ON THE OSCILLATION OF SOLUTIONS OF THIRD ORDER LINEAR DIFFERENCE EQUATIONS OF NEUTRAL TYPE

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Abstract. In this note we consider the third order linear difference equations of neutral type

$$\Delta^3[x(n) - p(n)x(\sigma(n))] + \delta q(n)x(\tau(n)) = 0, \quad n \in N(n_0), \tag{E}$$

where $\delta = \pm 1$, $p, q\colon N(n_0) \to \mathbb{R}_+$; $\sigma, \tau\colon N(n_0) \to \mathbb{N}$, $\lim\limits_{n\to\infty} \sigma(n) = \lim\limits_{n\to\infty} \tau(n) = \infty$. We examine the following two cases:

$$\{0 < p(n) \leqslant 1,\ \sigma(n) = n + k,\ \tau(n) = n + l\},$$
$$\{p(n) > 1,\ \sigma(n) = n - k,\ \tau(n) = n - l\},$$

where k, l are positive integers and we obtain sufficient conditions under which all solutions of the above equations are oscillatory.

Keywords: neutral type difference equation, nonoscillatory solution, asymptotic behavior

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1. Introduction

Consider the third order neutral difference equations

$$\Delta^3[x(n) - p(n)x(\sigma(n))] + \delta q(n)x(\tau(n)) = 0, \quad n \in N(n_0), \tag{E}$$

where $\delta = \pm 1$, $N(n_0) = \{n_0, n_0 + 1, \ldots\}$, n_0 is fixed in $\mathbb{N} = \{1, 2, \ldots\}$ and Δ is the forward difference operator defined by $\Delta x(n) = x(n+1) - x(n)$, $\Delta^{i+1}x(n) = \Delta(\Delta^i x(n))$ for $i = 1, 2, \ldots$, $\Delta^0 x(n) = x(n)$. For $k \in \mathbb{N}$ we use the usual factorial notation

$$n^{\underline{k}} = n(n-1)\ldots(n-k+1) \quad \text{with } n^{\underline{0}} = 1.$$

The following hypotheses are always assumed to hold:

(H1) $p, q\colon N(n_0) \longrightarrow \mathbb{R}_+$;

(H2) $\sigma\colon N(n_0) \longrightarrow \mathbb{N}$, σ is strictly increasing and $\sigma(N(n_0)) = N(n_*)$ for some $n_* \in \mathbb{N}$.

(H3) $\tau\colon N(n_0) \longrightarrow \mathbb{N}$, $\lim\limits_{n \longrightarrow \infty} \tau(n) = \infty$.

By a solution of equation (E) we mean a real sequence which is defined for $n \geqslant \min\limits_{i \geqslant n_0}\{\tau(i), \sigma(i)\}$ and which satisfies equation (E) for all $n \geqslant n_0$. We consider only such solutions which are nontrival for all large n. As usual a solution x of equation (E) is called oscillatory if for any $M \geqslant n_0$ there exists $n \geqslant M$ such that $x_n x_{n+1} \leqslant 0$. Otherwise it is called nonoscillatory.

In recent years there has been increasing interest in the study of the oscillation of neutral difference equations. For example, the first order linear difference equation of neutral type

$$\Delta(y_n + p_n y_{n-k}) + q_n y_{n-l} = 0, \quad n = 0, 1, 2, \ldots,$$

and its special cases, have been investigated in [5], [9]–[11] and the nonlinear case has been considered in [6], [12], [14], [16], [17], [19], see also the monographs of Agarwal [1] and Agarwal, Grace and O'Regan [2]. Compared to the study of first order neutral type difference equations, the study of higher order equations, and in particular third order neutral difference equations, has received considerably less attention (see, for example [7], [13], [15], [20], and the references contained therein). The purpose of this paper is to obtain sufficient conditions for oscillation of all solutions of equations (E). The results in this paper have been motivated by results in [3], [4]. Observe that a similar problem has been investigated for a third order differential equation in [8], and in [15] for the third order difference equation

$$\Delta(c_n \Delta(d_n \Delta(y_n + p_n y_{n-k}))) + q_n f(y_{n-m}) = e_n$$

where $0 \leqslant p_n < 1, q_n \geqslant 0$.

2. Some basic lemmas

To prove our results we need the following lemmas which can be found in [12].

Lemma 1. *Suppose that the conditions* (H1), (H2) *and*

$$0 < p(n) \leqslant 1 \quad \text{for } n \geqslant n_0$$

hold. Let x be a nonoscillatory solution of the inequality

$$x(n)[x(n) - p(n)x(\sigma(n))] < 0$$

defined in a neighbourhood of the infinity.

(i) *Suppose that $\sigma(n) < n$, for $n \geqslant n_0$. Then x is bounded. If, moreover,*

$$(1) \qquad 0 < p(n) \leqslant \lambda < 1 \qquad \text{for } n \geqslant n_0$$

for some positive constant λ, then $\lim_{n \longrightarrow \infty} x(n) = 0$.

(ii) *Suppose that $\sigma(n) > n$ for $n \geqslant n_0$. Then x is bounded away from zero. If, moreover,* (1) *holds, then $\lim_{n \longrightarrow \infty} |x(n)| = \infty$.*

Lemma 2. *Suppose that conditions* (H1), (H2) *and*

$$p(n) \geqslant 1 \text{ for } n \geqslant n_0$$

hold. Let x be a nonoscillatory solution of the inequality

$$x(n)[x(n) - p(n)x(\sigma(n)] > 0$$

defined in a neighbourhood of the infinity.

(i) *Suppose that $\sigma(n) > n$ for $n \geqslant n_0$. Then x is bounded. If, moreover,*

$$(2) \qquad 1 < v \leqslant p(n) \text{ for } n \geqslant n_0,$$

for some positive constant v, then $\lim_{n \longrightarrow \infty} x(n) = 0$.

(ii) *Suppose that $\sigma(n) < n$ for $n \geqslant n_0$. Then x is bounded away from zero. If, moreover* (2) *holds, then $\lim_{n \longrightarrow \infty} |x(n)| = \infty$.*

The next lemma can be found in [1], [14].

Lemma 3. *Assume g is a positive real sequence and m is a positive integer. If*

$$\liminf_{n \to \infty} \sum_{i=n}^{n+m-1} g(i) > \left(\frac{m}{m+1}\right)^{m+1},$$

then

(i) *the difference inequality*

$$\Delta u(n) - g(n)u(n+m) \geqslant 0$$

has no eventually positive solution,

(ii) *the difference inequality*

$$\Delta u(n) - g(n)u(n+m) \leqslant 0$$

has no eventually negative solution.

3. Main results

In this section we establish oscillation theorems for equations (E). We begin by classifying all possible nonoscillatory solutions of equations (E) on the basis of a well known lemma of Kiguradze [18] (also see [1, Theorem 1.8.11]).

Lemma 4. *Let y be a sequence of real numbers and let $y(n)$ and $\Delta^m y(n)$ be of constant sign with $\Delta^m y(n)$ not eventually identically zero. If*

$$\delta y(n)\Delta^m y(n) < 0, \tag{3}$$

then there exist integers $l \in \{0, 1, \ldots, m\}$ and $N > 0$ such that $(-1)^{m+l-1}\delta = 1$ and

$$\begin{aligned} y(n)\Delta^j y(n) &> 0 \quad \text{for } j = 0, 1, \ldots, l, \\ (-1)^{j-l} y(n)\Delta^j y(n) &> 0 \quad \text{for } j = l+1, \ldots, m, \end{aligned} \tag{4}$$

for $n \geqslant N$.

A sequence y satisfying (4) is called a Kiguradze sequence of degree l.

Let x be a nonoscillatory solution of equation (E) and let

$$u(n) = x(n) - p(n)x(\sigma(n)), \quad n \in N(n_0). \tag{5}$$

It is clear that u is eventually of one sign, so that either

$$x(n)[x(n) - p(n)x(\sigma(n))] > 0 \tag{6}$$

or

$$x(n)[x(n) - p(n)x(\sigma(n))] < 0 \tag{7}$$

for all sufficiently large n.

Let $\mathcal{N}_l^+$ [or $\mathcal{N}_l^-$] denote the set of solutions x of equation (E) satisfying (6) [or (7)] for which $u(n) = x(n) - p(n)x(\sigma(n))$ is of degree l. Then we have the following classification of the set $\mathcal{N}$ of all nonoscillatory solutions of equation (E):

$$\begin{aligned} \mathcal{N} &= \mathcal{N}_1^+ \cup \mathcal{N}_3^+ \cup \mathcal{N}_0^- \cup \mathcal{N}_2^- \quad \text{for } \delta = -1;\\ \mathcal{N} &= \mathcal{N}_0^+ \cup \mathcal{N}_2^+ \cup \mathcal{N}_1^- \cup \mathcal{N}_3^- \quad \text{for } \delta = 1. \end{aligned} \tag{8}$$

In addition to the hypothesis (H1)–(H3), we assume that p, σ and τ are subject to one of the following two cases:
(I) $0 < p(n) \leqslant 1$, $\sigma(n) = n + k$, $\tau(n) = n + l$,
(II) $p(n) > 1$, $\sigma(n) = n - k$, $\tau(n) = n - l$,
where k, l are positive integers. For simplicity, equation (E) subject to the case (I) or (II) will be referred to as equation

$$\Delta^3(x(n) - p(n)x(n+k)) - q(n)x(n+l) = 0, \quad n \in N(n_0), \tag{EI}$$

or

$$\Delta^3(x(n) - p(n)x(n-k)) + q(n)x(n-l) = 0, \quad n \in N(n_0). \tag{EII}$$

Theorem 1. *Let $k + 2 \geqslant l \geqslant 3$. If*

$$\limsup_{n\to\infty} \sum_{i=n-l}^{n-3} (n-i-1)^{\underline{2}} \quad q(i) > 2, \tag{9}$$

$$\sum_{i=n_0}^{\infty} iq(i) = \infty, \tag{10}$$

$$\limsup_{n\to\infty} n^{\underline{2}} \sum_{i=n+k-l+1}^{\infty} \frac{q(i)}{p(i-k+l)} > 2, \tag{11}$$

and

$$\liminf_{n\to\infty} \sum_{i=n}^{n+l-1} \sum_{j=i}^{\alpha(i)} (j-i+1)q(j) > \Big(\frac{l}{l+1}\Big)^{l+1}, \tag{12}$$

for some $\alpha\colon \mathbb{N} \to \mathbb{N}$ such that $\alpha(n) \geqslant n$, then all solutions of equation (EI) *are oscillatory.*

P r o o f. Assume, aiming at contradiction, that x is an eventually positive solution of equation (EI). Then there exists an integer $n_1 \geqslant n_0$, such that $x(n) > 0$ for all $n \geqslant n_1$. By (8), there are four cases to consider:

(A-I) $u(n) > 0$, $\Delta u(n) > 0$, $\Delta^2 u(n) > 0$, $\Delta^3 u(n) > 0$,
(B-I) $u(n) < 0$, $\Delta u(n) < 0$, $\Delta^2 u(n) < 0$, $\Delta^3 u(n) > 0$,
(C-I) $u(n) < 0$, $\Delta u(n) > 0$, $\Delta^2 u(n) < 0$, $\Delta^3 u(n) > 0$,
(D-I) $u(n) > 0$, $\Delta u(n) > 0$, $\Delta^2 u(n) < 0$, $\Delta^3 u(n) > 0$,
eventually.

Case (A-I). Let us take $n_2 \geqslant n_1$ so large that

$$u(n) > 0, \quad \Delta u(n) > 0, \quad \Delta^2 u(n) > 0, \quad \Delta^3 u(n) > 0, \quad \text{for } n \geqslant n_2.$$

Equation (EI) can be written in the form

$$\Delta^3 u(n) = q(n)x(n+l). \tag{13}$$

From discrete Taylor's formula (see [1, Theorem 1.8.5]), we have

$$u(n) = \sum_{i=0}^{2} \frac{(n-n_2)^{\underline{i}}}{i!} \Delta^i [u(n_2)] + \frac{1}{2} \sum_{j=n_2}^{n-3} (n-j-1)^{\underline{2}} \Delta^3 u(j), \quad n \geqslant n_2. \tag{14}$$

Therefore, we obtain

$$u(n) \geqslant \frac{1}{2} \sum_{j=n_2}^{n-3} (n-j-1)^{\underline{2}} \Delta^3 u(j)$$

and by (13), we have

$$u(n) \geqslant \frac{1}{2} \sum_{j=n_2}^{n-3} (n-j-1)^{\underline{2}} [q(j)x(j+l)].$$

By (5)

$$x(n) = u(n) + p(n)x(n+k) \geqslant u(n).$$

Therefore

$$u(n) \geqslant \frac{1}{2} \sum_{j=n_2}^{n-3} (n-j-1)^{\underline{2}} [q(j)u(j+l)] \geqslant \frac{1}{2} \sum_{j=n-l}^{n-3} (n-j-1)^{\underline{2}} [q(j)u(j+l)] \text{ for } n \geqslant n_2 + l.$$

Since u is increasing, one can see that

$$u(n) \geqslant \frac{1}{2} u(n) \sum_{j=n-l}^{n-3} (n-j-1)^{\underline{2}} q(j).$$

Dividing both sides of the above inequality by $u(n)$ we obtain

$$1 \geqslant \frac{1}{2} \sum_{j=n-l}^{n-3} (n-j-1)^{\underline{2}} q(j),$$

for all large n, which is a contradiction to (9).

Case (B-I). Let us take $n_3 \geqslant n_1$ so large that

$$u(n) < 0, \quad \Delta u(n) < 0, \quad \Delta^2 u(n) < 0, \quad \Delta^3 u(n) > 0 \quad \text{for } n \geqslant n_3.$$

Summing both sides of (13) from n to $s-1$ we obtain

$$\Delta^2 u(s) - \Delta^2 u(n) = \sum_{i=n}^{s-1} q(i)x(i+l).$$

Since $\Delta^2 u(n) < 0$, letting $s \to \infty$ we get

$$-\Delta^2 u(n) \geqslant \sum_{i=n}^{\infty} q(i)x(i+l). \tag{15}$$

Because

$$u(n-k+l) = x(n-k+l) - p(n-k+l)x(n+l),$$

we have

$$\frac{u(n-k+l)}{p(n-k+l)} = \frac{x(n-k+l)}{p(n-k+l)} - x(n+l),$$

and

$$x(n+l) \geqslant \frac{-1}{p(n-k+l)} u(n-k+l). \tag{16}$$

Substituting (16) into (15), we obtain

$$-\Delta^2 u(n) \geqslant -\sum_{i=n}^{\infty} \frac{q(i)u(i-k+l)}{p(i-k+l)}. \tag{17}$$

Now, we consider the identity

$$\sum_{i=N}^{n-1} i\Delta^3 u(i) = n\Delta^2 u(n) - N\Delta^2 u(N) - \Delta u(n+1) + \Delta u(N+1), \quad N \in N(n_0). \tag{18}$$

From (13) we have

$$\sum_{i=N}^{n-1} i\Delta^3 u(i) = \sum_{i=N}^{n-1} iq(i)x(i+l).$$

Hence, by (16), we obtain

$$\sum_{i=N}^{n-1} iq(i)x(i+l) \geqslant -\sum_{i=N}^{n-1} i\frac{q(i)}{p(i-k+l)}u(i-k+l),$$

and then

$$n\Delta^2 u(n) - N\Delta^2 u(N) - \Delta u(n+1) + \Delta u(N+1) \geqslant -\sum_{i=N}^{n-1} i\frac{q(i)}{p(i-k+l)}u(i-k+l).$$

From the above inequalities we get

$$\begin{aligned}\Delta u(n+1) - n\Delta^2 u(n) + N\Delta^2 u(N) &\leqslant \sum_{i=N}^{n-1} \frac{iq(i)}{p(i-k+l)}u(i-k+l)\\ &\leqslant \sum_{i=N}^{n-1} iq(i)u(i-k+l),\end{aligned}$$

and using (10) and letting $n \to \infty$, we have

$$\begin{aligned}\lim_{n\to\infty}\left[\Delta u(n+1) - n\Delta^2 u(n) + N\Delta^2 u(N)\right] &\leqslant \sum_{i=N}^{\infty} iq(i)u(i-k+l)\\ &\leqslant u(N-k+l)\sum_{i=N}^{\infty} iq(i) = -\infty.\end{aligned}$$

Therefore

$$\lim_{n\to\infty}\left[\Delta u(n+1) - n\Delta^2 u(n)\right] = -\infty. \tag{19}$$

Thus

$$\Delta u(n+1) \leqslant n\Delta^2 u(n) \tag{20}$$

for sufficiently large n. Since

$$\sum_{i=N}^{n-1}[\Delta u(i+1) - i\Delta^2 u(i)] = 2u(n+1) - n\Delta u(n) - 2u(N+1) + N\Delta u(N),$$

from (19), we obtain

$$\lim_{n\to\infty}[2u(n+1)-n\Delta u(n)] = \lim_{n\to\infty}\sum_{i=N}^{n-1}[\Delta u(i+1)-i\Delta^2 u(i)] = -\infty.$$

Thus

$$u(n+1) \leqslant \frac{1}{2}n\Delta u(n)$$

and, by (20) and (17), we get

$$\begin{aligned} u(n+1) &\leqslant \frac{1}{2}n^{\underline{2}}\Delta^2 u(n-1) = \frac{1}{2}n^{\underline{2}}\sum_{i=n-1}^{\infty}\frac{q(i)u(i-k+l)}{p(i-k+l)} \\ &\leqslant \frac{1}{2}n^{\underline{2}}\sum_{i=n+k-l+1}^{\infty}\frac{q(i)u(i-k+l)}{p(i-k+l)} \leqslant \frac{1}{2}n^{\underline{2}}u(n+1)\sum_{i=n+k-l+1}^{\infty}\frac{q(i)}{p(i-k+l)}. \end{aligned}$$

Hence

$$2 \geqslant n^{\underline{2}}\sum_{i=n+k-l+1}^{\infty}\frac{q(i)}{p(i-k+l)}$$

which contradicts (11).

C a s e (C-I) Let us take $n_4 \geqslant n_1$ so large that

$$u(n)<0, \quad \Delta u(n)>0, \quad \Delta^2 u(n)<0, \quad \Delta^3 u(n)>0 \quad \text{for } n \geqslant n_4.$$

From the equality (cf. [1], Problem 1.9.35 p. 43)

$$\begin{aligned} \Delta^\nu u(n) = &\sum_{i=\nu}^{m-1}(-1)^{(i-\nu)}\frac{(s-n+i-\nu-1)^{\underline{i-\nu}}}{(i-\nu)!}\Delta^i u(s) \\ &+(-1)^{(m-\nu)}\frac{1}{(m-\nu-1)!}\sum_{j=n}^{s-1}(j-n+m-\nu-1)^{\underline{m-\nu-1}}\Delta^m u(j), \end{aligned} \tag{21}$$

where $n_4 \leqslant n \leqslant s$, $0 \leqslant \nu \leqslant m-1$, with regard to equation (EI) for $\nu = 1$ and $m = 3$, we get

$$\Delta u(n) = \sum_{i=1}^{2}(-1)^{(i-1)}\frac{(s-n+i-2)^{\underline{i-1}}}{(i-1)!}\Delta^i u(s) + \sum_{j=n}^{s-1}(j-n+1)q(j)x(j+l), \tag{22}$$

for $n_4 \leqslant n \leqslant s$.

Therefore, we have

$$\Delta u(n) \geqslant \sum_{j=n}^{s-1}(j-n+1)q(j)x(j+l), \quad \text{for } n \geqslant n_4.$$

Since $u(n) < 0$, from Lemma 1(ii), it follows that x is bounded away from zero, so there exists a constant $c > 0$ such that $x(n) \geqslant c$, for $n \geqslant n_5 \geqslant n_4$.

Hence, from the above inequality we get

$$\Delta u(n) \geqslant c \sum_{j=n}^{s-1} (j-n+1)q(j) \quad \text{for } n \geqslant n_5.$$

Then

$$\Delta u(n_5) \geqslant c \sum_{i=n_5}^{s-1} (i-n_5+1)q(j).$$

Letting $s \to \infty$, we get a contradiction with (10).

Case (D-I). Let us take $n_6 \geqslant n_1$ so large that

$$u(n) > 0, \quad \Delta u(n) > 0, \quad \Delta^2 u(n) < 0, \quad \Delta^3 u(n) > 0 \quad \text{for } n \geqslant n_6.$$

From (22), we have

$$\Delta u(n) = \Delta u(s) - (s-n)\Delta^2 u(s) + \sum_{j=n}^{s-1} (j-n+1)q(j)x(j+l).$$

Since $x(n) > u(n)$ we have

$$\Delta u(n) \geqslant \sum_{j=n}^{s-1} (j-n+1)q(j)u(j+l) \text{ for all } s \geqslant n.$$

Thus

$$\Delta u(n) \geqslant \sum_{j=n}^{\alpha(n)} (j-n+1)q(j)u(j+l) \geqslant u(n+l) \sum_{j=n}^{\alpha(n)} (j-n+1)q(j)$$

for every $\alpha\colon \mathbb{N} \to \mathbb{N}$ such that $\alpha(n) \geqslant n$. Hence

$$\Delta u(n) - u(n+l) \sum_{j=n}^{\alpha(n)} (j-n+1)q(j) \geqslant 0.$$

By Lemma 3, with regard to (12) the last inequality can not have an eventually positive solution, which is a contradiction. This completes the proof. $\square$

Theorem 2. *Let $k \geqslant l+3$. If*

$$\limsup_{n\to\infty} \sum_{i=n-k+l}^{n-3} (n-i-1)^{\underline{2}} \frac{q(i)}{p(i+k-l)} > 2, \tag{23}$$

$$\sum_{i=n_0}^{\infty} i q(i) = \infty, \tag{24}$$

$$\limsup_{n\to\infty} n^{\underline{2}} \sum_{i=n+l+1}^{\infty} q(i) > 2, \tag{25}$$

and

$$\liminf_{n\to\infty} \sum_{i=n}^{n+k-l-1} \sum_{j=i}^{\alpha(i)} \frac{(j-i+1)q(j)}{p(j+k-l)} > \Big(\frac{k-l}{k-l+1}\Big)^{k-l+1} \tag{26}$$

for some $\alpha\colon \mathbb{N} \to \mathbb{N}$ such that $\alpha(n) \geqslant n$, then all solutions of equation (EII) *are oscillatory.*

Proof. Assume, aiming at contradiction, that x is an eventually positive solution of equation (EII). Then, there exists an integer $n_1 \geqslant n_0$, such that $x(n-l) > 0$ for all $n \geqslant n_1$. By (8), there are four cases to consider:

(A-II) $u(n) < 0$, $\Delta u(n) < 0$, $\Delta^2 u(n) < 0$, $\Delta^3 u(n) < 0$,
(B-II) $u(n) > 0$, $\Delta u(n) > 0$, $\Delta^2 u(n) > 0$, $\Delta^3 u(n) < 0$,
(C-II) $u(n) > 0$, $\Delta u(n) < 0$, $\Delta^2 u(n) > 0$, $\Delta^3 u(n) < 0$,
(D-II) $u(n) < 0$, $\Delta u(n) < 0$, $\Delta^2 u(n) > 0$, $\Delta^3 u(n) < 0$,

eventually.

Case (A-II). Let us take $n_2 \geqslant n_1$ so large that

$$u(n) < 0, \quad \Delta u(n) < 0, \quad \Delta^2 u(n) < 0, \quad \Delta^3 u(n) < 0 \quad \text{for } n \geqslant n_2.$$

From (14), we have

$$u(n) \leqslant \frac{1}{2} \sum_{j=n_2}^{n-3} (n-j-1)^{\underline{2}} \Delta^3 u(j),$$

and hence

$$-u(n) \geqslant \frac{1}{2} \sum_{j=n_2}^{n-3} (n-j-1)^{\underline{2}} [q(j)x(j-l)]. \tag{27}$$

From (5), for $\sigma(n) = n-k$, we obtain

$$u(n) > -p(n)x(n-k).$$

Hence

$$x(n) > -\frac{u(n+k)}{p(n+k)}$$

and

$$q(n)x(n-l) > \frac{-q(n)u(n+k-l)}{p(n+k-l)}.$$

Using the above inequality in (27), we get

$$-u(n) \geqslant -\frac{1}{2}\sum_{j=n_2}^{n-3}(n-j-1)^{\underline{2}}\ \frac{q(j)u(j+k-l)}{p(j+k-l)}.$$

Thus, for $n \geqslant n_2 + k - l$, we have

$$-u(n) \geqslant -\frac{1}{2}u(n)\sum_{j=n-k+l}^{n-3}(n-j-1)^{\underline{2}}\ \frac{q(j)}{p(j+k-l)}.$$

Therefore

$$1 \geqslant \frac{1}{2}\sum_{j=n-k+l}^{n-3}(n-j-1)^{\underline{2}}\ \frac{q(j)}{p(j+k-l)},$$

which contradicts (23).

Case (B-II). Let us take $n_3 \geqslant n_1$ so large that

$$u(n) > 0, \quad \Delta u(n) > 0, \quad \Delta^2 u(n) > 0, \quad \Delta^3 u(n) < 0 \quad \text{for } n \geqslant n_3.$$

Summing equation (EII) from n to ∞, we get

$$\Delta^2 u(n) \geqslant \sum_{i=n}^{\infty} q(i)x(i-l).$$

Since $x(n-l) \geqslant u(n-l)$, we have

$$\Delta^2 u(n) \geqslant \sum_{i=n}^{\infty} q(i)u(i-l).$$

From the identity (18) and (EII), we get

$$-\sum_{i=N}^{n-1} iq(i)x(i-l) = n\Delta^2 u(n) - N\Delta^2 u(N) - \Delta u(n+1) + \Delta u(N+1),$$

and thus

$$\sum_{i=N}^{n-1} iq(i)u(i-l) \leqslant -n\Delta^2 u(n) + N\Delta^2 u(N) + \Delta u(n+1).$$

Since u is an increasing sequence, it follows from (24) that

$$\lim_{n\to\infty} [\Delta u(n+1) - n\Delta^2 u(n)] = \infty.$$

Thus

$$\Delta u(n+1) \geqslant n\Delta^2 u(n) \quad \text{for } n \geqslant N \geqslant n_2,$$

where N is sufficienly large. Then, similarly as in the proof of Theorem 1, case (B-I), we get

$$u(n+1) \geqslant \frac{1}{2} n^{\underline{2}} u(n+1) \sum_{i=n+l+1}^{\infty} q(i).$$

Hence

$$2 \geqslant n^{\underline{2}} \sum_{i=n+l+1}^{\infty} q(i),$$

which contradicts (24).

Case (C-II). For $n \geqslant n_1$ we have

$$u(n) > 0, \quad \Delta u(n) < 0, \quad \Delta^2 u(n) > 0, \quad \Delta^3 u(n) < 0.$$

From Lemma 2(ii), it follows that x is bounded away from zero. So, there exists $c > 0$, such that $x(n) \geqslant c$ for $n \geqslant n_3 \geqslant n_1$. From equality (21), with regard to equation (EII), for $\nu = 1$ and using

$$\Delta^3 u(n) = -q(n)x(n-l),$$

we get

$$\Delta u(n) \leqslant -\sum_{j=n}^{s-1} (j-n+1)q(j)x(j-l), \tag{28}$$

for $s \geqslant n \geqslant n_3 + l = n_4$. Therefore, we have

$$\Delta u(n_4) \leqslant -c \sum_{j=n_4}^{s-1} (j-n_4+1)q(j),$$

hence

$$\frac{-\Delta u(n_4)}{c} \geqslant \sum_{j=n_4}^{s-1} (j - n_4 + 1)q(j).$$

Letting $s \to \infty$, we get a contradiction with (24).

C a s e (D-II). Let us take $n_5 \geqslant n_1$ so large that

$$u(n) < 0, \quad \Delta u(n) < 0, \quad \Delta^2 u(n) > 0, \quad \Delta^3 u(n) < 0 \quad \text{for } n \geqslant n_5.$$

From (28) and using

$$q(n)x(n-l) \geqslant \frac{-q(n)u(n+k-l)}{p(n+k-l)},$$

we get

$$\Delta u(n) \leqslant \sum_{j=n}^{s-1} (j-n+1)\frac{q(j)u(j+k-l)}{p(j+k-l)}.$$

Then

$$\Delta u(n) - \left[\sum_{j=n}^{\alpha(n)} (j-n+1)\frac{q(j)u(j+k-l)}{p(j+k-l)}\right] \leqslant 0,$$

for any $\alpha\colon \mathbb{N} \to \mathbb{N}$ such that $\alpha(n) \geqslant n$, and since u is decreasing

$$\Delta u(n) - u(n+k-l)\left[\sum_{j=n}^{\alpha(n)} (j-n+1)\frac{q(j)}{p(j+k-l)}\right] \leqslant 0.$$

By Lemma 3(ii) with regard to (26) the last inequality cannot have an eventually negative solution, which is a contradiction. This completes this proof. $\square$

R e m a r k 1. It is easy to extend the above results to nonlinear equations of the form

$$\Delta^3[x(n) - p(n)x(\sigma(n))] + \delta q(n) f(x(\tau(n))) = 0, \quad n \in N(n_0),$$

where f is a real valued function satisfying $xf(x) > 0$ for $x \neq 0$, under the condition that there exists a constant $B > 0$ such that $|f(x)| \geqslant B\,|x|$ for all x.

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