\in\mathcal{F}_{w^*}(Y^*,X^*)\} \tag{20}$$

We claim that (19) and (20) are equivalent conditions. Indeed, it suffices to note that taking the adjoints $S\to S^*$ makes an isometry from $\mathcal{F}(X,Y)$ onto $\mathcal{F}_{w^*}(Y^*,X^*)$, and thus a reformulation of (19)

$$Ker(i^*)=\{z=\sum_{i=1}^{\infty} y_i^*\otimes x_i : \langle z,S\rangle=\sum_{i=1}^{\infty}\langle S^*(y_i^*),x_i\rangle=0,\ \text{for all } S\in\mathcal{F}(X,Y)\} \tag{21}$$

is precisely (20). Since i^* is assumed to be injective, so is J. It is clear by the Banach open mapping theorem that J is an isometry. This proves that Y^* has indeed the AP. The opposite implication follows from Theorem 2.4. □

There exist Banach spaces with the AP whose dual fails the AP. The construction of such spaces relies of course on the fundamental result of Enflo [Enf73], and is shown e.g. in [LT77] (Thm. 1.e.7.) (using the method of [J60] and [L71]). Alternatively, one can use the space constructed in [FJ73].

Therefore we obtain a negative solution to the problem of Defant and Floret. Using the information from the proof of Theorem 2.5, we get the next corollary.

Corollary 2.6. *Let Y be a Banach space with the AP, whose dual Y^* fails the AP. Then $i^*:Y^*\otimes_\pi Y^{**}\to(\mathcal{L}(Y^{**},Y),\tau)^*$ is not injective.*

3. Duality of the injective tensor product $X\otimes_\varepsilon Y$

In this section we are going to investigate the Banach space dual to the injective tensor product space $X\otimes_\varepsilon Y$. A fundamental result is the following.

Theorem 3.1. *(Grothendieck) Let X,Y be Banach spaces. There is an isometry*

$$(X\otimes_\varepsilon Y)^*=\mathcal{I}(X,Y^*). \tag{22}$$

The notion of integral operators is rather abstract. We are going to investigate two special cases of the above theorem, namely the case when Y is an Asplund space (equivalently Y^* has the RNP) and the more general case when $\ell_1\not\hookrightarrow Y$. Our approach is to use the theory of vector integration (in the sense of Bochner, resp. Pettis) to obtain new proofs and new results giving a more concrete description of $\mathcal{I}(X,Y^*)$. We refer to [DU77] and [Tal84] for definitions and background on Bochner and Pettis integrals. The dual balls B_{X^*},B_{Y^*} are assumed to be equipped with the w^*-topology, unless specified otherwise. We will rely on the following results.

Theorem 3.2. *(Schwartz,* [Bou83] *Cor. 7.8.7)*
Let X be an Asplund space (equivalently, X^* be a RNP space). Then for every w^*-Radon measure μ on B_{X^*}, $Id:B_{X^*}\to B_{X^*}$ is μ-Bochner integrable.

The following result follows from [Tal84], Corollary 7-3-8.

Theorem 3.3. *(Talagrand) Assume that $\ell_1 \not\hookrightarrow X$. Then for every w^*-Radon measure μ on B_{X^*}, $Id : B_{X^*} \to B_{X^*}$ is Pettis integrable.*

Corollary 3.4. *Let Y be an Asplund space, resp. $\ell_1 \not\hookrightarrow Y$, and X be an arbitrary Banach space. Let μ be a Radon measure on $(B_{X^*}, w^*) \times (B_{Y^*}, w^*)$. Then $I : B_{X^*} \times B_{Y^*} \to Y^*$, $I(x^*, y^*) = y^*$ is μ-Bochner integrable, resp. μ-Pettis integrable.*

Proof. Let $P : X^* \times Y^* \to Y^*$ be the (w^*-continuous) projection. Denote by $\eta = P\mu$ the image measure. Since $Id : B_{Y^*} \to B_{Y^*}$ is η-Bochner integrable by Theorem , there exist simple functions $f_n : Y^* \to Y^*$ such that $\lim_{n\to\infty} \int_{B^*} \|f_n - Id\| d\eta = 0$. Let $g_n : B_{X^*} \times B_{Y^*} \to Y^*$ be simple functions $g_n = P \circ f_n$. Clearly, $\lim_{n\to\infty} \int_{B^*} \|g_n - I\| d\mu = 0$ so I is Bochner integrable.
Pettis case is similar. By the same argument with compositions, we see that I is Dunford integrable. Now given any w^*-Borel set $E \subset B_{X^*} \times B_{Y^*}$, let $\eta = P(\mu \restriction_S)$ be a Radon measure on B_{Y^*}. The Dunford integral satisfies $\int Id\, d\eta \in Y^*$, and so $\int_E Id\mu = \int Id\, d\eta \in Y^*$. Thus I is Pettis integrable. □

We also need a fact, to be found e.g. in [DU77] Lemma VI.3.

Lemma 3.5. *Let (S, Σ, μ) be a finite positive measure space and $f : S \to X$ be Bochner integrable. For each $\varepsilon > 0$ there are sequences $\{x_n\}_{n=1}^\infty$ in X and $\{E_n\}_{n=1}^\infty$ in Σ (not necessarily disjoint), such that*

$$\sum_{n=1}^{\infty} \chi_{E_n} x_n \text{ converges to } f \quad \text{absolutely } \mu - a.e. \tag{23}$$

$$\int \|f\| d\mu - \varepsilon \le \sum_{n=1}^{\infty} \|x_n\| \mu(E_i) \le \int \|f\| d\mu + \varepsilon. \tag{24}$$

The following simple but important results ([DU77] Theorem VIII.5) will be needed.

Lemma 3.6. *There is a canonical isometric embedding*

$$j : \overline{\mathcal{F}_{w^*}}(X^*, Y) = X \otimes_\varepsilon Y \hookrightarrow C(B_{X^*} \times B_{Y^*}) \tag{25}$$

given by

$$j(S)(x^*, y^*) = \langle y^*, S(x^*) \rangle \tag{26}$$

Theorem 3.7. *(Grothendieck)*
Let $j : X \otimes_\varepsilon Y \hookrightarrow C(B_{X^*} \times B_{Y^*})$ be the isometric embedding from Lemma 3.6. Then every $\phi \in (X \otimes_\varepsilon Y)^*$ has a representation as a positive w^*-Radon measure μ on $(B_{X^*} \times B_{Y^*}, w^* \times w^*)$, so that for $z \in X \otimes_\varepsilon Y$

$$\langle \phi, z \rangle = \int\limits_{B_{X^*} \times B_{Y^*}} j(z)(x^*, y^*) d\mu = \int\limits_{B_{X^*} \times B_{Y^*}} \langle x^* \otimes y^*, z \rangle d\mu. \tag{27}$$

Moreover, $\|\phi\| = |\mu|(B_{X^*} \times B_{Y^*})$.

We now proceed with a new proof of the following classical Grothendick duality theorem.

Theorem 3.8. *(Grothendieck) Let Y^* be an RNP space. Then there is an isometry*

$$(X \otimes_\varepsilon Y)^* = \mathcal{N}(X, Y^*). \tag{28}$$

More precisely, every $\phi \in (X \otimes_\varepsilon Y)^*$, $\|\phi\| < 1$, is represented by a nuclear operator $T \in \mathcal{N}(X, Y^*)$, $T = \sum_{n=1}^\infty x_n^* \otimes y_n^*$, $\sum_{n=1}^\infty \|x_n^*\| \|y_n^*\| < 1$ so that for every $S \in \overline{\mathcal{F}_{w^*}}(X^*, Y) = X \otimes_\varepsilon Y$ we have

$$\langle T, S\rangle = \sum_{n=1}^{\infty} \langle y_n^*, S(x_n^*)\rangle \tag{29}$$

Proof. By Theorem 3.7, every $\phi \in (X \otimes_\varepsilon Y)^*$, $\|\phi\| < 1$ is represented by a positive w^*-Radon measure μ on $B_{X^*} \times B_{Y^*}$, $|\mu| < 1$. Our goal is to represent ϕ as a nuclear operator $T \in \mathcal{N}(X, Y^*)$. We are going to define T by using the commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{\quad T \quad} & Y^* \\ \big\downarrow{\scriptstyle i_1} & & \big\uparrow{\scriptstyle i_3} \\ C(B_{X^*} \times B_{Y^*}) & \xrightarrow{\quad i_2 \quad} & L_1(\mu) \end{array} \tag{30}$$

where the mappings are defined as follows:
$i_1(x)(x^*, y^*) = x^*(x)$. Clearly, $\|i_1\| = 1$.
i_2 is the formal identity mapping from $C(B_{X^*} \times B_{Y^*})$ to $L_1(\mu)$. Thus $\|i_2\| = |\mu| < 1$.
$i_3 : L_1(\mu) \to Y^*$ is defined by the formula

$$i_3(f) = \int_{B_{X^*} \times B_{Y^*}} f(x^*, y^*) y^* d\mu. \tag{31}$$

The integrated function is a product of an integrable scalar function with the mapping $(x^*, y^*) \to y^*$. Due to Corollary 3.4 the later is μ-Bochner integrable. Again, we have $\|i_3\| < 1$. Thus $T = i_3 \circ i_2 \circ i_1$ is well-defined. Next we claim that the linear operator $i_3 \circ i_2 : C(B_{X^*} \times B_{Y^*}) \to Y^*$ is nuclear. Using Lemma 3.5, for $\varepsilon > 0$ small enough, there are sequences $\{y_n\}_{n=1}^{\infty}$ in Y^* and $\{E_n\}_{n=1}^{\infty}$ of w^*-Borel subsets of $B_{X^*} \times B_{Y^*}$, so that

$$\int \|y^*\| d\mu - \varepsilon \le \sum_{n=1}^{\infty} \|y_n^*\| \mu(E_i) \le \int \|y^*\| d\mu + \varepsilon < 1. \tag{32}$$

and moreover

$$i_3 \circ i_2(f) = \int f(x^*, y^*) y^* d\mu = \int f(x^*, y^*) \sum_{n=1}^{\infty} \chi_{E_n} y_n^* d\mu = \sum_{n=1}^{\infty} (\int_{E_n} f d\mu) y_n^*. \tag{33}$$

Note that $l_n(f) = \int_{E_n} f d\mu \in C(B_{X^*} \times B_{Y^*})^*$, $\|l_n\| = \mu(E_n)$. By (32), we see that $i_3 \circ i_2 = \sum_{n=1}^{\infty} l_n \otimes y_n^*$ is a nuclear operator with $N(i_3 \circ i_2) < 1$. Therefore, putting $x_n^* = i_1^*(l_n)$, we get that $T = \sum_{n=1}^{\infty} x_n^* \otimes y_n^*$ is a nuclear operator of norm less than one. Equation (33) yields

$$T(x) = \int x(x^*, y^*) y^* d\mu = \int x^*(x) y^* d\mu = \sum_{n=1}^{\infty} x_n^*(x) y_n^*. \tag{34}$$

Given $z = \sum_{i=1}^{k} u_i \otimes v_i \in X \otimes_\varepsilon Y$, by (27) and (34)

$$\langle \phi, z\rangle = \int_{B_{X^*} \times B_{Y^*}} \sum_{i=1}^{k} y^*(v_i) x^*(u_i) d\mu = \sum_{i=1}^{k} \langle \int_{B_{X^*} \times B_{Y^*}} x^*(u_i) y^* d\mu, v_i\rangle = \tag{35}$$

$$= \sum_{i=1}^{k} \langle T(u_i), v_i\rangle = \sum_{n=1}^{\infty} \sum_{i=1}^{k} x_n^*(u_i) y_n^*(v_i) = \sum_{n=1}^{\infty} \langle y_n^*, \sum_{i=1}^{k} (u_i \otimes v_i)(x_n^*)\rangle = \langle T, z\rangle, \tag{36}$$

and (29) follows.

$\square$

We proceed by giving a new proof to a slight improvement of Theorem 1 in Rosenthal [R].

Theorem 3.9. *Let Y be a Banach space. The following conditions are equivalent.*
1. $\ell_1 \not\hookrightarrow Y$
2. $\mathcal{I}(X, Y^*) \subset \overline{\mathcal{F}}(X, Y^*)$ *for every Banach space* X*.*

Proof. To prove 1. $\Rightarrow$ 2. we follow the proof of Theorem 3.8 with no changes until the diagram (30) has been obtained. Next we observe that for every f the integrand in (31) is Pettis integrable by Corollary 3.4. By Theorem 4-1-6 and Proposition 1-3-2 in [Tal84] the Dunford operator associated with I is compact. Thus $i_3 \circ i_2$ is a compact operator, so every operator in $\mathcal{I}(X, Y^*)$ is compact. By a result of K. John in [Jo90] we conclude that $\mathcal{I}(X, Y^*) \subset \overline{\mathcal{F}}(X, Y^*)$ for every Banach space X.
The opposite implication. If $\ell_1 \hookrightarrow Y$, by Pelczynski's theorem $L_1[0,1] \hookrightarrow Y^*$. Now $\iota : C[0,1] \to L_1[0,1]$ is an integral operator which is noncompact (e.g. check images of r_n Rademachers). $\square$

We remark that the duality assumption on Y^* cannot be removed. Indeed, $\iota : C[0,1] \to L_1[0,1] \to c_0$, where $f \to (\int f r_n dt)$ is a factorization witnessing that ι is an integral operator. But again it is not compact.

4. BAP in duals

In the last part of our note we give a new proof of another classical result of Grothendieck. The proof simply combines two dualities for tensor products.

Theorem 4.1. *(Grothendieck)*
Let X be a dual Banach space with the RNP. Then X has the 1*-BAP whenever X has the AP.*

Proof. Let Y be a Banach space, $X = Y^*$ be its dual with the AP, and $z \in X^* \otimes_\pi X$. By Proposition 1.1 we have $(X^* \otimes_\pi X)^* = \mathcal{L}(X^*)$, so

$$\pi(z) = \sup_{\|T\| \leq 1, T \in \mathcal{L}(X^*)} \langle T, z \rangle \geq \sup_{\|T\| \leq 1, T \in \mathcal{L}(X)} \langle T^*, z \rangle \tag{37}$$

On the other hand, by combining Theorem 3.8 and Theorem 1.2 we have $\mathcal{K}(Y)^* = X^* \otimes_\pi X$, so:

$$\pi(z) = \sup_{\|T\| \leq 1, T \in \mathcal{K}(Y)} \langle z, T \rangle \leq \sup_{\|T\| \leq 1, T \in \mathcal{K}(X)} \langle T^*, z \rangle = \sup_{\|T\| \leq 1, T \in \mathcal{F}(X)} \langle T^*, z \rangle \tag{38}$$

The last equality follows from condition 5. of Theorem 1.2, since X has the AP. Combining (37) with (38), we obtain that

$$\pi(z) = \sup_{\|T\| \leq 1, T \in \mathcal{L}(X)} \langle T^*, z \rangle = \sup_{\|T\| \leq 1, T \in \mathcal{F}(X)} \langle T^*, z \rangle \tag{39}$$

Given $z = \sum_{i=1}^{\infty} x_i^* \otimes x_i \in X^* \otimes_\pi X$ and $T \in \mathcal{L}(X)$, we have the equality

$$\langle T^*, z \rangle = \sum_{i=1}^{\infty} \langle T^*(x_i^*), x_i \rangle = \sum_{i=1}^{\infty} \langle x_i^*, T(x_i) \rangle. \tag{40}$$

By Theorem 2.3, the mapping $i^* : X^* \otimes_\pi X \to (\mathcal{L}(X), \tau)^*$ is surjective. Thus applying the Hahn-Banach theorem to the set $\{T \in \mathcal{F}(X) : \|T\| \leq 1\} \subset (\mathcal{L}(X), \tau)$, and using (39), we see that no operator $T \in \mathcal{L}(X)$, $\|T\| < 1$ can be separated by a τ-continuous hyperplane. This is clearly a reformulation of the 1-BAP. $\square$

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