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AN OPTIMAL CONDITION FOR THE UNIQUENESS OF PERIODIC SOLUTION FOR LINEAR FUNCTIONAL DIFFERENTIAL SYSTEMS

S. MUKHIGULASHVILI, I. GRYTSAY

ABSTRACT. Unimprovable effective sufficient conditions are established for the unique solvability of the periodic problem

$$u_i'(t) = \sum_{j=2}^{i+1} \ell_{i,j}(u_j)(t) + q_i(t) \qquad \text{for} \quad 1 \le i \le n-1,$$

$$u_n'(t) = \sum_{j=1}^{n} \ell_{n,j}(u_j)(t) + q_n(t),$$

$$u_j(0) = u_j(\omega) \qquad \text{for} \quad 1 \le j \le n,$$

where $\omega > 0$, $\ell_{ij} : C([0,\omega]) \to L([0,\omega])$ are the linear bounded operators, and $q_i \in L([0,\omega])$.

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1. Statement of Problem and Formulation of Main Results

Consider on $[0,\omega]$ the system

$$\begin{aligned} u_i'(t) &= \sum_{j=2}^{i+1} \ell_{i,j}(u_j)(t) + q_i(t) \qquad \text{for} \quad 1 \le i \le n-1, \\ u_n'(t) &= \sum_{j=1}^{n} \ell_{n,j}(u_j)(t) + q_n(t), \end{aligned} \tag{1.1}$$

with the periodic boundary conditions

$$u_j(0) = u_j(\omega) \qquad \text{for} \quad 1 \le j \le n, \tag{1.2}$$

where $\omega > 0$, $\ell_{i,j} : C([0,\omega]) \to L([0,\omega])$ are linear bounded operators and $q_i \in L([0,\omega])$.

By a solution of the problem (1.1), (1.2) we understand a vector function $u = (u_i)_{i=1}^n$ with $u_i \in \widetilde{C}([0,\omega])$ $(i = \overline{1,n})$ which satisfies the

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system (1.1) almost everywhere on $[0,\omega]$ and satisfies the conditions (1.2).

Much work has been carried out on the existence and uniqueness of the solution for a periodic boundary value problem for systems of ordinary differential equations and many interesting results have been obtained (see, for instance, [1–3,5,10] and the references therein). However, an analogous problem for functional differential equations, even in the case of linear equations remains less detailed investigated.

Thus, in the present paper, we study the problem (1.1) (1.2) under the assumptions that $\ell_{n,1}, \ell_{i,i+1}$ $(i=\overline{1,n-1})$ are monotone linear operators. We establish new unimprovable, integral, sufficient conditions of unique solvability of the problem (1.1),(1.2) which from one hand generalize the well-known results of A. Lasota and Z. Opial (see the Remark 1.1) obtained for ordinary differential equations in [6], and from the other hand the results obtained for linear functional differential equations in our works (see [7–9]). These results are new also if (1.1) is the following system of ordinary differential equations

$$
\begin{aligned}
u_i'(t) &= \sum_{j=2}^{i+1} p_{i,j}(t)u_j(t) + q_i(t) \qquad \text{for} \quad 1 \le i \le n-1, \\
u_n'(t) &= \sum_{j=1}^{n} p_{n,j}(t)u_j(t) + q_n(t),
\end{aligned} \tag{1.3}
$$

where $q_i, p_{i,j} \in L([0,\omega])$. The method used for the investigation of the considered problem is based on the method developed in our previous papers (see [7–9]) for functional differential equations.

The following notation is used throughout: $N(R)$ is the set of all natural (real) numbers; R^n is the space of n-dimensional column vectors $x=(x_i)_{i=1}^n$ with elements $x_i \in R$ $(i=\overline{1,n})$; $R_+=[0,+\infty[$; $C([0,\omega])$ is the Banach space of continuous functions $u:[0,\omega]\to R$ with the norm $||u||_C = \max\{|u(t)| : 0\le t\le \omega\}$; $C([0,\omega];R^n)$ is the space of continuous functions $u:[0,\omega]\to R^n$; $\widetilde{C}([0,\omega])$ is the set of absolutely continuous functions $u:[0,\omega]\to R$; $L([0,\omega])$ is the Banach space of Lebesgue integrable functions $p:[0,\omega]\to R$ with the norm $\|p\|_L = \int_0^\omega |p(s)|ds$; if $\ell : C([0,\omega]) \to L([0,\omega])$ is a linear operator, then $\|\ell\| = \sup_{0<||x||_C\le 1} \|\ell(x)\|_L$.

Definition 1.1. We will say that a linear operator $\ell : C([0,\omega]) \to L([0,\omega])$ is *nonnegative* (*nonpositive*), if for any nonnegative $x \in C([0,\omega])$ the inequality $\ell(x)(t) \ge 0$ $\;(\ell(x)(t) \le 0)$ for $0 \le t \le \omega$ is satisfied. We will say that an operator ℓ is *monotone* if it is nonnegative or nonpositive.

Definition 1.2. For the system (1.1) we define the matrix $A_1 = (a^{(1)}_{i,j})^n_{i,j=1}$ by the equalities

$$a^{(1)}_{1,1} = -1, \quad a^{(1)}_{n,1} = \frac{1}{4}||\ell_{n,1}||, \quad a^{(1)}_{i,1} = 0 \quad \text{for} \quad 2 \le i \le n-1,$$

$$\text{(1.4)} \quad a^{(1)}_{i+1,i+1} = ||\ell_{i+1,i+1}|| - 1, \quad a^{(1)}_{i,i+1} = \frac{1}{4}||\ell_{i,i+1}|| \quad \text{for} \quad 1 \le i \le n-1,$$

$$a^{(1)}_{i,j} = 0 \ \text{ for } \ i+2 \le j \le n, \quad a^{(1)}_{i,j} = ||\ell_{i,j}|| \ \text{ for } \ 3 \le j+1 \le i \le n.$$

and the matrices $A_k = (a^{(k)}_{i,j})^n_{i,j=1}, \ \ (k = \overline{2,n})$ by the recurrent relations

$$\text{(1.5)} \qquad A_2 = A_1,$$

$$\text{(1.6)} \qquad a^{(k+1)}_{i,j} = a^{(k)}_{i,j} \qquad \text{for} \quad i \le k \quad \text{or} \quad j \notin \{k, k+1\},$$

$$\text{(1.7)} \quad a^{(k+1)}_{i,j} = a^{(k)}_{i,j} + \frac{a^{(k)}_{k,j}}{|a^{(k)}_{k,k}|} a^{(k)}_{i,k} \qquad \text{for} \quad k+1 \le i \le n, \quad k \le j \le k+1.$$

Theorem 1.1. *Let $\ell_{n,1}, \ell_{i,i+1} : C([0,\omega]) \to L([0,\omega])$ $(i = \overline{1, n-1})$ be linear monotone operators,*

$$\text{(1.8)} \quad \int_0^\omega \ell_{n,1}(1)(s)ds \ne 0, \quad \int_0^\omega \ell_{i,i+1}(1)(s)ds \ne 0 \quad \textit{for} \quad 1 \le i \le n-1,$$

the matrices A_k be defined by the relations (1.4)-(1.7), *and*

$$\text{(1.9)} \qquad a^{(k)}_{k,k} < 0 \quad \textit{for} \quad 2 \le k \le n.$$

Let moreover

$$\text{(1.10)} \qquad \int_0^\omega |\ell_{n,1}(1)(s)|ds \prod_{j=1}^{n-1} \int_0^\omega |\ell_{j,j+1}(1)(s)|ds < 4^n \prod_{j=2}^{n} |a^{(j)}_{j,j}|.$$

Then the problem (1.1), (1.2) *has a unique solution.*

Definition 1.3. For the system (1.3) we define the matrix $A_1 = (a^{(1)}_{i,j})^n_{i,j=1}$ by the equalities (1.4)-(1.7) with

$$\text{(1.11)} \qquad \ell_{i,j}(x)(t) = p_{i,j}(t)x(t) \quad \text{for} \quad i, j \in \overline{1,n}, \quad x \in C([0,\omega]).$$

Corollary 1.1. *Let*

$$\text{(1.12)} \quad 0 \le \sigma_n p_{n,1}(t) \not\equiv 0, \ \ 0 \le \sigma_i p_{i,i+1}(t) \not\equiv 0 \qquad \textit{for} \quad 1 \le i \le n-1$$

where $\sigma_i \in \{-1, 1\}$ $(i = \overline{1,n})$, the matrices A_k be defined by the relations (1.5)-(1.7), (1.11) *and*

$$\text{(1.13)} \qquad a^{(k)}_{k,k} < 0 \qquad \textit{for} \quad 2 \le k \le n.$$

Let moreover

$$\int_0^\omega |p_{n,1}(s)|ds \prod_{j=1}^{n-1} \int_0^\omega |p_{j,j+1}(s)|ds < 4^n \prod_{j=2}^{n} |a_{j,j}^{(j)}|. \tag{1.14}$$

Then the problem (1.3), (1.2) *has a unique solution.*

Now, assume that

$$\begin{gathered} \ell_{1,j} \equiv 0 \ \text{ for } \ j \neq 2, \quad \ell_{i,j} \equiv 0 \ \text{ for } \ j \notin \{i, i+1\}, \ \ i = \overline{2, n-1}, \\ \ell_{n,j} = 0 \ \text{ for } \ j = \overline{2, n-1}. \end{gathered} \tag{1.15}$$

Then the system (1.1) is of the following type

$$\begin{aligned} &u_1'(t) = \ell_{1,2}(u_2)(t) + q_1(t), \\ &u_i'(t) = \ell_{i,i}(u_i)(t) + \ell_{i,i+1}(u_{i+1})(t) + q_i(t) \quad \text{for } \ 2 \le i \le n-1, \\ &u_n'(t) = \ell_{n,1}(u_1)(t) + \ell_{n,n}(u_n)(t) + q_n(t), \end{aligned} \tag{1.16}$$

and from Theorem 1.1 we obtain

Corollary 1.2. *Let $\ell_{n,1}, \ell_{i,i+1}$ ($i = \overline{1, n-1}$) be linear monotone operators, the conditions* (1.8) *hold and*

$$\int_0^\omega |\ell_{k,k}(1)(s)|ds < 1 \quad \textit{for} \quad 2 \le k \le n. \tag{1.17}$$

Let moreover

$$\begin{gathered} \int_0^\omega |\ell_{n,1}(1)(s)|ds \prod_{j=1}^{n-1} \int_0^\omega |\ell_{j,j+1}(1)(s)|ds < \\ < 4^n \prod_{j=2}^{n} \Big(1 - \int_0^\omega |\ell_{j,j}(1)(s)|ds\Big). \end{gathered} \tag{1.18}$$

Then the problem (1.16), (1.2) *has a unique solution.*

It is clear that for the cyclic feedback system

$$\begin{aligned} &u_i'(t) = \ell_i(u_{i+1})(t) + q_i(t) \qquad \text{for} \quad 1 \le i \le n-1, \\ &u_n'(t) = \ell_n(u_1)(t) + q_n(t), \end{aligned} \tag{1.19}$$

from Corollary 1.2 we get

Corollary 1.3. *Let $\ell_i : C([0, \omega]) \to L([0, \omega])$ ($i = \overline{1, n}$) be the linear monotone operators,*

$$||\ell_i|| \neq 0 \quad \textit{for} \quad i = \overline{1, n}, \tag{1.20}$$

and

$$\prod_{i=1}^{n} ||\ell_i|| < 4^n. \tag{1.21}$$

Then the problem (1.19), (1.2) *has a unique solution.*

Remark 1.1. It is clear that the problem

$$u''(t) = p(t)u(t) + q(t), \;\; u(0) = u(\omega), \; u'(0) = u'(\omega), \tag{1.22}$$

is equivalent to the problem (1.19), (1.2) with $n = 2$, $\ell_1(x)(t) = x(t)$, $\ell_2(x)(t) = p(t)x(t)$, $q_1(t) \equiv 0$ and $q_2(t) = q(t)$.

Then if $p, q \in L([0, \omega])$, $p(t) \le 0$ and $\int_0^\omega p(s)ds \ne 0$ from the corollary 1.3 it follows that the problem (1.19), (1.2) and then the problem (1.22), has a unique solution if the condition $\int_0^\omega |p(s)|ds < \frac{16}{\omega}$ is fulfilled, which follows from the well-known result of A. Lasota and Z. Opial (see [6]).

Example 1.1. The example below shows that condition (1.21) in Corollary 1.3 is optimal and it cannot be replaced by the condition

$$\prod_{i=1}^{n} ||\ell_i|| \le 4^n. \tag{1.21$_1$}$$

Define the functions $u_0 \in \widetilde{C}([0, 1])$ on $[0, 1/2]$, and $[1/2, 1]$ by the equalities

$$u_0(t) = u_0(1 - t) \quad \text{for} \quad 1/2 < t \le 1,$$

$$u_0(t) = \begin{cases} 1 & \text{for } 0 \le t \le 1/8 \\ \sin \pi(1 - 4t) & \text{for } 1/8 < t \le 3/8 \\ -1 & \text{for } 3/8 < t \le 1/2 \end{cases}.$$

Now, let a measurable functions $\tau_i : [0, 1] \to [0, 1]$, and the linear nonnegative operators $\ell_i : C([0, 1]) \to L([0, 1]), (i = \overline{1, n})$ be given by the equalities:

$$\tau_i(t) = \begin{cases} 1/8i & \text{for } 0 \le u_0'(t) \\ 1/2 - 1/8i & \text{for } 0 > u_0'(t) \end{cases}, \;\; \ell_i(x)(t) = |u_0'(t)|x(\tau_i(t)).$$

Then it is clear that $u_0(0) = u_0(1)$, $\ell_i \ne \ell_j$ if $i \ne j$, and $||\ell_i|| = \int_0^1 |\ell_i(1)(s)|ds = 16\pi \int_{1/8}^{1/4} \cos \pi(1 - 4s)ds = 4$ for $i = \overline{1, n}$. Thus, all the assumptions of Corollary 1.3 are satisfied except (1.21), instead of which the condition (1.21$_1$) is fulfilled with $\omega = 1$. On the other hand, from the relations $u_0'(t) = |u_0'(t)|u_0(\tau_i(t)) = \ell_i(u_0)(t)$ $(i = \overline{1, n})$, it follows that the vector function $(u_i(t))_{i=1}^n$ if $u_i(t) \equiv u_0(t)$ $(i = \overline{1, n})$ is the nontrivial solution of the problem (1.1), (1.2) with $\omega = 1$, $q(t) \equiv 0$, which contradicts the conclusion of Corollary 1.3.

2. Auxiliary Propositions

Lemma 2.1. *Let the matrices A_k $(k = \overline{1, n})$ be defined by the equalities (1.4)-(1.7). Then the following relations*

$$a_{i,j}^{(m)} \ge 0 \qquad \textit{for} \quad i \ne j, \;\; m = \overline{1, n}, \tag{2.1$_m$}$$

$$a_{n,1}^{(1)} = a_{n,1}^{(n)} \tag{2.2$_0$}$$

$$(2.2_m) \qquad a_{i,j}^{(\lambda)} \le a_{i,j}^{(m)} \qquad \textit{for} \quad i \ge m \ge 2, \;\; j \ge m, \;\; \lambda \le m,$$

hold.

Proof. From the definition of A_1, A_2 immediately follows that the inequalities (2.1_1) and (2.2_2) are true. Now, we assume that (2.1_m) holds for $m = 3, .., m_0$ $(m_0 < n)$ and prove (2.1_{m_0+1}). If $i \le m_0$ or $j \notin \{m_0, \; m_0+1\}$, from (1.6) there immediately follows the inequality (2.1_{m_0+1}), and if $i \ge m_0+1, \;\; j \in \{m_0, \; m_0+1\}$, then (2.1_{m_0+1}) follows from (1.7).

Now we prove the inequality (2.2_m). First assume that $j \ge m+1$. Then from (1.6) it is clear that

$$(2.3) \quad a_{i,j}^{(\lambda)} = a_{i,j}^{(\lambda+1)} = ... = a_{i,j}^{(m)} \qquad \text{for} \quad j \ge m+1, \; i \ge m, \; \lambda \le m.$$

Now, let $j = m$. Then from (1.6) we get $a_{i,m}^{(\lambda)} = a_{i,m}^{(\lambda+1)} = ... = a_{i,m}^{(m-1)}$ for $i \ge m, \; \lambda \le m$. By the last equalities and (2.1_m), from (1.7) it follows

$$a_{i,m}^{(m)} = a_{i,m}^{(m-1)} + \frac{a_{m-1,m}^{(m-1)}}{|a_{m-1,m-1}^{(m-1)}|} a_{i,m-1}^{(m-1)} \ge a_{i,m}^{(m-1)} = a_{i,m}^{(\lambda)} \quad \text{for} \quad i \ge m, \; \lambda \le m,$$

From this inequality and (2.3) we conclude that (2.2_m) is fulfilled for all $j \ge m$ and $i \ge m$. The equality (2.2_0) immediately follows from (1.5) and (1.6). □

Also we need the following simple lemma proved in the paper [10].

Lemma 2.2. *Let $\sigma \in \{-1, 1\}$ and $\sigma\ell : C([0,\omega]) \to L([0,\omega])$ be a nonnegative linear operator. Then*

$$-m|\ell(1)(t)| \le \sigma\ell(x)(t) \le M|\ell(1)(t)| \quad \textit{for} \quad 0 \le t \le \omega, \quad x \in C([0,\omega]),$$

where $m = -\min_{0\le t\le\omega}\{x(t)\}, \; M = \max_{0\le t\le\omega}\{x(t)\}.$

Now, consider on $[0,\omega]$ the homogeneous problem

$$(2.4_i) \qquad v_i'(t) = \sum_{j=2}^{i+1} \ell_{i,j}(v_j)(t) \qquad \text{for} \quad 1 \le i \le n,$$

$$(2.5) \qquad v_j(0) = v_j(\omega) \qquad \text{for} \quad 1 \le j \le n,$$

where the operator $\ell_{n,n+1}$ and function v_{n+1} are defined by the equalities $\ell_{n,n+1} \equiv \ell_{n,1}$ and $v_{n+1} \equiv v_1$. Also define the functional $\Delta_i : C([0,\omega]; R^n) \to R_+$ by the equality $\Delta_i(v) = \max_{0\le t\le\omega}\{v_i(t)\} - \min_{0\le t\le\omega}\{v_i(t)\}$ $(i = \overline{1,n})$ for any vector function $v = (v_i)_{i=1}^n$ and let $\Delta_{n+1} \equiv \Delta_1$.

Lemma 2.3. *Let $\ell_{i,i+1} : C([0,\omega]) \to L([0,\omega])$ $(i = \overline{1,n})$ be linear monotone operators,*

$$(2.6) \qquad \int_0^\omega \ell_{i,i+1}(1)(s)ds \ne 0 \qquad \textit{for} \quad 1 \le i \le n,$$

the matrices A_k be defined by the equalities (1.4)-(1.7) *and*

$$a^{(k)}_{k,k} < 0 \qquad \text{for} \quad 2 \le k \le n. \tag{2.7}$$

Let, moreover $v = (v_i)^n_{i=1}$ be a nontrivial solution of the problem $((2.4_i))^n_{i=1}$, (2.5) *for which such $k_1 \in \{2, ..., n\}$ exists that $v_{k_1} \not\equiv 0$. Then if*

$$k_0 = \min\{k \in \{2, ..., n\} : v_k \not\equiv 0\}, \tag{2.8}$$

the inequalities

$$0 < ||v_k||_C \le \Delta_k(v) \qquad \text{for} \quad k = 1, \;\; k_0 \le k \le n, \tag{2.9_k}$$

$$0 \le a^{(k)}_{k,k}\Delta_k(v) + a^{(k)}_{k,k+1}\Delta_{k+1}(v) \qquad \text{for} \quad k_0 \le k \le n, \tag{2.10_k}$$

with $a^{(1)}_{n,n+1} = a^{(1)}_{n,1}$, hold.

Proof. Define the numbers $M_k, m_k \in R$, $t'_k, t''_k \in [0, \omega]$ by the relations

$$M_k = v_k(t'_k) = \max_{0\le t\le\omega}\{v_k(t)\}, \quad -m_k = v_k(t''_k) = \min_{0\le t\le\omega}\{v_k(t)\}, \tag{2.11_k}$$

and if $t'_k < t''_k$, the sets $I^{(1)}_k = [t'_k,\ t''_k]$, $I^{(2)}_k = I \setminus I^{(1)}_k$. From (2.8) it is clear

$$v_{k_0} \not\equiv 0. \tag{2.12}$$

On the other hand, from (2.4_{k_0-1}) by (2.8) we obtain

$$\int_0^\omega \ell_{k_0-1,k_0}(v_{k_0})(s)ds = 0. \tag{2.13}$$

From (2.13), in view of (2.6) and Lemma 2.2 there follows the existence of such $t_0 \in [0,\ \omega]$ that $v_{k_0}(t_0) = 0$. Then from (2.12) we get (2.9_{k_0}).

Let the numbers $M_{k_0}, m_{k_0} \in R$, $t'_{k_0}, t''_{k_0} \in [0, \omega]$ be defined by the relations (2.11_{k_0}) and $t'_{k_0} < t''_{k_0}$ (the case $t''_{k_0} < t'_{k_0}$ can be proved analogously). The integration of (2.4_{k_0}) on $I^{(r)}_{k_0}$, by virtue of (2.5) and (2.8) results in

$$\Delta_{k_0}(v) = (-1)^r\Big[\int\limits_{I^{(r)}_{k_0}} \ell_{k_0,k_0}(v_{k_0})(s)ds + \int\limits_{I^{(r)}_{k_0}} \ell_{k_0,k_0+1}(v_{k_0+1})(s)ds\Big] \tag{2.14}$$

for $r = 1, 2$. From the last equality, by virtue of (1.4), (2.7), (2.9_{k_0}) and (2.2_{k_0}) with $\lambda = 1$, $i = j = k_0$ we get

$$0 < -a^{(k_0)}_{k_0,k_0}\Delta_{k_0}(v) \le (-1)^r \int_{I^{(r)}_{k_0}} \ell_{k_0,k_0+1}(v_{k_0+1})(s)ds \tag{2.15_r}$$

for $r = 1, 2$. Assume that v_{k_0+1} is a constant sign function. Then in view of the fact that the operator ℓ_{k_0,k_0+1} is monotone we get the contradiction with (2.15_1) or (2.15_2), i.e., v_{k_0+1} changes its sign. Then

$$M_{k_0+1} > 0, \quad m_{k_0+1} > 0, \tag{2.16}$$

and the inequality (2.9_{k_0+1}) holds ((2.9_1) if $k_0 = n$). If ℓ_{k_0,k_0+1} is a nonnegative operator, from (2.15_r) $(r = 1, 2)$ in view of (2.16) by Lemma

2.2 we get $0 < -a^{(k_0)}_{k_0,k_0}\Delta_{k_0}(v) \leq m_{k_0+1}\int_{I^{(1)}_{k_0}}|\ell_{k_0,k_0+1}(1)(s)|ds$, $0 < -a^{(k_0)}_{k_0,k_0}\Delta_{k_0}(v) \leq M_{k_0+1}\int_{I^{(2)}_{k_0}}|\ell_{k_0,k_0+1}(1)(s)|ds$. By multiplying these estimates and applying the numerical inequality $4AB \leq (A+B)^2$, in view of the notations (1.4) we obtain $0 \leq a^{(k_0)}_{k_0,k_0}\Delta_{k_0}(v) + \frac{1}{4}(M_{k_0+1} + m_{k_0+1})\Big(\int_{I^{(1)}_{k_0}}|\ell_{k_0,k_0+1}(1)(s)|ds + \int_{I^{(2)}_{k_0}}|\ell_{k_0,k_0+1}(1)(s)|ds\Big) = a^{(k_0)}_{k_0,k_0}\Delta_{k_0}(v) + a^{(1)}_{k_0,k_0+1}\Delta_{k_0+1}(v)$, $(0 \leq a^{(n)}_{n,n}\Delta_n(v) + a^{(1)}_{n,1}\Delta_1(v)$ if $k_0 = n)$, from which by (2.2_0) if $k_0 = n$ and (2.2_{k_0}) with $\lambda = 1,\ i = k_0,\ j = k_0+1$ if $k_0 < n$, follows (2.10_{k_0}). Analogously from (2.15_r) we get (2.10_{k_0}), in the case when the operator ℓ_{k_0,k_0+1} is non-positive.

Consequently we proved the proposition:

i. Let $2 \leq k_0 \leq n$, then the inequalities (2.9_{k_0}), (2.9_{k_0+1}) ((2.9_1) if $k_0 = n$) and (2.10_{k_0}) hold.

Now, we shall prove the following proposition:

ii. Let $k_1 \in \{k_0, ..., n-1\}$ be such that the inequalities (2.9_k),(2.10_k) for $(k = \overline{k_0, k_1})$, and (2.9_{k_1+1}) hold. Then the inequalities (2.9_{k_1+2}) if $k_1 \leq n-2$, (2.9_1) if $k_1 = n-1$ and (2.10_{k_1+1}) hold too.

Define the numbers $M_{k_1+1}, m_{k_1+1} \in R$, $t'_{k_1+1}, t''_{k_1+1} \in [0,\omega]$ by the relations (2.11_{k_1+1}) and let $t'_{k_1+1} < t''_{k_1+1}$ (the case $t''_{k_1+1} < t'_{k_1+1}$ can be proved analogously). The integration of (2.4_{k_1+1}) on $I^{(r)}_{k_1+1}$, by virtue of (2.5) and (2.8) results in

$$\Delta_{k_1+1}(v) = (-1)^r \sum_{j=k_0}^{k_1+2} \int_{I^{(r)}_{k_1+1}} \ell_{k_1+1,j}(v_j)(s)ds \tag{2.17}$$

for $r = 1,2$. From this equality, by the conditions (1.4),(2.7),(2.9_k) with $k = k_0, ..., k_1+1$, and (2.2_{k_0}) with $\lambda = 1,\ i = k_1+1,\ j = k_0, ..., k_1+1$ we get

$$0 \leq \sum_{j=k_0}^{k_1+1} a^{(k_0)}_{k_1+1,j}\Delta_j(v) + (-1)^r \int_{I^{(r)}_{k_1+1}} \ell_{k_1+1,k_1+2}(v_{k_1+2})(s)ds \tag{2.18}$$

for $r = 1,2$. By multiplying (2.10_k) with $a^{(k)}_{k_1+1,k}/|a^{(k)}_{k,k}|$ for $k \in \{k_0, ..., k_1\}$ in view of the inequalities (2.7) we obtain

$$0 \leq -a^{(k)}_{k_1+1,k}\Delta_k(v) + \frac{a^{(k)}_{k,k+1}}{|a^{(k)}_{k,k}|}a^{(k)}_{k_1+1,k}\Delta_{k+1}(v). \tag{2.19_k}$$

Now, summing (2.18) and (2.19_{k_0}) by virtue of (1.7) with $k = k_0,\ i = k_1+1,\ j = k_0+1$, we get

$$0 \leq a^{(k_0+1)}_{k_1+1,k_0+1}\Delta_{k_0+1}(v) + \sum_{j=k_0+2}^{k_1+1} a^{(k_0)}_{k_1+1,j}\Delta_j(v) +$$

$$+(-1)^r \int_{I^{(r)}_{k_1+1}} \ell_{k_1+1,k_1+2}(v_{k_1+2})(s)ds,$$

from which by (2.2_{k_0+1}) with $i = k_1 + 1,\ j \geq k_0 + 2,\ \lambda = k_0$, we obtain

$$0 \leq \sum_{j=k_0+1}^{k_1+1} a^{(k_0+1)}_{k_1+1,j} \Delta_j(v) + (-1)^r \int_{I^{(r)}_{k_1+1}} \ell_{k_1+1,k_1+2}(v_{k_1+2})(s)ds \tag{2.20}$$

for $r = 1, 2$. Analogously, by summing (2.20) and the inequalities (2.19_k) for all $k = k_0 + 1, ..., k_1$ we get

$$0 < -a^{(k_1+1)}_{k_1+1,k_1+1} \Delta_{k_1+1}(v) \leq (-1)^r \int_{I^{(r)}_{k_1+1}} \ell_{k_1+1,k_1+2}(v_{k_1+2})(s)ds \tag{2.21}$$

for $r = 1, 2$. By the same way as the inequality (2.9_{k_0+1}) and (2.10_{k_0}) follow from (2.15_r), the inequalities (2.9_{k_1+2}) ((2.9_1) if $k_0 = n - 1$) and (2.10_{k_1+1}) follow from (2.21).

From the propositions i. and ii. by the the method of mathematical induction we obtain that the inequalities (2.9_1), (2.9_k) and (2.10_k) $(k = \overline{k_0, n})$ hold. $\square$

3. Proofs

Proof of Theorem 1.1. It is known from the general theory of boundary value problems for functional differential equations that if $\ell_{i,j}$ $(i, j = \overline{1, n})$ are strongly bounded linear operators, then the problem (1.1), (1.2) has the Fredholm property (see [4]). Thus, the problem (1.1), (1.2) is uniquely solvable iff the homogeneous problem $(2.4_i)_{i=1}^n$, (2.5) has only the trivial solution.

Assume that, on the contrary, the problem $(2.4_i)_{i=1}^n$,(2.5) has a non-trivial solution $v = (v_i)_{i=1}^n$. Let

$$v_1 \not\equiv 0, \quad v_i \equiv 0 \quad \text{for} \quad 2 \leq i \leq n. \tag{3.1}$$

Thus from (2.4_1) and (2.4_n) it follows that $v_1'(t) \equiv 0$ and $\ell_{n,1}(v_1)(t) \equiv 0$, i.e., in view of the fact that the operator $\ell_{n,1}$ satisfies (1.8) we obtain that $v_1 \equiv 0$, which contradicts (3.1). Consequently such $k_0 \in \{2, ..., n\}$ exists that $v_{k_0} \not\equiv 0$. Then all the conditions of Lemma 2.3 are satisfied, from which it follows that $0 < ||v_1||_C \leq \Delta_1(v)$, i.e., $v_1 \not\equiv Const$ and in view of the condition (2.5) the function v_1' changes its sign. Thus from (2.4_1) by the monotonicity of the operator $\ell_{1,2}$, we get that v_2 changes its sign too. Consequently if M_2, m_2 are the numbers defined by the equalities (2.11_2) then

$$M_2 > 0, \ \ m_2 > 0, \tag{3.2}$$

and if k_0 is the number defined by the equality (2.8), then $k_0 = 2$. Thus from Lemma 2.3 it follows that the inequalities (2.9_1), (2.9_k) and (2.10_k) $(k = \overline{2, n})$ hold.

Now, assume that the numbers M_1, m_1, and $t_1', t_1'' \in [0,\omega[$ are defined by the equalities (2.11_1) and $t_1' < t_1''$ (the case $t_1'' < t_1'$ can be proved analogously). By integration of (2.4_1) on the set $I_1^{(r)}$ we obtain

$$\Delta_1(v) = (-1)^r \int_{I_1^{(r)}} \ell_{1,2}(v_2)(s)ds \tag{3.3}$$

for $r = 1, 2$. First assume that the operator $\ell_{1,2}$ is non-negative (the case of non-positive $\ell_{1,2}$ can be proved analogously), then from (3.3) by (2.9_1), (3.2) and the Lemma 2.2 we obtain

$$0 < \Delta_1(v) \le m_2 \int_{I_1^{(1)}} |\ell_{1,2}(1)(s)|ds, \;\; 0 < \Delta_1(v) \le M_2 \int_{I_1^{(2)}} |\ell_{1,2}(1)(s)|ds.$$

By multiplying these estimates and applying the numerical equality $4AB \le (A+B)^2$ and the equalities (1.4) we get $0 \le a_{1,1}^{(1)}\Delta_1(v) + \frac{1}{4}(m_2 + M_2)\Big(\int_{I_1^{(1)}} |\ell_{1,2}(1)(s)|ds + \int_{I_1^{(2)}} |\ell_{1,2}(1)(s)|ds\Big) = a_{1,1}^{(1)}\Delta_1(v) + a_{1,2}^{(1)}\Delta_2(v)$, i.e., all the inequalities (2.10_k) $(k = \overline{1,n})$ are satisfied.

On the other hand from (1.4)–(1.6) and Lemma 2.1 it is clear that

$$a_{1,1}^{(1)} = -1, \;\; a_{n,1}^{(n)} = a_{n,1}^{(1)}, \;\; a_{k,k+1}^{(k)} = a_{k,k+1}^{(1)} = \frac{1}{4}||\ell_{k,k+1}|| \tag{3.4}$$

for $1 \le k \le n-1$. By multiplying all the estimates (2.10_k) $(k = \overline{1,n})$ and applying (3.4) we get the contradiction to the condition (1.10). Thus our assumption fails, and $v_i \equiv 0$ $(i = \overline{1,n})$. □

Proof of Corollary 1.1. From (1.11) and (1.12) it is clear that $\ell_{n,1}$ and $\ell_{i,i+1}$ are monotone operators and (1.8) holds. Also, from (1.13) and (1.14), the conditions (1.9) and (1.10) follow. Consequently all the conditions of Theorem 1.1 for the system (1.3) are fulfilled. □

Proof of Corollary 1.2. From (1.4), (1.6), and (1.15) it is clear that

$$a_{k,k}^{(k-1)} = a_{k,k}^{(k-2)} = ... = a_{k,k}^{(1)} = ||\ell_{k,k}|| - 1 \quad \text{for} \quad 2 \le k \le n, \tag{3.5}$$

and

$$\begin{gathered} a_{k,k-i}^{(k-i-1)} = a_{k,k-i}^{(k-i-2)} = ... = a_{k,k-i}^{(1)} = 0 \quad \text{for} \quad 3 \le k-i \le n, \\ a_{2,1}^{(1)} = 0. \end{gathered} \tag{3.6}$$

From (1.7), (1.15) and the first equality of (3.6) we get

$$\begin{aligned} a_{k,k-1}^{(k-1)} &= a_{k,k-1}^{(k-2)} + \frac{a_{k-2,k-1}^{(k-2)}}{|a_{k-2,k-2}^{(k-2)}|} a_{k,k-2}^{(k-2)} = \frac{a_{k-2,k-1}^{(k-2)}}{|a_{k-2,k-2}^{(k-2)}|} a_{k,k-2}^{(k-2)} = \\ &= \frac{a_{k-2,k-1}^{(k-2)}}{|a_{k-2,k-2}^{(k-2)}|} \frac{a_{k-3,k-2}^{(k-3)}}{|a_{k-3,k-3}^{(k-3)}|} a_{k,k-3}^{(k-3)} = ... = a_{k,2}^{(2)} \prod_{j=2}^{k-2} \frac{a_{j,j+1}^{(j)}}{|a_{j,j}^{(j)}|} = 0 \end{aligned} \tag{3.7}$$

for $k \ge 3$. From (3.7) and the second equality of (3.6) it is clear that

$$a_{k,k-1}^{(k-1)} = 0 \qquad \text{for} \qquad 2 \le k \le n \tag{3.8}$$

Then from (1.7) by (3.5) and (3.8) we obtain

$$a_{k,k}^{(k)} = a_{k,k}^{(k-1)} + a_{k-1,k}^{(k-1)} a_{k,k-1}^{(k-1)} / |a_{k-1,k-1}^{(k-1)}| = ||\ell_{k,k}|| - 1.$$

Thus from the conditions (1.17) and (1.18) it follows that (1.9) and (1.10) hold. Consequently all the conditions of Theorem 1.1 for the system (1.16) are fulfilled. □

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References

[1] F. Bates , Y. Ward, *Periodic solutions of higer order systems,* pacific J. Math. **84** (1979), 275–282.

[2] A. Capietto, D. Qian, F. Zanolin, *Periodic solutions for differential systems of cyclic feedback type,* Differential Equations Dyn. Systems **7**(1)(1999), 99–120.

[3] I. Kiguradze, B. Půža, *On some boundary value problems for systems of ordinary differential equations,* Differentsial'nye Uravnenija **12** (1996), 2139–2148 (in Russian).

[4] I. Kiguradze, B. Půža, *On boundary value problems for systems of linear functional differential equations,* Czechoslovak Math. J. **47** (1997), 341–373.

[5] I. Kiguradze, S. Mukhigulashvili, *On periodic solutions of two-dimensional nonautonomous differential systems,* Nonlinear Analysis **60** A(2005), 241–256.

[6] A. Lasota, Z. Opial, *Sur les solutions periodiques des equations differentielles ordinaires,* Ann. Polon. Math. **16** (1964), 69–94.

[7] A. Lomtatidze, S. Mukhigulashvili, *On periodic solutions of second order functional differential equations,* Mem. Differential Equations Math. Phys. **5** (1995), 125–126.

[8] S. Mukhigulashvili, *On a periodic solutions of second order functional differential equations,* Italian J. of Pure and Appl. Math. (2006), No 20, 29-50.

[9] S. Mukhigulashvili, *On a periodic boundary value problem for cyclic feedback type linear functional differential systems,* Arch. Math.**87** (2006), 255-260.

[10] A. Ronto and M. Ronto, *On certain symmetry properties of periodic solutions, nonlinear oscillations* **6** (2003), No.1, 82-107.

Authors' addresses:

Sulkhan Mukhigulashvili

1. Mathematical Institute, Academy of Sciences of the Czech Republic, Žižkova 22, 616 62 Brno, Czech Republic.

2. I. Chavchavadze State University, Faculty of physics and mathematics, I. Chavchavadze Str. No.32, 0179 Tbilisi, Georgia.

E-mail: `mukhig@ipm.cz`

Iryna Grytsay

1. Taras Shevchenko National University of Kyiv, Faculty of Cybernetics, Department of Mathematical Analysis, Vladimirskaya Street, 64, 01033, Kyiv, Ukraine

E-mail: `grytsay@mail.univ.kiev.ua`