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A boundary value problem for the spherically symmetric motion of a pressureless gas with a temperature-dependent viscosity.

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SUMMARY

We consider an initial boundary value problem for the equations of spherically symmetric motion of a pressureless gas with temperature-dependent viscosity $\mu(\theta)$ and conductivity $\kappa(\theta)$. We prove that this problem admits a unique weak solution, assuming the Belov's functional relation between $\mu(\theta)$ and $\kappa(\theta)$ and we give the behaviour of the solution for large times.

Keywords: spherically symmetric motion, pressureless gas,temperature - dependent viscosity

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1 Introduction

Pressureless gas have been the object of various mathematical studies in recent years [5, 6, 4, 8, 6, 15, 7, 5]. Physically, these models (which may be considered as generalization of the popular Burgers model (see [16, 27, 17, 18])) have been introduced in astrophysics [28, 26] to describe sticky particles in interstellar madium, galaxy gases or rarefied cold plasmas. Also in some recent high-energy works [25, 24] it has been shown that classical decay of unstable higher-dimensional objects in string theories produces pressureless gas with non-zero energy density.

In the present work we are interested in the compressible case of a pressureless gas with non-constant transport coefficients (viscosity and conductivity) in spherical symmetry. If the density dependent viscosity case has been the object of a number of works in recent years (see for example [19, 23, 9] and references therein for the 1D and spherical symmetries), the temperature dependent-viscosity is much less known. After the pioneering article by C. Dafermos and L. Hsiao [6] in the incompressible case, to our knowledge, only the paper by S. Ya. Belov [2] deals with the compressible case. Our purpose in the following is to test the robustness of the model in [2] on the spherically symmetric geometry. We would like to mention that in 3d case the situation is different and the existence and asymptotic behavior of full system of the Navier-Stokes- Fourier system in 3D with nonideal gas (including pressure) were proved in the works of Feireisl and his coworkers [11, 13, 12]. With ideal polytropic gas and density dependent viscosity the existence of solution was proved by D. Bresch and B. Desjardins [3].

We consider the following model of compressible Navier-Stokes system for a spherical symmetric flow of a pressureless gas

$$\begin{cases} \rho_t + (\rho v)_r + \dfrac{2\rho v}{r} = 0, \\[2ex] \rho(v_t + v v_r) = \left(\mu\left(v_r + \dfrac{2v}{r}\right)\right)_r, \\[2ex] \rho(\theta_t + v\theta_r) = q_r + \dfrac{2q}{r} + \mu\left(v_r + \dfrac{2v}{r}\right)^2, \end{cases} \tag{1}$$

in the domain $\Omega \times \mathbf{R}^+$ with $\Omega := (R_0, R_1)$, for the density $\rho(r,t)$, the velocity $v(r,t)$ and the temperature $\theta(r,t)$. The heat flux q is given by the Fourier law $q(\theta) := \kappa(\theta)\theta_r$.

Writing the system in the lagrangian (mass) coordinates (x,t), with

$$r(x,t) := r_0(x) + \int_0^t v(x,s)\, ds, \tag{2}$$

where

$$r_0(x) := \left[R_0^3 + 3\int_0^x \eta^0(y)\, dy\right]^{1/3}, \quad \text{for } x \in \Omega,$$

we get

$$\begin{cases} \eta_t = (r^2 v)_x, \\ v_t = r^2 \left(\dfrac{\mu}{\eta} (r^2 v)_x \right)_x, \\ \theta_t = q_x + \left(\dfrac{\mu}{\eta} (r^2 v)_x \right) (r^2 v)_x, \\ r_t = v, \end{cases} \tag{3}$$

in the domain $Q := \Omega \times \mathbf{R}^+$ with $\Omega := (0, M)$, where the specific volume η (with $\eta := \frac{1}{\rho}$), the velocity v, the temperature θ and the radius r depend on the lagrangian mass coordinates.

For our pressureless model, the stress σ is only viscous

$$\sigma(\eta, \theta) := \frac{\mu(\theta)}{\eta} (r^2 v)_x,$$

the energy is normalized $e = \theta$, and the heat flux is $q(\theta) := \frac{\kappa(\theta) r^4}{\eta} \theta_x$.

We consider the boundary conditions

$$\begin{cases} v|_{x=0,M} = 0, \\ \pi|_{x=0,M} = 0, \end{cases} \tag{4}$$

for $t > 0$, and initial conditions

$$\eta|_{t=0} = \eta^0(x), \quad v|_{t=0} = v^0(x), \quad r|_{t=0} = r^0(x), \quad \theta|_{t=0} = \theta^0(x) \qquad \text{on } \Omega. \tag{5}$$

The viscosity coefficient μ is such that $\mu \in C^2(\mathbf{R}^+)$ and satisfy the conditions

$$\frac{d}{d\xi} \mu(\xi) \leqslant 0, \quad \mu(\xi) \geqslant \underline{\mu} > 0. \tag{6}$$

The thermal conductivity satisfies the Belov's condition [2]

$$\kappa(\xi) = -\Lambda \, \frac{d}{d\xi} (\log \mu(\xi)) \;\; \text{for } \xi \geqslant 0, \tag{7}$$

where Λ is a positive constant.

We study weak solutions for the above problem with properties

$$\begin{cases} \eta \in L^\infty(Q_T), \quad \eta_t \in L^\infty([0,T], L^2(\Omega)), \quad \sqrt{\rho}\ (r^2 v)_x \in L^\infty([0,T], L^2(\Omega)), \\ v \in L^\infty([0,T], L^4(\Omega)), \quad v_t \in L^\infty([0,T], L^2(\Omega)),\ \sigma_x \in L^\infty([0,T], L^2(\Omega)), \\ \theta \in L^\infty([0,T], L^2(\Omega)), \quad \sqrt{\rho}\ \theta_x \in L^\infty([0,T], L^2(\Omega)). \end{cases} \tag{8}$$

and

$$r \in C(Q) \text{ and for all } t \in [0,T], x \to r(x,t) \text{ is strictly increasing on } \Omega, \tag{9}$$

where $Q_T := \Omega \times (0,T)$.

We also assume the following conditions on the data:

$$\begin{cases} \eta^0 > 0 \text{ on } \Omega,\ \eta^0 \in L^1(\Omega), \\ v_0 \in L^2(\Omega), \quad \sqrt{\rho^0}\ v_x^0 \in L^2(\Omega), \\ \theta^0 \in L^2(\Omega), \quad \inf_\Omega \theta^0 > 0. \end{cases} \tag{10}$$

We look for a weak solution (η, v, θ) such that

$$\eta(x,t) = \eta^0(x) + \int_0^t \left(r^2 v_x + \frac{2\eta v}{r} \right)(x,s)\ ds, \tag{11}$$

for a.e. $x \in \Omega$ and any $t > 0$, and such that for any test function $\phi \in L^2([0,T], H^1(\Omega))$ with $\phi_t \in L^1([0,T], L^2(\Omega))$ such that $\phi(\cdot, T) = 0$

$$\int_Q \left[\phi_t v + \left(r^2 \phi_x + \frac{2\eta\phi}{r} \right) p - \frac{\mu \phi_x r^4}{\eta} v_x - 2\mu \frac{\phi \eta v}{r^2} \right]\ dx\ dt$$
$$= \int_\Omega \phi(0,x)\ v^0(x)\ dx, \tag{12}$$

and

$$\int_Q \left[\phi_t e + \frac{\kappa r^4 \theta_x}{\eta} \phi_x - r^2 v \sigma \phi_x - r^2 v \sigma_x \phi \right]\ dx\ dt = \int_\Omega \phi(0,x)\ \theta^0(x)\ dx. \tag{13}$$

The aim of the present paper is to prove the following result

Theorem 1 *Suppose that the initial data satisfy (10) and that T is an arbitrary positive number.*

Then the problem (3)(4)(5) possesses a global weak solution satisfying (8) and (9) together with properties (11), (12) and (13).

For that purpose, we first prove a classical existence result in the Hölder category. We denote by $C^\alpha(\Omega)$ the Banach space of real-valued functions on Ω which are uniformly Hölder continuous with exponent $\alpha \in \Omega$, and $C^{\alpha,\alpha/2}(Q_T)$ the Banach space of real-valued functions on $Q_T := \Omega \times (0, T)$ which are uniformly Hölder continuous with exponent α in x and $\alpha/2$ in t. The norms of $C^\alpha(\Omega)$ (resp. $C^{\alpha,\alpha/2}(Q_T)$) will be denoted by $\|\cdot\|_\alpha$ (resp. $|||\cdot|||_\alpha$).

Theorem 2 *Suppose that the initial data satisfy*

$$\left(\eta^0, \eta^0_x, v^0, v^0_x, v^0_{xx}, \theta^0, \theta^0_x, \theta^0_{xx}\right) \in (C^\alpha(\Omega))^8 ,$$

for some $\alpha \in \Omega$. Suppose also that $\eta^0(x) > 0$ and $\theta^0(x) > 0$ on Ω, and that the compatibility conditions

$$\theta^0_x(0) = \theta^0_x(M) = 0, \quad v^0(0) = v^0(M) = 0,$$

hold. Then, there exists a unique solution $(\eta(x,t), v(x,t), \theta(x,t))$ with $\eta(x,t) > 0$ and $\theta(x,t) > 0$ to the initial-boundary value problem (3)(4)(5) on $Q = \Omega \times \mathbb{R}_+$ such that for any $T > 0$

$$(\eta, \eta_x, \eta_t, \eta_{xt}, v, v_x, v_t, v_{xx}, \theta, \theta_x, \theta_t, \theta_{xx}) \in (C^\alpha(Q_T))^{12} ,$$

and

$$(\eta_{tt}, v_{xt}, \theta_{xt}) \in \left(L^2(Q_T)\right)^3 .$$

Then Theorem 1 will be obtained from Theorem 2 through a regularization process.

The plan of the article is as follows: in section 2 we give a priori estimates sufficient to prove in section 3 global existence of a solution, then we gives in section 4 the asymptotic behaviour of the solution for large time. In the last section we briefly study the case of constant transport coefficients.

2 A priori estimates

In the spirit of [21], we first suppose that the solution is classical in the following sense

$$\left\{\begin{array}{c} \eta \in C^1(Q_T), \quad \rho > 0, \\ v, \theta \in C^1([0,T], C^0(\Omega)) \cap C^0([0,T], C^2(\Omega)), \quad \theta > 0, \end{array}\right. \tag{14}$$

and

$$r > 0 \text{ for all } t \in [0,T]. \tag{15}$$

Our first task is to prove the following regularity result

Theorem 3 *Suppose that the initial-boundary value problem (3)(4)(5) has a classical solution described by Theorem 2. Then the solution* $(\eta, v, v_x, \theta, \theta_x)$ *is bounded in the Hölder space* $C^{1/3,1/6}(Q_T)$

$$|||\eta|||_{1/3} + |||v|||_{1/3} + |||v_x|||_{1/3} + |||\theta|||_{1/3} + |||\theta_x|||_{1/3} \leqslant C(T),$$

where C *depends on* T*, the physical data of the problem and the initial data. Moreover*

$$0 < \underline{\eta} \leqslant \eta \leqslant \overline{\eta}, \qquad 0 < \underline{\theta} \leqslant \theta \leqslant \overline{\theta}.$$

Let N and T be arbitrary positive numbers In all the following, we denote by $C = C(N)$ or $K = K(N)$ various positive non-decreasing functions of N, which may possibly depend on the physical constants M etc., but not on T. We also denote by Ψ the elementary positive function: $\Psi(s) := s - \log s - 1$, for any $s > 0$.

Lemma 1 *Under the following condition on the data*

$$\left\|v^0\right\|_{L^2(\Omega)} + \left\|\eta^0\right\|_{L^1(\Omega)} + \left\|\theta^0\right\|_{L^1(\Omega)} \leqslant N, \tag{16}$$

1. *The following mass-energy equality holds*

$$\int_\Omega \left[\frac{1}{2}v^2 + \eta + e\right] dx = \int_\Omega \left[\frac{1}{2}(v^0)^2 + \eta^0 + e^0\right] dx. \tag{17}$$

2. The following "entropy" inequality holds

$$\int_\Omega \Psi(\theta)\ dx + \int_0^T \int_\Omega \left(\frac{\kappa(\theta) r^4}{\eta\theta^2}\ \theta_x^2 + \frac{\mu(\theta)}{\eta\theta}\ [(r^2 v)_x]^2 \right) dx\ dt \leqslant K(N). \tag{18}$$

3. The following estimates hold

$$\|\eta\|_{L^\infty(0,T;L^1(\Omega))} + \|v\|_{L^\infty(0,T;L^2(\Omega))} + \|\theta\|_{L^\infty(0,T;L^1(\Omega))} \leqslant K(N). \tag{19}$$

Proof: 1. Multiplying the second equation (3) by v, adding the result to the first and third equations (3), integrating on Ω and using (4), (5), one gets the energy identity (17).

2. Computing the time-derivative $(\log\theta)_t$ we get

$$(\log\theta)_t = \left(\frac{\kappa(\theta) r^4}{\eta\theta}\ \theta_x \right)_x + \frac{\kappa(\theta) r^4}{\eta\theta^2}\ \theta_x^2 + \frac{\mu(\theta)}{\eta\theta}\ [(r^2 v)_x]^2.$$

Integrating on Ω and using (17) we get (18).

3. The estimate (19) follows from (17) $\square$

Proposition 1 *The following uniform bound holds on Q*

$$|v(x,t)| \leqslant \|v^0\|_{C(\Omega)}. \tag{20}$$

Proof: Applying the strong maximum principle to the second equation (3) gives (20) $\square$

Proposition 2 *The following uniform lower bound holds on Q*

$$\theta(x,t) \geqslant \underline{\theta} > 0, \tag{21}$$

where $\underline{\theta} = \left(\|\frac{1}{\theta^0}\|_{C(\Omega)} \right)^{-1}$.

Proof: Multiplying, as in [1], the third equation (3) by θ^{-2}, we get

$$\omega_t = \left(\kappa \frac{r^4}{\eta} \omega_x \right)_x - 2\kappa \frac{r^4}{\eta\theta^3} \theta_x^2 - \frac{\mu}{\eta\theta^2} [(r^2 v)_x]^2 \leqslant \left(\kappa \frac{r^4}{\eta} \omega_x \right)_x,$$

where $\omega := \theta^{-1}$. Multiplying by $2p\omega^{2p-1}$, we get

$$\left(\omega^{2p}\right)_t \leqslant \left(\kappa \frac{r^4}{\eta}(\omega^{2p})_x\right)_x - \kappa \frac{r^4}{\eta}\ 2p\omega^{2p-2}\omega_x^2,$$

which implies

$$\frac{d}{dt}\left(\int_\Omega \omega^{2p}\ dx\right) \leqslant 0.$$

Integrating in t and letting $p \to \infty$ gives $\|\omega(\cdot,t)\|_\infty \leqslant \|\omega^0\|_\infty$, which implies (21) $\square$

Lemma 2 *One has the kinetic energy bound*

$$\left\| \sqrt{\frac{\mu}{\eta}}(r^2 v)_x \right\|_{L^1(0,T,L^2(\Omega)} \leqslant K, \tag{22}$$

and the improved thermal bound

$$\left\| \sqrt{\frac{\kappa^2 r^4}{\eta}}\theta_x \right\|_{L^1(0,T,L^2(\Omega)} \leqslant K. \tag{23}$$

Proof: 1. Multiplying the second equation (3) by v and integrating by parts, we get

$$\frac{d}{dt}\int_\Omega \frac{1}{2}v^2\ dx = \int_\Omega r^2\sigma_x v\ dx = -\int_\Omega \frac{\mu}{\eta}[(r^2v)_x]^2\ dx,$$

which gives (22) by integrating in t.

2. Multiplying the third equation (3) by $\mathcal{K}(\theta) := \int_{\theta_0}^\theta \kappa(s)\ ds$, for $\theta_0 > 0$ arbitrary, and integrating by parts, we get

$$\begin{aligned}\int_\Omega \mathcal{K}\theta_t &= \int_\Omega \mathcal{K}\left(\kappa\frac{r^4}{\eta}\theta_x\right)_x\ dx + \int_\Omega \mathcal{K}\frac{\mu}{\eta}[(r^2v)_x]^2\ dx \\ &= -\int_\Omega \mathcal{K}_x\ \kappa\frac{r^4}{\eta}\theta_x\ dx + \int_\Omega \mathcal{K}\frac{\mu}{\eta}[(r^2v)_x]^2\ dx.\end{aligned}$$

Then

$$\frac{d}{dt}\int_\Omega\left(\int_1^\theta \mathcal{K}(s)\ ds\right)dx + \int_\Omega\ \kappa^2\frac{r^4}{\eta}\theta_x^2\ dx = \int_\Omega \mathcal{K}(\theta)\ \frac{\mu}{\eta}[(r^2v)_x]^2\ dx. \tag{24}$$

After the growth property (7) of κ and the lower bound (21) of θ, we get

$$\mathcal{K}(\theta) = -\Lambda \int_{\theta_0}^{\theta} \frac{d}{ds}(\log \mu(s))\ ds \leqslant K,$$

which gives (23) by plugging into (24) after integrating in t, and using (22) $\square$

Lemma 3 *One has the bounds*

$$\left\| \sqrt{\frac{\mu}{\eta}}\ (r^2 v)_x \right\|_{L^\infty(0,T,L^2(\Omega))} \leqslant K, \quad \left\| \sqrt{\frac{\kappa}{\eta}}\ r^4\ \theta_x^2 \right\|_{L^\infty(0,T,L^2(\Omega))} \leqslant K, \tag{25}$$

and

$$\left\| \left(\frac{\mu}{\eta} \left(r^2 v\right)_x \right)_x \right\|_{L^1(0,T,L^2(\Omega))} \leqslant K. \tag{26}$$

Proof: All along the proof, we denote by C a generic positive constant, possibly depending on the various physical constants of the problem, but which do not depend on T.

1. Observing that the second equation (3) rewrites $(r^2 v)_t = r^4 \sigma_x + 2rv^2$, multiplying by σ_x and integrating on Ω, we get

$$\int_\Omega \sigma_x (r^2 v)_t\ dx = \int_\Omega r^4 \sigma_x^2 dx + 2 \int_\Omega r v^2 \sigma_x\ dx.$$

Integrating by parts

$$\frac{d}{dt} \int_\Omega \frac{\mu}{\eta}\ [(r^2 v)_x]^2\ dx + \int_\Omega r^4 \sigma_x^2 dx = -\int_\Omega r^2 v\ \sigma_{xt} dx - 2 \int_\Omega r v^2 \sigma_x\ dx := A_1 + A_2. \tag{27}$$

Rewriting A_1, we have

$$A_1 = \int_\Omega (r^2 v)_x \sigma_t dx = \frac{d}{dt} \int_\Omega \frac{1}{2}\ \frac{\eta}{\mu}\ \sigma^2\ dx - \frac{1}{2} \int_\Omega \left(\frac{\eta}{\mu}\right)_t \sigma^2\ dx$$

$$= \frac{1}{2} \frac{d}{dt} \int_\Omega \frac{\mu}{\eta}\ [(r^2 v)_x]^2\ dx - \frac{1}{2} \int_\Omega \frac{\mu}{\eta^2}\ [(r^2 v)_x]^3\ dx + \frac{1}{2} \int_\Omega \frac{\mu'}{\eta}\ [(r^2 v)_x]^2 \theta_t\ dx$$

$$= \frac{1}{2} \frac{d}{dt} \int_\Omega \frac{\mu}{\eta}\ [(r^2 v)_x]^2\ dx - \frac{1}{2} \int_\Omega \frac{\mu}{\eta^2}\ [(r^2 v)_x]^3\ dx + \frac{1}{2} \int_\Omega \frac{\mu'}{\eta}\ [(r^2 v)_x]^2 \left(\kappa \frac{r^4}{\eta} \theta_x \right)_x\ dx$$

$$+\frac{1}{2}\int_\Omega \frac{\mu\mu'}{\eta^2}\ [(r^2v)_x]^4\ dx.$$

In the same stroke

$$A_2 = 2\int_\Omega (rv^2)_x\sigma\ dx = 4\int_\Omega \frac{\mu v}{r\eta}[(r^2v)_x]^2\ dx - 6\int_\Omega \frac{\mu v^2}{r^2}(r^2v)_x\ dx.$$

Plugging into (27), we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \frac{\mu}{\eta}\ [(r^2v)_x]^2\ dx + \int_\Omega r^4\sigma_x^2 dx$$

$$= -\frac{1}{2}\int_\Omega \frac{\mu}{\eta^2}\ [(r^2v)_x]^3\ dx + \frac{1}{2}\int_\Omega \frac{\mu'}{\eta}\ [(r^2v)_x]^2 \left(\kappa\frac{r^4}{\eta}\theta_x\right)_x\ dx$$

$$+\frac{1}{2}\int_\Omega \frac{\mu\mu'}{\eta^2}\ [(r^2v)_x]^4\ dx + 4\int_\Omega \frac{\mu v}{r\eta}[(r^2v)_x]^2\ dx - 6\int_\Omega \frac{\mu v^2}{r^2}(r^2v)_x\ dx =: \sum_{j=1}^{5} B_j. \tag{28}$$

Let us estimate the contributions in the right-hand side.

One observes first that, after the boundary conditions (4)

$$\forall t \in [0,T],\ \exists\xi(t):\ (r^2v)_x(\xi(t),t) = 0.$$

So splitting Ω accordingly, we have

$$B_1 = -\frac{1}{2}\int_0^\xi \frac{\mu}{\eta^2}\ [(r^2v)_x]^3\ dx - \frac{1}{2}\int_\xi^M \frac{\mu}{\eta^2}\ [(r^2v)_x]^3\ dx.$$

Integrating by part, we find first

$$-\frac{1}{2}\int_0^\xi \frac{\mu}{\eta}(r^2v)_x\ \frac{1}{\eta}[(r^2v)_x]^2\ dx = \frac{1}{2}\int_0^\xi \left(\frac{\mu}{\eta}(r^2v)_x\right)_x \int_0^x \frac{1}{\eta}[(r^2v)_y]^2\ dy\ dx.$$

So

$$\left|\frac{1}{2}\int_0^\xi \frac{\mu}{\eta}(r^2v)_x\ \frac{1}{\eta}[(r^2v)_x]^2\ dx\right| \leqslant \frac{1}{2}\int_0^\xi r^2 \left|\left(\frac{\mu}{\eta}(r^2v)_x\right)_x\right| \left(\frac{1}{r^2}\int_0^x \frac{1}{\eta}[(r^2v)_y]^2\ dy\right)\ dx,$$

and by Cauchy-Schwarz

$$\left|\frac{1}{2}\int_0^\xi \frac{\mu}{\eta}(r^2v)_x\ \frac{1}{\eta}[(r^2v)_x]^2\ dx\right| \leqslant \frac{\epsilon_1}{6}\int_\Omega r^4\sigma_x^2\ dx + C\left(\int_\Omega \frac{\mu}{\eta}[(r^2v)_x]^2\ dx\right)^2,$$

for any $\epsilon_1 > 0$, and a $C(\epsilon_1, \underline{\mu}, R_0)$.

As the same bound clearly holds for $\frac{1}{2}\int_\xi^M \frac{\mu}{\eta^2}\, [(r^2v)_x]^3\, dx$, we have

$$|B_1| \leqslant \frac{1}{3}\epsilon_1 \int_\Omega r^4 \sigma_x^2\, dx + C\left(\int_\Omega \frac{\mu}{\eta}[(r^2v)_x]^2\, dx\right)^2. \tag{29}$$

By Cauchy-Schwarz in B_2, we have

$$|B_2| \leqslant -\frac{1}{4}\epsilon_2 \int_\Omega \frac{\mu\mu'}{\eta^2}\, [(r^2v)_x]^4\, dx + \frac{1}{4\epsilon_2}\int_\Omega \frac{1}{\mu\kappa}\, \kappa \left[\left(\kappa\frac{r^4}{\eta}\theta_x\right)_x\right]^2\, dx. \tag{30}$$

Using the same splitting: $\Omega = (0,\xi)\cup(\xi, M)$ (as in B_1) for B_4 and integrating by parts, we get

$$B_4 = 4\int_\Omega \frac{\mu(r^2v)_x}{\eta}\, \frac{v(r^2v)_x}{r}\, dx = -4\int_\Omega r^2\left(\frac{\mu(r^2v)_x}{\eta}\right)_x \left(\frac{1}{r^2}\int_0^x \frac{v(r^2v)_y}{r}\, dy\right) dx.$$

So by Cauchy-Schwarz

$$\begin{aligned}|B_4| &\leqslant 4\int_\Omega r^2 \left|\left(\frac{\mu(r^2v)_x}{\eta}\right)_x\right| \left|\frac{1}{r^2}\int_0^x \frac{v(r^2v)_y}{r}\, dy\right| dx \\ &\leqslant \frac{1}{3}\epsilon_1 \int_\Omega r^4\sigma_x^2\, dx + C\int_\Omega \left(\int_0^x \frac{v(r^2v)_x}{r}\, dy\right)^2 dx. \\ &\leqslant \frac{1}{3}\epsilon_1 \int_\Omega r^4\sigma_x^2\, dx + C\left(\int_\Omega \frac{\mu}{\eta}\, [(r^2v)_x]^2\, dx\right)\left(\int_\Omega \frac{\eta v^2}{\mu}\, dx\right).\end{aligned}$$

Using the energy estimate, Proposition 1 and (6) the last integral is bounded, so

$$|B_4| \leqslant \frac{1}{3}\epsilon_1 \int_\Omega r^4\sigma_x^2\, dx + C\int_\Omega \frac{\mu}{\eta}\, [(r^2v)_x]^2\, dx. \tag{31}$$

Using Cauchy-Schwarz in B_5 gives

$$B_5 \leqslant C\int_\Omega \frac{\mu}{\eta}(r^2v)_x^2\, dx + C\int_\Omega \mu\eta v^4\, dx.$$

But after energy estimate

$$v^2 \leqslant C\max_\Omega (r^2v)^2 \leqslant C\left(\int_\Omega \frac{\mu}{\eta}(r^2v)_x^2\, dx\right)^{1/2},$$

so

$$|B_5| \leqslant C \int_\Omega \frac{\mu}{\eta}(r^2 v)_x^2 \, dx. \tag{32}$$

Plugging (29), (30), (31) and (32) into (28), we get

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega \frac{\mu}{\eta}\ [(r^2v)_x]^2\ dx + \int_\Omega r^4\sigma_x^2 dx - \frac{1}{2}\int_\Omega \frac{\mu\mu'}{\eta^2}\ [(r^2v)_x]^4\ dx \leqslant \epsilon_1 \int_\Omega r^4\sigma_x^2\ dx \\
+C\left(\int_\Omega \frac{\mu}{\eta}[(r^2v)_x]^2\ dx\right)^2 \\
-\frac{1}{4}\epsilon_2 \int_\Omega \frac{\mu\mu'}{\eta^2}\ [(r^2v)_x]^4\ dx - \frac{1}{4\epsilon_2}\int_\Omega \frac{\mu'}{\mu\kappa}\ \kappa \left[\left(\kappa\frac{r^4}{\eta}\theta_x\right)_x\right]^2\ dx.
\end{aligned} \tag{33}$$

2. Multiplying now the third equation (3) by $\alpha\kappa\left(\kappa\frac{r^4}{\eta}\theta_x\right)_x$, where $\alpha > 0$ will be defined later, we find

$$\alpha\kappa\left(\kappa\frac{r^4}{\eta}\theta_x\right)_x \theta_t = \alpha\kappa\left[\left(\kappa\frac{r^4}{\eta}\theta_x\right)_x\right]^2 + \alpha\frac{\mu}{\eta}\ [(r^2v)_x]^2\ \kappa\left(\kappa\frac{r^4}{\eta}\theta_x\right)_x.$$

As the left-hand side rewrites $\alpha\mathcal{K}_t\left(\frac{r^4}{\eta}\ \mathcal{K}_x\right)_x$, we easily compute

$$\begin{aligned}
\alpha\mathcal{K}_t\left(\frac{r^4}{\eta}\ \mathcal{K}_x\right)_x &= \alpha\left(\mathcal{K}_t\frac{r^4}{\eta}\ \mathcal{K}_x\right)_x - \alpha\mathcal{K}_{tx}\frac{r^4}{\eta}\ \mathcal{K}_x \\
&= \alpha\left(\mathcal{K}_t\frac{r^4}{\eta}\ \mathcal{K}_x\right)_x - \frac{1}{2}\alpha\left(\mathcal{K}_x^2\frac{r^4}{\eta}\right)_t + \frac{1}{2}\alpha\mathcal{K}_x^2\left(\frac{r^4}{\eta}\right)_t. \\
&= \alpha\left(\mathcal{K}_t\frac{r^4}{\eta}\ \mathcal{K}_x\right)_x - \frac{1}{2}\alpha\left(\mathcal{K}_x^2\frac{r^4}{\eta}\right)_t + 2\alpha\mathcal{K}_x^2\frac{r^3v}{\eta} - \frac{1}{2}\alpha\mathcal{K}_x^2\frac{r^4(r^2v)_x}{\eta^2}.
\end{aligned}$$

So integrating on Ω and using (4)

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega \alpha\frac{r^4\mathcal{K}_x^2}{\eta}\ dx + \alpha\int_\Omega \kappa\left[\left(\kappa\frac{r^4}{\eta}\theta_x\right)_x\right]^2 dx \\
= 2\alpha\int_\Omega \mathcal{K}_x^2\frac{r^3v}{\eta}\ dx - \frac{\alpha}{2}\int_\Omega \mathcal{K}_x^2\frac{r^4(r^2v)_x}{\eta^2}\ dx - \alpha\int_\Omega \frac{\mu}{\eta}\ [(r^2v)_x]^2\ \kappa\left(\kappa\frac{r^4}{\eta}\theta_x\right)_x dx =: \sum_{j=1}^3 C_j.
\end{aligned} \tag{34}$$

In order to estimate the contributions on the right-hand side, we first integrate by parts in C_1

$$C_1 = 2\alpha \int_\Omega \frac{r^4\mathcal{K}_x^2}{\eta}\frac{r^2 v}{r^3}\, dx = -2\alpha \int_\Omega \left(\frac{r^2 v}{r^3}\right)_x \int_0^x \frac{r^4\mathcal{K}_y^2}{\eta}\, dy\, dx$$

$$= -2\alpha \int_\Omega \frac{(r^2v)_x}{r^3}\left(\int_0^x \frac{r^4\mathcal{K}_y^2}{\eta}\, dy\right) dx + 6\alpha \int_\Omega \frac{\eta v}{r^4}\left(\int_0^x \frac{r^4\mathcal{K}_y^2}{\eta}\, dy\right) dx.$$

The first integral gives by Cauchy-Schwarz

$$\left|2\alpha \int_\Omega \frac{(r^2v)_x}{r^3}\left(\int_0^x \frac{r^4\mathcal{K}_y^2}{\eta}\, dy\right) dx\right| \leqslant 2\alpha \int_\Omega \sqrt{\frac{\mu}{\eta}}\frac{|(r^2v)_x|}{r^3}\sqrt{\frac{\eta}{\mu}}\left(\int_0^x \frac{r^4\mathcal{K}_y^2}{\eta}\, dy\right) dx$$

$$\leqslant \frac{\alpha}{2}\int_\Omega \frac{\mu}{\eta}[(r^2v)_x]^2|\, dx + 2\alpha \int_\Omega \frac{\eta}{\mu}\left(\int_0^x \frac{r^4\mathcal{K}_y^2}{\eta}\, dy\right)^2 dx,$$

so, using energy estimate

$$\left|2\alpha \int_\Omega \frac{(r^2v)_x}{r^3}\left(\int_0^x \frac{r^4\mathcal{K}_y^2}{\eta}\, dy\right) dx\right| \leqslant \frac{1}{2}\int_\Omega \frac{\mu}{\eta}[(r^2v)_x]^2|\, dx + \frac{1}{2}\, C\left(\int_\Omega \frac{r^4\mathcal{K}_x^2}{\eta}\, dx\right)^2,$$

for a positive constant C.

As the second integral gives clearly the same estimate, one gets

$$|C_1| \leqslant \int_\Omega \frac{\mu}{\eta}[(r^2v)_x]^2\, dx + C\left(\int_\Omega \frac{r^4\mathcal{K}_x^2}{\eta}\, dx\right)^2. \tag{35}$$

In the same way, we get

$$C_2 = -\frac{1}{2}\alpha \int_\Omega \frac{\kappa r^4}{\eta}\theta_x \frac{\kappa}{\eta}\theta_x (r^2v)_x\, dx = \frac{1}{2}\int_\Omega \sqrt{\kappa}\left(\frac{\kappa r^4}{\eta}\theta_x\right)_x \frac{1}{\sqrt{\kappa}}\int_0^x \frac{\kappa}{\eta}\theta_y (r^2v)_y\, dy\, dx.$$

Using once more Cauchy-Schwarz, we get

$$|C_2| \leqslant \frac{1}{2}\,\epsilon_3 \int_\Omega \kappa\left[\left(\frac{\kappa r^4}{\eta}\theta_x\right)_x\right]^2 dx + C\int_\Omega \frac{1}{\kappa}\left(\int_0^x \frac{\kappa}{\eta}\theta_y (r^2v)_y\, dy\right)^2 dx.$$

So

$$|C_2| \leqslant \frac{1}{2}\,\epsilon_3 \int_\Omega \kappa\left[\left(\frac{\kappa r^4}{\eta}\theta_x\right)_x\right]^2 dx + C\left(\int_\Omega \frac{\kappa^2 r^4}{\eta}\theta_x^2\, dx\right)\left(\int_\Omega \frac{\mu}{\eta}[(r^2v)_x]^2\, dx\right). \tag{36}$$

Finally by Cauchy-Schwarz in C_3, we have

$$|C_3| \leqslant -\frac{1}{4}\epsilon_3 \int_\Omega \frac{\mu\mu'}{\eta^2}\ [(r^2v)_x]^4\ dx + \frac{\alpha^2}{4\epsilon_3}\int_\Omega \frac{\mu\kappa}{\mu'}\ \kappa\left[\left(\kappa\frac{r^4}{\eta}\theta_x\right)_x\right]^2\ dx. \tag{37}$$

Plugging (35), (36) and (37) into (34), we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega \alpha\frac{r^4\mathcal{K}_x^2}{\eta}\ dx + \alpha\int_\Omega \kappa\left[\left(\kappa\frac{r^4}{\eta}\theta_x\right)_x\right]^2 dx \leqslant \int_\Omega \frac{\mu}{\eta}[(r^2v)_x]^2\ dx \\ +C\left(\int_\Omega \frac{r^4\mathcal{K}_x^2}{\eta}\ dx\right)^2 + \frac{1}{2}\ \epsilon_3\int_\Omega \kappa\left[\left(\frac{\kappa r^4}{\eta}\theta_x\right)_x\right]^2\ dx \\ +C\left(\int_\Omega \frac{\kappa^2 r^4}{\eta}\theta_x^2\ dx\right)\left(\int_\Omega \frac{\mu}{\eta}[(r^2v)_x]^2\ dx\right) \\ -\frac{1}{4}\epsilon_3\int_\Omega \frac{\mu\mu'}{\eta^2}\ [(r^2v)_x]^4\ dx + \frac{\alpha^2}{4\epsilon_3}\int_\Omega \frac{\mu\kappa}{\mu'}\ \kappa\left[\left(\kappa\frac{r^4}{\eta}\theta_x\right)_x\right]^2\ dx.\end{aligned} \tag{38}$$

Now adding the inequalities (38) and (33), we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega \left[\alpha\frac{r^4\mathcal{K}_x^2}{\eta} + \frac{\mu}{\eta}\ [(r^2v)_x]^2\right]\ dx + \alpha\int_\Omega \kappa\left[\left(\kappa\frac{r^4}{\eta}\theta_x\right)_x\right]^2 dx \\ +\int_\Omega r^4\sigma_x^2 dx - \frac{1}{2}\int_\Omega \frac{\mu\mu'}{\eta^2}\ [(r^2v)_x]^4\ dx \\ \leqslant \int_\Omega \frac{\mu}{\eta}[(r^2v)_x]^2\ dx + C\left(\int_\Omega \frac{r^4\mathcal{K}_x^2}{\eta}\ dx\right)^2 + \frac{1}{2}\ \epsilon_3\int_\Omega \kappa\left[\left(\frac{\kappa r^4}{\eta}\theta_x\right)_x\right]^2\ dx \\ +C\left(\int_\Omega \frac{\kappa^2 r^4}{\eta}\theta_x^2\ dx\right)\left(\int_\Omega \frac{\mu}{\eta}[(r^2v)_x]^2\ dx\right) \\ +C\left(\int_\Omega \frac{\mu}{\eta}[(r^2v)_x]^2\ dx\right)^2 + \epsilon_1\int_\Omega r^4\sigma_x^2\ dx \\ -\frac{1}{4}\left(\epsilon_2+\epsilon_3\right)\int_\Omega \frac{\mu\mu'}{\eta^2}\ [(r^2v)_x]^4\ dx - \frac{1}{4}\int_\Omega \left(\frac{\mu'}{\epsilon_2\mu\kappa} + \alpha^2\frac{\mu\kappa}{\epsilon_3\mu'}\right)\ \kappa\left[\left(\kappa\frac{r^4}{\eta}\theta_x\right)_x\right]^2\ dx.\end{aligned} \tag{39}$$

Under the conditions

$$\begin{cases} \epsilon_2+\epsilon_3 \leqslant 2, \\ \dfrac{\mu'}{\epsilon_2\mu\kappa} + \alpha^2\dfrac{\mu\kappa}{\epsilon_3\mu'} \leqslant 2\alpha, \end{cases} \tag{40}$$

the two last contributions are absorbed by the left-hand side. One checks that for this system to have a solution it is necessary that $\epsilon_2 = \epsilon_3 = 1$. The second inequality then rewrites $x + \frac{\alpha^2}{x} \leqslant 2\alpha$, with $x = -\mu'/(\mu\kappa)$, and has the unique solution $x = \alpha$. Choosing then $\alpha = \Lambda$ after (7), inequality (39) implies the following

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \left[\frac{r^4\mathcal{K}_x^2}{\eta} + \frac{\mu}{\eta}\,[(r^2v)_x]^2\right]\,dx \leqslant \int_\Omega \frac{\mu}{\eta}[(r^2v)_x]^2\,dx + C\left(\int_\Omega \frac{r^4\mathcal{K}_x^2}{\eta}\,dx\right)^2$$
$$+C\left(\int_\Omega \frac{\kappa^2 r^4}{\eta}\theta_x^2\,dx\right)\left(\int_\Omega \frac{\mu}{\eta}[(r^2v)_x]^2\,dx\right)$$
$$+C\left(\int_\Omega \frac{\mu}{\eta}[(r^2v)_x]^2\,dx\right)^2.$$

If we define $X(t) := \int_\Omega \frac{r^4\mathcal{K}_x^2}{\eta}\,dx$ and $Y(t) := \int_\Omega \frac{\mu}{\eta}\,[(r^2v)_x]^2\,dx$, we observe that, as the functions X, Y and $\int_\Omega \eta v^2\,dx$ are $L^1(0,T)$ for any $T > 0$, the previous inequality is easily rewriten as

$$\frac{d}{dt}(X+Y) \leqslant f(t)(X+Y) + g(t),$$

where $f, g \in L^1(0,T)$. Applying Gronwall's lemma ends the proof $\square$

Lemma 4 *Under the previous condition on the data, there exists two positive constants $\underline{\eta}$ and $\overline{\eta}$ independent of T such that*

$$0 < \underline{\eta} \leqslant \eta(x,t) \leqslant \overline{\eta} \;\; \textit{for } (t,x) \in Q_T. \qquad (41)$$

Proof: The second equation (3) rewrites

$$(\log\eta)_{tx} = \frac{\mu'}{\mu}\theta_x(\log\eta)_t + \left(\frac{v}{r^2\mu}\right)_t + \frac{2}{r^3\mu}\,v^2 + \frac{\mu'}{r^2\mu^2}\,v\theta_t. \qquad (42)$$

Using the first equation (3) and (4), there exists for any $t \in [0,T]$ a $\xi(t) \in \Omega$ such that

$$\eta_t(\xi(t),t) = 0.$$

Integrating (42) on $[x, \xi(t)] \times [0,t]$, we find

$$\int_0^t \int_x^\xi \left[(\log\eta)_s\right]_y\,dy\,ds = \int_0^t \int_x^\xi \frac{\mu'}{\mu}\theta_y(\log\eta)_s\,dy\,ds$$

$$+\int_0^t\int_x^\xi \left(\frac{v}{r^2\mu}\right)_s \, dy\, ds + \int_0^t\int_x^\xi \frac{2v^2}{r^3\mu}\, dy\, ds + \int_0^t\int_x^\xi \frac{\mu'}{r^2\mu^2}\, v\theta_s\, dy\, ds.$$

Then using (7) we get

$$|\log\eta(x,t)| \leqslant C + \Lambda\int_{Q_T} \kappa|\theta_x|\frac{|(r^2v)_x|}{\eta}\, dx\, dt.$$

Applying Cauchy-Schwarz inequality and Lemma 3, we obtain

$$\left|\log\frac{\eta(x,t)}{\eta^0(x)}\right| \leqslant C + C\int_{Q_T}\left[\kappa r^4\frac{\mathcal{K}_x^2}{\eta} + \frac{\mu}{\eta}\,[(r^2v)_x]^2\right] dx\, dt \leqslant C \quad \square$$

Lemma 5 *Under the previous condition on the data, there exists a positive constant $\overline{\theta}$ independent of T such that*

$$\theta(x,t) \leqslant \overline{\theta} \;\; \textit{for } (t,x) \in Q_T. \tag{43}$$

Proof: Multiplying, as in [1], the third equation (3) by $n\theta^{n-1}$ for $n \geqslant 1$, we get

$$(\theta^n)_t = \left(n\theta^{n-1}\kappa\frac{r^4}{\eta}\theta_x\right)_x - n(n-1)\kappa\frac{r^4}{\eta}\theta^{n-2}\theta_x^2 + n\theta^{n-1}\frac{\mu}{\eta}[(r^2v)_x]^2.$$

Integrating on Ω

$$\frac{d}{dt}\int_\Omega \theta^n\, dx + n(n-1)\int_\Omega \kappa\frac{r^4}{\eta}\theta^{n-2}\theta_x^2 = n\int_\Omega \theta^{n-1}\frac{\mu}{\eta}[(r^2v)_x]^2.$$

Then

$$\frac{d}{dt}\int_\Omega \theta^n\, dx \leqslant \frac{n}{\underline{\theta}}\int_\Omega \theta^n \left\|\frac{\mu}{\eta}[(r^2v)_x]^2\right\|_{L^\infty(\Omega)}.$$

Using the inequality

$$\left\|\frac{\mu}{\eta}[(r^2v)_x]^2\right\|_{L^\infty(\Omega)} \leqslant C\int_\Omega r^4\sigma_x^2 dx,$$

after Lemma 3 and Gronwall's lemma, we get

$$\|\theta\|^n_{L^n(\Omega)} \leqslant \|\theta^0\|^n_{L^n(\Omega)} \exp\left(\frac{n}{\underline{\theta}}\left\|\frac{\mu}{\eta}[(r^2v)_x]^2\right\|_{L^\infty(0,T;L^2(\Omega))}\right).$$

Finally taking the $1/n$-power and passing to the limit $n \to \infty$ ends the proof $\quad\square$

Corollary 1 *For any $T > 0$*

$$\max_{[0,T]} \int_\Omega \left[(r^2 v)_x\right]^2 \; dx \leqslant K, \qquad \max_{[0,T]} \int_\Omega \theta_x^2 \; dx \leqslant K, \tag{44}$$

and

$$\max_\Omega \left[(r^2 v)_x\right]^2 \in L^1(0,T), \qquad \max_\Omega \theta_x^2 \in L^1(0,T). \tag{45}$$

Proof:

1. Inequalities (44) follow directly from Lemma 3.
2. As $(r^2 v)_x = \frac{\eta}{\mu}\sigma$, after Lemma 4 and 5, one gets

$$\left[(r^2 v)_x\right]^2 \leqslant C\sigma^2 \leqslant C \int_\Omega r^4 \sigma_x^2 dx,$$

implying the first inequality (45), after Lemma 3.

After Lemma 3

$$\frac{1}{2}\int_\Omega \frac{r^4 \mathcal{K}_x^2}{\eta} \; dx + \int_{Q_T} \kappa \left[\left(\kappa \frac{r^4}{\eta}\theta_x\right)_x\right]^2 dx \; dt \leqslant K,$$

which implies directly the second inequality(45), by using Lemma 4 and 5 □

Proposition 3 *For any $T > 0$, the following uniform bounds hold*

$$\max_{[0,T]} \|v_x\|_{L^2(\Omega)} \leqslant K, \quad \max_{[0,T]} \|\theta_x\|_{L^2(\Omega)} \leqslant K, \tag{46}$$

and the T-dependent bound holds

$$\max_{[0,T]} \|\eta_x\|_{L^2(\Omega)} \leqslant C(T). \tag{47}$$

Proof: Bounds (46) follows from Lemma 3.

To prove (47), we observe that the first equation (3) rewrites

$$(\log \eta)_t = \frac{\sigma}{\mu}.$$

Derivating with respect to x, multiplying by $(\log\eta)_x$ and integrating on Ω, we get

$$\frac{d}{dt}\left(\frac{1}{2}\int_\Omega [(\log\eta)_x]^2\,dx\right) = \int_\Omega (\log\eta)_x \left(\frac{\sigma}{\mu}\right)_x dx = -\int_\Omega (\log\eta)_x \frac{\mu'}{\mu^2}\theta_x\sigma\ dx + \int_\Omega (\log\eta)_x\ \frac{\sigma_x}{\mu}\ dx.$$

Then using Cauchy-Schwarz inequality together with Lemma 5, we find

$$\frac{d}{dt}\left(\frac{1}{2}\int_\Omega [(\log\eta)_x]^2\,dx\right) \leqslant \sup_\Omega \sigma^2 \int_\Omega [(\log\eta)_x]^2\ dx + C\int_\Omega \theta_x^2 dx$$

$$+\frac{1}{2}\int_\Omega r^2\sigma_x^2\ dx + \frac{1}{2}\int_\Omega [(\log\eta)_x]^2\,dx.$$

As, after Corollary 1, $\sup_\Omega \sigma^2(\cdot,t) \in L^1(0,T)$ for any $T>0$, this implies (47) by applying Gronwall's lemma $\square$

Proposition 4 *For any $T>0$, the following uniform bounds hold*

$$\max_{[0,T]} \|v_t\|_{L^2(\Omega)} \leqslant C, \quad \max_{[0,T]} \|\theta_t\|_{L^2(\Omega)} \leqslant C, \tag{48}$$

$$\|(r^2 v)_{xt}\|_{L^1(0,T;L^2(\Omega))} \leqslant C, \quad \|\theta_{xt}\|_{L^1(0,T;L^2(\Omega))} \leqslant C, \tag{49}$$

and the (non uniform) ones

$$\max_{[0,T]} \|(r^2 v)_{xx}\|_{L^2(\Omega)} \leqslant C(T), \quad \max_{[0,T]} \|\theta_{xx}\|_{L^2(\Omega)} \leqslant C(T). \tag{50}$$

Proof:

1. The first equation (3) rewrites

$$w_t = r^4\left(\frac{\mu}{\eta}\ w_x\right)_x + \frac{2w^2}{r^3},$$

with $w := r^2 v$.

We derivate formally this equation with respect to t (this can be made rigorous by taking finite difference and passing to the limit (see [1])), multiply by w_t and integrate by parts in x

$$\frac{d}{dt}\left(\int_\Omega \frac{1}{2}w_t^2\ dx\right) + \int_\Omega r^4\frac{\mu}{\eta}\ w_{xt}^2\ dx = \int_\Omega 4rww_t\left(\frac{\mu}{\eta}\ w_x\right)_x\ dx + \int_\Omega 4r\mu w_t w_{xt}\ dx$$

$$+\int_\Omega r^4\frac{\mu'}{\eta}\ \theta_t w_{xt}\ w_x\ dx+\int_\Omega r^4\frac{\mu}{\eta^2}\ w_x^2 w_{xt}\ dx-\int_\Omega 4r\mu'\theta_t\ w_t\ w_x\ dx+\int_\Omega 4r\frac{\mu}{\eta}\ w_t w_x^2\ dx$$

$$+\int_\Omega \frac{4}{r^3}\ w w_t^2\ dx-\int_\Omega \frac{6}{r^6}\ w^3 w_t\ dx=:\sum_{j=1}^{8} D_j.$$

Let us estimate all of these terms.

$$|D_1|\leqslant C\int_\Omega |w w_t\sigma_x|\ dx\leqslant C\int_\Omega w^2 w_t^2\ dx+\int_\Omega r^4\sigma_x^2\ dx$$

$$\leqslant C\max_\Omega v^2\int_\Omega w_t^2\ dx+\int_\Omega r^4\sigma_x^2\ dx.$$

$$|D_2|\leqslant C\int_\Omega |w_t w_{xt}\ dx\leqslant \frac{\epsilon}{3}\int_\Omega r^4\frac{\mu}{\eta}\ w_{xt}^2\ dx+C\int_\Omega w_t^2\ dx.$$

$$|D_3|\leqslant C\int_\Omega |\theta_t w_{xt} w_x|\ dx\leqslant \frac{\epsilon}{3}\int_\Omega r^4\frac{\mu}{\eta}\ w_{xt}^2\ dx+C\int_\Omega \theta_t^2\ dx,$$

where we used Proposition 3.

$$|D_4|\leqslant C\int_\Omega w_x^2|w_{xt}|\,dx\leqslant C\max_\Omega w_x^2\int_\Omega |w_{xt}|\,dx\leqslant C\max_\Omega w_x^2\left(1+\int_\Omega w_{xt}^2\ dx\right).$$

$$|D_5|\leqslant C\int_\Omega |w_t\theta_t w_x|\ dx\leqslant \frac{C}{2}\left(\int_\Omega w_t^2\ dx+\int_\Omega \theta_t^2\ dx\right),$$

where we used Proposition 3.

$$|D_6|\leqslant C\int_\Omega |w_t|w_x^2\ dx\leqslant C\max_\Omega w_x^2\int_\Omega |w_t|\ dx\leqslant C\max_\Omega w_x^2\left(1+\int_\Omega w_t^2\ dx\right).$$

$$|D_7|\leqslant C\int_\Omega w_t^2|w|\ dx\leqslant C\max_\Omega |v^0|\int_\Omega w_t^2\ dx.$$

$$|D_8|\leqslant C\int_\Omega |w_t w^3|\ dx\leqslant C(\max_\Omega |v^0|)^3\int_\Omega w_t^2\ dx.$$

So finally

$$\frac{d}{dt}\left(\int_\Omega \frac{1}{2}w_t^2\ dx\right)+\int_\Omega r^4\frac{\mu}{\eta}\ w_{xt}^2\ dx\leqslant f(t)+g(t)\int_\Omega \left(w_t^2+\theta_t^2\right)\ dx, \qquad (51)$$

where $f,g\in L^1(0,T)$, for any $T>0$.

2. We derivate formally the third equation (3) with respect to t (this can be made rigorous as previously), and multiply by θ_t

$$\left(\frac{1}{2}\ \theta_t^2\right)_t = \theta_t q_{xt} + \theta_t \left(\frac{\mu}{\eta}\ w^2\right)_t .$$

Integrating by parts in x, we get

$$\frac{d}{dt}\int_\Omega \frac{1}{2}\theta_t^2\ dx + \int_\Omega \kappa\frac{r^4}{\eta}\ \theta_{xt}^2\ dx - \int_\Omega \frac{\mu'}{\eta}\ \theta_t^2 w^2\ dx = -\int_\Omega \frac{r^4\kappa'}{\eta}\ \theta_x\theta_t\theta_{xt}\ dx$$

$$-\int_\Omega \frac{4r\kappa}{\eta}\ w\theta_x\theta_{xt}\ dx + \int_\Omega \frac{r^4\kappa}{\eta^2}\ w_x\theta_x\theta_{xt}\ dx - \int_\Omega \frac{\mu}{\eta^2}\ w^2 w_x\theta_t\ dx + \int_\Omega \frac{2\mu}{\eta}\ ww_t\theta_t\ dx =: \sum_{j=1}^5 E_j.$$

Let us estimate all of these terms.

$$|E_1| \leqslant C\int_\Omega |\theta_x\theta_t\theta_{xt}|\ dx \leqslant \frac{\epsilon}{3}\int_\Omega \frac{r^4\kappa}{\eta}\theta_{xt}^2\ dx + C\int_\Omega \theta_x^2\theta_t^2\ dx$$

$$\leqslant \frac{\epsilon}{3}\int_\Omega \frac{r^4\kappa}{\eta}\theta_{xt}^2\ dx + C\max_\Omega \theta_x^2 \int_\Omega \theta_t^2\ dx.$$

$$|E_2| \leqslant C\int_\Omega |w\theta_x\theta_{xt}|\ dx \leqslant \frac{\epsilon}{3}\int_\Omega \frac{r^4\kappa}{\eta}\theta_{xt}^2\ dx + C\int_\Omega v^2\theta_x^2\ dx$$

$$\leqslant \frac{\epsilon}{3}\int_\Omega \frac{r^4\kappa}{\eta}\theta_{xt}^2\ dx + C\max_\Omega \theta_x^2.$$

$$|E_3| \leqslant C\int_\Omega |w_x\theta_x\theta_{xt}|\ dx \leqslant \frac{\epsilon}{3}\int_\Omega \frac{r^4\kappa}{\eta}\theta_{xt}^2\ dx + C\int_\Omega w_x^2\theta_x^2\ dx$$

$$\leqslant \frac{\epsilon}{3}\int_\Omega \frac{r^4\kappa}{\eta}\theta_{xt}^2\ dx + C\max_\Omega \theta_x^2\ \max_{[0,T]}\int_\Omega w_x^2\ dx.$$

$$|E_4| \leqslant C\int_\Omega |w^2\theta_t w_x|\ dx \leqslant -\epsilon\int_\Omega \frac{\mu'}{\eta} w^2\theta_t^2\ dx + C\int_\Omega w^2 w_x^2\ dx$$

$$\leqslant -\epsilon\int_\Omega \frac{\mu'}{\eta} w^2\theta_t^2\ dx + C\int_\Omega w^2\ dx,$$

after Proposition 3.

$$|E_5| \leqslant C\int_\Omega |w\theta_t w_t|\ dx \leqslant C\int_\Omega \left(w_t^2 + \theta_t^2\right)\ dx.$$

Finally, collecting all of the previous estimates, we get

$$\frac{d}{dt}\int_\Omega \frac{1}{2}\theta_t^2\ dx + \int_\Omega \kappa\frac{r^4}{\eta}\ \theta_{xt}^2\ dx + \int_\Omega \frac{\mu'}{\eta}\ \theta_t^2 w^2\ dx \leqslant g(t)\left(1+\int_\Omega \left(w_t^2+\theta_t^2\right)\ dx\right), \tag{52}$$

where $g \in L^1(0,T)$, for any $T>0$.

Summing (51) and (52), we have

$$\frac{d}{dt}\int_\Omega \frac{1}{2}\left(w_t^2+\theta_t^2\right)\ dx + \int_\Omega \left(w_{xt}^2+\theta_{xt}^2\right)\ dx \leqslant g(t)\left(1+\int_\Omega \left(w_t^2+\theta_t^2\right)\ dx\right), \tag{53}$$

which implies estimates (48) by Gronwall's Lemma. Bounds (49) then follows.

3. The second equation (3) rewrites

$$(r^2v)_{xx} = \frac{\eta}{r^2\mu}\ v_t + \frac{\mu'}{\mu}\ \theta_x(r^2v)_x - \frac{1}{\eta}\ \eta_x(r^2v)_x.$$

Taking the square and integrating on Ω, we get

$$\int_\Omega (r^2v)_{xx}^2\ dx \leqslant C\int_\Omega \left(v_t^2+\theta_x^2[(r^2v)_x]^2+\eta_x^2[(r^2v)_x]^2\right)\ dx.$$

$$\leqslant C\int_\Omega v_t^2\ dx + C\max_\Omega[(r^2v)_x]^2\ \int_\Omega \left(\theta_x^2+\eta_x^2\right)\ dx.$$

So

$$\int_\Omega (r^2v)_{xx}^2\ dx \leqslant C\int_\Omega v_t^2\ dx + C(T)\max_\Omega[(r^2v)_x]^2, \tag{54}$$

after Corollary 1 and Proposition 3. But

$$|(r^2v)_x| \leqslant \int_\Omega |(r^2v)_{xx}|\ dx,$$

then

$$[(r^2v)_x]^2 \leqslant C + \frac{\epsilon}{2}\int_\Omega [(r^2v)_{xx}]^2\ dx.$$

Plugging into (54) and taking $\epsilon>0$ small enough gives the first estimate (50).

The third equation (3) rewrites

$$\theta_{xx} = -\frac{\eta\kappa'}{\kappa}\ \theta_x^2 + \frac{4\eta}{r^3}\ \theta_x - \frac{\mu}{\kappa r^4}[(r^2v)_x]^2 + \frac{\mu}{\kappa r^4}\theta_t + \frac{1}{\eta}\ \eta_x\theta_x.$$

Taking the square and integrating on Ω, we get

$$\int_\Omega \theta_{xx}^2 \, dx \leqslant C \int_\Omega \left(\theta_x^4 + \theta_x^2 + [(r^2v)_x]^4 + \theta_t^2 + \eta_x^2\theta_x^2 \right) \, dx.$$

Using the inequality $[(r^2v)_x]^4 \leqslant 4 \int_\Omega [(r^2v)_x]^2 \, dx \cdot \int_\Omega [(r^2v)_{xx}]^2 \, dx$, and Corollary 1, together with Proposition 3 and the first bound (50), we can bound the right-hand side, which provide us with the last estimate (50) □

Proof of Theorem 3

1. From the proof of Lemmal3 we have

$$|\eta(x,t) - \eta(x,t')| \leqslant |t-t'|^{1/2} \left(\int_0^T [(r^2v)_x]^2 \, dt \right)^{1/2}$$

$$\leqslant C|t-t'|^{1/2} \left(\int_0^T \int_\Omega r^4\sigma_x^2 \, dx \, dt \right)^{1/2} \leqslant C|t-t'|^{1/2}.$$

After Proposition 3

$$|\eta(x,t) - \eta(x',t)| \leqslant C|x-x'|^{1/2} \left(1 + \int_\Omega \eta_x^2 \, dx \right) \leqslant C|x-x'|^{1/2},$$

so we find that $\eta \in C^{1/2,1/4}(Q_T)$.

2. From the proof of Lemmal3 we have

$$|\theta(x,t) - \theta(x,t')| \leqslant |t-t'|^{1/2} \left(\int_0^T \theta_t^2 \, dt \right)^{1/2}$$

$$\leqslant C|t-t'|^{1/2} \left(\int_0^T \int_\Omega 2|\theta_t\theta_{xt} \, dx \, dt \right)^{1/2} \leqslant C|t-t'|^{1/2}.$$

After Propositions 3 and 4

$$|\eta(x,t) - \eta(x',t)| \leqslant C|x-x'|^{1/2} \left(T \cdot \max_{[0,T]} \int_\Omega \theta_t^2 \, dx + \int_0^T \int_\Omega \theta_{xt}^2 \, dx \right) \leqslant C|x-x'|^{1/2},$$

so we find that $\theta \in C^{1/2,1/4}(Q_T)$. As we have also after Propositions 4

$$|\theta_x(x,t) - \theta_x(x',t)| \leqslant |x-x'|^{1/2} \left(\int_\Omega \theta_{xx}^2 \, dt \right)^{1/2} \leqslant |x-x'|^{1/2},$$

we deduce as in [21], using an interpolation argument of [22], that $\theta_x \in C^{1/3,1/6}(Q_T)$.

The same arguments holding verbatim for r^2v and $(r^2v)_x$, we have that $v, v_x \in C^{1/3,1/6}(Q_T)$, which ends the proof □

3 Existence and uniqueness of solutions

To complete the proof of strong solution locally in time we apply the idea of Dafermos and Hisao [6] together with using the crucial Theorem 3. To get the existence of weak solution we apply method of [14].

3.1 Proof of existence

Theorem 4 *Let the conditions on the data*

$$v_0, \theta_0 \in C^{2+\nu}(\Omega),\ \ \eta_0 \in C^{1+\nu}(\Omega)\ \textit{with}\ \nu = 1/3,$$

$$\inf_{\Omega} \eta_0(x) > 0, \inf_{\Omega} \theta_0(x) > 0,$$

and the following extra condition of compatibility

$$v_0|_{x=0,M} = 0,$$

be satisfied.

The system of equations (1) together with conditions (3)-(7), where r is defined in (2) then for $\bar{t} \in (0, \infty)$, has a solution v, η, θ such that

$$v, \theta \in C^{2+\nu, 1+\frac{\nu}{2}}(\Omega \times (0, T^*)),\ \ \rho \in C^{1+\nu, 1+\frac{\nu}{2}}(\Omega \times (0, T^*)).$$

Proof:
We can rewrite our system (3) by the following way

$$\begin{aligned} w_t &= a_1(x,t) w_{xx} + b_1 w_x + c_1(x,t) \\ \theta_t &= a_2(x,t)\theta_{xx} + b_2(x,t)\theta_x + C_2(x,t) \\ \eta_t &= w_x, \end{aligned} \tag{55}$$

where

$$\begin{aligned} & w = r^2 v \\ & a_1(x,t) = r^4 \tfrac{\mu}{\eta} \\ & b_1(x,t) = r^4 \left(\tfrac{\mu' \theta_x}{\eta} - \tfrac{\mu \eta_x}{\eta^2} \right) \\ & c_1(x,t) = -\tfrac{2}{r^3} w^2 \\ & a_2(x,t) = r^4 \tfrac{\kappa}{\eta} \\ & b_2(x,t) = \tfrac{\kappa' \theta_x}{r}^4 \eta + 4 r \kappa + r^4 \tfrac{\kappa \eta_x}{\eta^2} \\ & c_2(x,t) = \tfrac{\mu}{\eta} (w_x)^2. \end{aligned} \tag{56}$$

From Theorem 3, it follows that

$$\begin{aligned} &\|a_i\|_{C^{1/3,1/6}} \leqslant N_1, \ \ \|c_i\|_{C^{1/3,1/6}} \leqslant N_2, \\ &\|b_i\|_{C^{1/3,1/6}} \leqslant N_3 + N_4\|\eta_x\|_{C^{1/3,1/6}}, \text{ for } i = 1, 2. \end{aligned} \tag{57}$$

Applying the Schauder estimates to the solutions $(55)_{1,2}$ gives

$$\begin{aligned} \|u\|_{C^{2+1/3,1+1/6}} &\leqslant N_5 + N_6\|\eta_x\|_{C^{1/3,1/6}}, \\ \|\eta\|_{C^{2+1/3,1+1/6}} &\leqslant N_7 + N_8\|\eta_x\|_{C^{1/3,1/6}}. \end{aligned} \tag{58}$$

Derivating $(55)_3$ with respect to x and integrating over $(0, T^*)$, $T^* < 1$ with respect to t, we get

$$\|\eta_x\|_{C^{1/3,1/6}} \leqslant N_9 T_*^{1-1/6}\|w_{xx}\|_{C^{1/3,1/6}} + N_{10}. \tag{59}$$

All of the previous estimates give us the following

$$\begin{aligned} \|w\|_{C^{2+1/3,1+1/6}(Q_{T^*})} &\leqslant N_{11}, \\ \|\theta\|_{C^{2+1/3,1+1/6}(Q_{T^*})} &\leqslant N_{12}, \end{aligned} \tag{60}$$

where $N_i,\ i = 1, ...12$ are constants.

From the previous arguments and a priori estimates, we know that there exist subsequences $(v_k, \eta_k, \theta_k, r_k)$ such that

- $v_k \to v$ in $L^p(0, T^*, C^0(\Omega))$ strongly and in $L^p(0, T^*, H^1(\Omega))$, weakly for any $1 < p < \infty$,
- $v_k \to v$ a.e. in $\Omega \times (0, T^*)$ and in $L^\infty(0, T^*, L^4(\Omega))$ * weakly,
- $(v_k)_t \to v_t$ in $L^2(0, T^*, L^2(\Omega))$ weakly,
- $\theta_k \to \theta$ in $L^2(0, T^*, C^0(\Omega))$ strongly and in $L^2(0, T^*, H^1\Omega)$ weakly,
- $\theta_k \to \theta$ a.e. in $\Omega \times (0, T^*)$ and in $L^\infty(0, T^*; L^2(\Omega))$ weakly,
- $r_k \to r$ in $C^0(\Omega \times (0, T^*))$,
- $r^2(\frac{\mu}{\eta_k}(r^2 v_k)_x)$ converge to A_1 in $L^2(0, T^*, H^1(\Omega))$ weakly,
- $\frac{\kappa(\eta,\theta)r^4}{\eta}(\theta_k)_x \to A_2$ in $L^2(0, T^*, L^2(\Omega))$ weakly,
- $\frac{\mu}{\bar{\eta}}\partial_x(r^2 u_k) \to A_3$ in $L^\infty(0, \bar{t}, L^2(\Omega))$ weakly *,

- $\frac{\kappa(\theta)r^4}{\underline{\eta}}\theta_x$ converge to A_4 in $L^2(0,T^*;L^2(\Omega))$ weakly.

After the definition of $r(x,t)$, one has

$$r(x,t) = r_0(x) + \int_0^t v(x,t')dt' \text{ a. e. } \Omega \times (0,T^*),$$

then

$$r_k(x,t) - r_k(y,t) = (\int_y^x \eta_k(s,t)ds)^{1/3}$$
$$\geqslant \epsilon(x-y) \ \bigvee(x,y,t) \in \Omega \times (0,x) \times (0,T^*).$$

Then from the previous computations we get

$$r(x,t) - r(y,t) \geqslant \epsilon(x-y) \ \bigvee(x,y,t) \in \Omega \times (0,x) \times (0,T^*),$$

and finally

$$f_k r_k \to f\ r \text{ in } C^0(\Omega \times (0,T^*)).$$

Moreover, it implies that

- $\eta_k \to \eta$ a.e. in $\Omega\times(0,T^*)$ and $L^s(\Omega\times(0,T^*))$ strongly for all $s \in (1,\infty)$,
- $A_1 = (\frac{\mu}{\eta}(r^2v)_x)$ in $L^2(0,T^*;H^1(\Omega))$,
- $A_2 = \frac{\kappa(\eta,\theta)r^4}{\eta}\theta_x$ in $L^2(0,T^*,L^2(\Omega))$,
- $A_3 = \frac{\mu}{\bar{\eta}}(r^2v)_x$ in $L^\infty(0,T^*,L^2(\Omega))$,
- $A_4 = \frac{\kappa r^4}{\bar{\eta}}(r^2v)_x$ in $L^\infty(0,T^*,L^2(\Omega))$.

So we can pass to the limit in the weak formulation of $(1)_2$ and $(1)_3$, and we get a weak solution of (3).

3.2 Proof of uniqueness

Let $\eta_i, v_i, \theta_i,\ i = 1,2$ be two solutions of (3), and let us consider the differences: $\eta = \eta_1 - \eta_2$, $\theta = \theta_1 - \theta_2$ and $v = v_1 - v_2$.

The following auxiliary result holds

Proposition 1

$$|r_2^m - r_1^m| \leqslant c\int_\Omega (\eta_2 - \eta_1)dx.$$

Proof. from the definition of $r(x,t)$, we see that

$$r_2^m - r_1^m = (r_2^4)^{m/2} - (r_1^3)^{m/3}$$
$$= \tfrac{m}{3} r_*^{m-3}(r_2^3 - r_1^3) = \tfrac{m}{3} r_*^{m-3} 3\int_0^x (\eta_2 - \eta_1)ds \leqslant c\int_0^1 (\eta_2 - \eta_1)dx,$$

where

$$1 \leqslant r_k \leqslant c, r_* = r_1 + \epsilon(r_2 - r_1) \quad \square$$

Now, we subtract $(3)_2$ for η_2, w_2, θ_2 from $(3)_2$ for η_1, w_1, θ_1 ($w_1 = r_1^2 v_1, w_2 = r_2^2 v_2$) in order to get

$$\int_\Omega (w_2 - w_1)_t \phi \, dx = -\Big\{ \int_\Omega \{(r_2^4 - r_1^4)\tfrac{\mu_1}{\eta_1} w_{1x} + r_2^4 \tfrac{\eta_1\mu_2 - \mu_1\eta_2}{\eta_2\eta_1} w_{1x}\}\phi_x dx \Big\} +$$
$$-\Big\{ \int_\Omega \{r_2^4(\tfrac{\mu_2}{\eta_2}(w_2 - w_1)_x) + \tfrac{2}{r_2^3}(w_2^2 - w_1^2) + 2(\tfrac{(r_1^3 - r_2^3)}{r_2^3 r_1^3} w_1^2\}\phi_x dx \Big\}. \tag{61}$$

Setting $\phi = w_2 - w_1$ we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega w^2 dx + \int_\Omega r_2^4 \frac{\mu_2}{\eta_2}(w_x)^2 = -\sum_{i=1}^4 I_i, \tag{62}$$

where

- $I_1 = \int_\Omega \{(r_2^4 - r_1^4)\frac{\mu_1}{\eta_1} w_{1x} w_x dx,$
- $I_2 = \int_\Omega r_2^4 \frac{\eta_1\mu_2 - \mu_1\eta_2}{\eta_2\eta_1} w_{1x} w_x dx,$
- $I_3 = \int_\Omega \frac{2}{r_2^3}(w_2^2 - w_1^2) w_x dx,$
- $I_4 = \int_\Omega 2(\frac{(r_1^3 - r_2^3)}{r_2^3 r_1^3} w_1^2 w_x dx.$

Then it follows that

$$\tfrac{1}{2}\tfrac{d}{dt}\int_\Omega |w^2| dx + \int_\Omega |r_2^4 \tfrac{\mu_2}{\eta_2}(w_x)^2| dx \leqslant$$
$$\leqslant \quad c\Big(\|\eta\|_2(\|(w_1)_x\|_2 + \|(w_1)_x\|_2 + \|(w_1)_{xx}\|_2)\|w_x\|_2 + (\|(w_1)_x\|_2 + \|(w_2)_x\|_2)\|w\|_2\|w_x\|_2\Big), \tag{63}$$

where c is a constant.

Now substracting $(3)_3$ for η_2, w_2, θ_2 from $(3)_3$ for η_1, w_1, θ_1 ($w_1 = r_1^2 v_1, w_2 = r_2^2 v_2$) in order to get

$$\int_\Omega c_v(\theta_2-\theta_1)_t\psi dx = -\Big\{ \int_\Omega\{\tfrac{\kappa(\theta_2)r_2}{\eta_2}(\theta_2-\theta_1)_x + \tfrac{\kappa(\theta_2)r_2}{\eta_1\eta_2}(\eta_1-\eta_2)(\theta_1)_x\}\psi_x dx +$$
$$\int_\Omega\{\tfrac{\kappa(\theta_2)}{\eta_1}(r_2-r_1)(\theta_1)_x + \tfrac{(\kappa(\theta_2)-\kappa(\theta_1))r_1}{\eta_1}(\theta_1)_x\}\psi_x dx\}+$$
$$+\int_\Omega\{\tfrac{\mu_2}{\eta_2}((w_2-w_1)_x(w_1+w_2)_x + \tfrac{\mu_2(\eta_1-\eta_2)}{\eta_2\eta_1}(w_1)_x^2 + \tfrac{\mu_2-\mu_1}{\eta_1}(w_1)_x^2\}\psi dx. \tag{64}$$

Setting $\psi = \theta_2 - \theta_1$ we get the following estimate

$$\tfrac{d}{dt}\tfrac{1}{2}\int_\Omega |\theta|^2 + dx\int_\Omega \tfrac{\kappa(\theta_2)r_2}{\eta-2}(\theta-x)^2 dx \leqslant \sum_{i=1}^6 |J_i|, \tag{65}$$

where

- $J_1 = \int_\Omega \frac{\kappa(\theta_2)r_2}{\eta_1\eta_2}(\eta_1-\eta_2)(\theta_1)_x\}\theta_x dx$
- $J_2 = \int_\Omega \frac{\kappa(\theta_2)}{\eta_1}(r_2-r_1)(\theta_1)_x\theta_x dx$
- $J_3 = \int_\Omega \frac{(\kappa(\theta_2)-\kappa(\theta_1))r_1}{\eta_1}(\theta_1)_x\theta_x dx$
- $J_4 = \int_\Omega \frac{\mu_2}{\eta_2}((w_2-w_1)_x(w_1+w_2)_x\theta dx$
- $J_5 = \int_\Omega \frac{\mu_2(\eta_1-\eta_2)}{\eta_2\eta_1}(w_1)_x^2\theta dx$
- $J_6 = \int_\Omega \frac{\mu_2-\mu_1}{\eta_1}(w_1)_x^2\theta dx$

Assuming that $\mu \in C^2(R^+)$ then

$$\tfrac{d}{dt}\tfrac{1}{2}\int_\Omega |\theta|^2 dx + \int_\Omega \tfrac{\kappa(\theta_2)r_2}{\eta-2}(\theta_x)^2 dx \leqslant \{d_1\|\eta\|_2\|\|(\theta_1)_{xx}\|_2 + d_2\|\eta\|_2\|(\theta_1)_x\|_2 + d_3\|\theta\|_2\|(\theta_1)_{xx}\|_2\}\|\theta_x\|_2 +$$
$$\{d_4\|w_x\|_2(\|(w_2)_x\|_2 + \|(w_1)_x\|_2) + d_5\|\eta\|_2\|(w_1)_{xx}\|_2 + d_6\|\eta\|_2\|(w_1)_{xx}\|_2\}\|\theta\|_2, \tag{66}$$

where d_i, $i = 1,..6$ are constants. From continuity equation it follows that

$$\tfrac{d}{dt}\|\eta\|_2^2 \leqslant \|w_x\|_2\|\eta\|_2. \tag{67}$$

Finally, $w_2 - w_1 = r^2(v_2 - v_1) + (r_2^2 - r_1^2)v_1$ and using $(55)_2$ it implies

$$\tfrac{1}{2}\tfrac{d}{dt}\int_\Omega |v^2|dx + \int_\Omega |r_2^2 \tfrac{\mu_2}{\eta_2}(r_2^2 v_x)^2|dx \leqslant D\|v^2\|_2 \qquad (68)$$

Putting together previous estimates it implies the uniqueness of the problem.

4 Asymptotic behaviour

We partially use the technique developped in [10].

Lemma 6 *There exists a positive function* $\Phi \in L^1(\mathbb{R}_+)$ *such that*

$$\frac{d}{dt}\int_\Omega \left(\frac{1}{2}\ v^2 + \theta\right)^2 dx \leqslant \Phi(t). \qquad (69)$$

Proof: Multiplying the second equation (3) by v, adding to the third equation (3), multiplying the result by the energy $\frac{1}{2}\ v^2 + \theta$ and integrating on Ω, we get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \left(\frac{1}{2}\ v^2 + \theta\right)^2 dx = \int_\Omega (q + r^2 v\sigma)_x \left(\frac{1}{2}\ v^2 + \theta\right)\ dx.$$

Integrating by parts

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \left(\frac{1}{2}\ v^2 + \theta\right)^2 dx + \int_\Omega \frac{\kappa r^4}{\eta}\ \theta_x^2\ dx + \int_\Omega \frac{\mu}{\eta}\ v^2[(r^2 v)_x]^2 dx$$

$$= -\int_\Omega q v v_x\ dx + 2\int_\Omega \mu\frac{v^3}{r}\ (r^2 v)_x\ dx - \int_\Omega \sigma\theta_x r^2 v\ dx =: \sum_{j=1}^{3} F_j.$$

Let us majorize the right-hand side.

By using Cauchy-Schwarz

$$|F_1| \leqslant C\int_\Omega \frac{\kappa r^4}{\eta}\ \theta_x^2\ dx + C\int_\Omega v_x^2\ dx$$

$$\leqslant C\int_\Omega \frac{\kappa r^4}{\eta}\ \theta_x^2\ dx + C\int_\Omega \frac{\mu}{\eta}\ [(r^2 v)_x]^2 dx + C\max_\Omega v_x^2,$$

and finally

$$|F_1| \leqslant C \int_\Omega \frac{\kappa r^4}{\eta} \ \theta_x^2 \ dx + C \int_\Omega \frac{\mu}{\eta} \ [(r^2 v)_x]^2 dx.$$

$$|F_2| \leqslant C \int_\Omega \frac{\mu}{\eta} \ [(r^2 v)_x]^2 dx + C \int_\Omega \mu \eta v^6 \ dx \leqslant C \int_\Omega \frac{\mu}{\eta} \ [(r^2 v)_x]^2 dx + C \max_\Omega (r^2 v)^2.$$

Then

$$|F_2| \leqslant C \int_\Omega \frac{\mu}{\eta} \ [(r^2 v)_x]^2 dx.$$

Finally

$$|F_3| \leqslant C \int_\Omega |v \sigma \theta_x| \ dx \leqslant C \int_\Omega \frac{\kappa r^4}{\eta} \ \theta_x^2 \ dx + C \int_\Omega \frac{\mu}{\eta} \ [(r^2 v)_x]^2 dx.$$

Applying Lemma 2 to these bounds ends the proof □

Theorem 5 *The solution of the problem (3)(4)(5) has the following properties*

1. *There exist a constant K_v depending only of the physical data of the problem and the initial data such that for any $t > 0$*

$$\|v(\cdot, t)\|_{L^2(\Omega)} \leqslant K_v e^{-\lambda_v t}, \tag{70}$$

   where $\lambda_v = \frac{2R_0^4 \mu(\underline{\theta})}{M^2 \overline{\eta}}$.
   Moreover when $t \to \infty$

$$\|v(\cdot, t)\|_{C(\Omega)} \to 0, \tag{71}$$

2. *When $t \to \infty$*

$$\|\theta(\cdot, t) - \theta_\infty\|_{C(\Omega)} \to 0, \tag{72}$$

   where $\theta_\infty = \frac{1}{M} \int_\Omega \left(\frac{1}{2} \ (v^0)^2 + \theta^0 \right) \ dx$.

3. *When $t \to \infty$*

$$\|\eta_t(\cdot, t)\|_{L^2(\Omega)} \to 0, \tag{73}$$

Proof:

1. From Lemma 1, 4 and 5

$$\frac{d}{dt}\int_\Omega v^2\ dx + \frac{2\mu(\underline{\theta})}{\overline{\eta}}\int_\Omega [(r^2v)_x]^2 dx \leqslant 0.$$

As $|r^2 v| \leqslant \int_\Omega |(r^2v)_x| dx$, we get

$$\int_\Omega [(r^2v)_x]^2 dx \geqslant \frac{R_0^4}{M^2}\ \int_\Omega v^2\ dx,$$

so

$$\frac{d}{dt}\int_\Omega v^2\ dx + K_v \int_\Omega v^2\ dx \leqslant 0,$$

which gives (70).

After Lemma 3, we know that

$$t \to \frac{d}{dt}\int_\Omega \frac{\mu}{\eta}\ [(r^2v)_x]^2\ dx \in L^1(\mathbb{R}_+),$$

which implies that $\|v(\cdot,t)\|_{H^1(\Omega)} \to 0$ and then (71).

2. Revisiting the proof of Lemma 6, we get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \left(\frac{1}{2}\ v^2 + \theta - \theta_\infty\right)^2 dx + \int_\Omega \frac{\kappa r^4}{\eta}\ \theta_x^2\ dx + \int_\Omega \frac{\mu}{\eta}\ v^2[(r^2v)_x]^2 dx$$

$$= -\int_\Omega q v v_x\ dx + 2\int_\Omega \mu\frac{v^3}{r}\ (r^2v)_x\ dx - \int_\Omega \sigma\theta_x r^2 v\ dx =: \sum_{j=1}^{3} F_j.$$

First we observe, after (70) and (71), we see that

$$F(t) := \int_\Omega \frac{\mu}{r^4\eta}\ [(r^2v)_x]^2 dx \leqslant C\int_\Omega [v^2 + v_x^2]\ dx \to 0,$$

as $t \to \infty$.

By using Cauchy-Schwarz

$$|F_1| \leqslant \frac{1}{3}\ \epsilon\int_\Omega \frac{\kappa r^4}{\eta}\ \theta_x^2\ dx + C_\epsilon\int_\Omega v_x^2\ dx.$$

$$|F_2| \leqslant F(t) + C\int_\Omega v^6\ dx \leqslant F(t) + C\max_\Omega v^4.$$

But as $v^2 \leqslant \int_\Omega 2|vv_x|\ dx \leqslant C\left(\int_\Omega v_x^2\ dx\right)^{1/2}$, we have

$$|F_2| \leqslant F(t) + C\int_\Omega v_x^2\ dx.$$

Finally

$$|F_3| \leqslant \frac{1}{3}\ \epsilon \int_\Omega \frac{\kappa r^4}{\eta}\ \theta_x^2\ dx + C_\epsilon F(t).$$

Collecting all of these bounds we find

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \left(\frac{1}{2}\ v^2 + \theta - \theta_\infty\right)^2 dx + \int_\Omega \frac{\kappa r^4}{\eta}\ \theta_x^2\ dx + \int_\Omega \frac{\mu}{r^4\eta}\ [(r^2 v)_x]^2 dx \leqslant G(t), \tag{74}$$

where $G(t) \to 0$, as $t \to \infty$.

Now integrating with respect to y the equality $\theta(x,t) - \theta(y,t) = \int_y^x \theta_x\ dx$, we get

$$\theta(x,t) - \theta_\infty \leqslant M\left(\int_\Omega \frac{\kappa r^4}{\eta}\theta_x\ dx\right)^{1/2},$$

which implies

$$\int_\Omega (\theta - \theta_\infty)^2\ \ dx \leqslant \frac{M^2\overline{\eta}}{R_0^4 \kappa(\underline{\theta})}\ \int_\Omega \frac{\kappa r^4}{\eta}\theta_x\ dx.$$

The left-hand side of (74) rewrites

$$\frac{1}{8}\frac{d}{dt}\int_\Omega v^4\ dx + \frac{1}{2}\frac{d}{dt}\int_\Omega v^2\left(\theta - \theta_\infty\right)\ dx + \frac{1}{2}\frac{d}{dt}\int_\Omega \left(\theta - \theta_\infty\right)^2\ dx.$$

Multiplying the second equation (3) by v^3 and integrating by parts, we have

$$\frac{d}{dt}\int_\Omega v^4\ dx = -4\int_\Omega (r^2 v^3)_x \sigma\ dx,$$

which gives

$$\left|\frac{d}{dt}\int_\Omega v^4\ dx\right| \leqslant 4\int_\Omega r^2 |v^3 \sigma_x|\ dx\ dx \leqslant C\int_\Omega \left(v^2 + |v_x| + v_x^2\right)\ dx,$$

then using (70) and (71), we have

$$\left|\frac{d}{dt}\int_\Omega v^4\ dx\right| \to 0,$$

as $t \to \infty$.

In the same stroke, multiplying the second equation (3) by $v\theta$ and integrating by parts, we have

$$\frac{d}{dt}\int_\Omega v^2\theta\ dx = \int_\Omega \left(-2vv_x q + \sigma v^2(r^2 v)_x - 2(r^2 v\theta)_x\sigma\right)\ dx.$$

Then

$$\left|\frac{d}{dt}\int_\Omega v^2\theta\ dx\right| \leqslant \frac{1}{3}\ \epsilon \int_\Omega \frac{\kappa r^4}{\eta}\theta_x\ dx + H(t).$$

Collecting all of the previous estimates, we get finally

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (\theta-\theta_\infty)^2\,dx + \frac{R_0^4\kappa(\underline{\theta})}{M^2\overline{\eta}}\int_\Omega (\theta-\theta_\infty)^2\,dx \leqslant \Psi(t), \tag{75}$$

where $\Psi \in L^1(\mathbb{R}_+)$ and $\Psi(t) \to 0$ as $t \to \infty$. Integrating this differential inequality, we get

$$\int_\Omega (\theta-\theta_\infty)^2\,dx \leqslant e^{-\frac{R_0^4\kappa(\underline{\theta})}{M^2\overline{\eta}}\ t}\int_\Omega \left(\theta^0-\theta_\infty\right)^2 dx + \int_0^t e^{-\frac{R_0^4\kappa(\underline{\theta})}{M^2\overline{\eta}}\ (t-s)}\Psi(s)\ ds.$$

As the last integral converges to zero when $t \to \infty$ due to the dominated convergence theorem, we get that $\|\theta(\cdot,t)-\theta_\infty\|_{L^2(\Omega)} \to 0$.

After Lemma 3, we know that

$$t \to \frac{d}{dt}\int_\Omega \frac{r^4\mathcal{K}_x^2}{\eta}\ dx \in L^1(\mathbb{R}_+),$$

which implies that $\|\theta(\cdot,t)-\theta_\infty\|_{H^1(\Omega)} \to 0$ and then (72).

3. Clearly (73) follows directly from (71) $\square$

Remark 1 *An asymptotic result for the specific volume η would easily follow from a uniform-in-time bound for the gradient $\|\eta_x\|_{L^2(\Omega)}$. Unfortunately the result of Proposition 3 is not sufficient for this purpose. This fact seems to be a consequence of the pressureless model with variable viscosity.*

5 The constant coefficient case

In order to check Remark 1, we briefly study the case where μ and κ are constant (after (7), notice that this case is not strictly included in the previous study).

1. One checks first that the energy estimates of Lemma 1 and the pointwise bounds of Propositions 1 and 2 for v and θ are valid. Lemma 2 also holds provided that the multiplicator $\mathcal{K}$ is replaced by θ.

2. The proof of Lemma 3 is modified as follows.

One checks fist the analogous of (33)

$$\frac{d}{dt}\int_\Omega \frac{\mu}{\eta}\,[(r^2v)_x]^2\,dx + \int_\Omega r^4\sigma_x^2 dx \leqslant \left(\int_\Omega \frac{\mu}{\eta}[(r^2v)_x]^2\,dx\right)^2, \tag{76}$$

which gives the first bound (25) and (26).

Inequality (38) is replaced by

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega \frac{r^4\mathcal{K}_x^2}{\eta}\,dx + \int_\Omega \kappa\left[\left(\kappa\frac{r^4}{\eta}\theta_x\right)_x\right]^2 dx \leqslant \int_\Omega \frac{\mu}{\eta}[(r^2v)_x]^2\,dx \\ +C\left(\int_\Omega \frac{r^4\mathcal{K}_x^2}{\eta}\,dx\right)^2 + \frac{1}{2}\,\epsilon_3\int_\Omega \kappa\left[\left(\frac{\kappa r^4}{\eta}\theta_x\right)_x\right]^2\,dx \\ +C\left(\int_\Omega \frac{\kappa^2r^4}{\eta}\theta_x^2\,dx\right)\left(\int_\Omega \frac{\mu}{\eta}[(r^2v)_x]^2\,dx\right) \\ -\frac{1}{2\epsilon_3}\int_\Omega \frac{\mu^2}{\kappa\eta^2}\,[(r^2v)_x]^4\,dx + \frac{1}{2}\,\epsilon_3\int_\Omega \frac{\mu\kappa}{\mu'}\,\kappa\left[\left(\kappa\frac{r^4}{\eta}\theta_x\right)_x\right]^2\,dx.\end{aligned} \tag{77}$$

As $[(r^2v)_x]^4 \leqslant C\int_\Omega \sigma^2\,dx\int_\Omega r^4\sigma_x^2\,dx$, using (76), we get the second bound (25) for ϵ_3 small enough.

3. Uniform bounds for η and θ (Lemma 4 5) and for $(r^2v)_x$ and θ_x (Corollary 1) are proved as previously and the bound for η_x may be improved as follows.

As the second equation (3) rewrites $\mu(\log\eta)_{xt} = \left(\frac{v}{r^2}\right)_t + \frac{2v^2}{r^3}$, we have

$$\frac{1}{2}\,\frac{d}{dt}\int_\omega \left[\mu(\log\eta)_x - \frac{v}{r^2}\right]^2\,dx = \int_\omega \frac{2v^2}{r^3}\left[\mu(\log\eta)_x - \frac{v}{r^2}\right]\,dx.$$

So if $X(t) := \int_\omega \left[\mu(\log\eta)_x - \frac{v}{r^2}\right]^2\,dx$, we find the differential inequality

$$\frac{d}{dt}\,Y(t) \leqslant F(t)(1+Y(t)), \tag{78}$$

where $F\in L^1(\mathbb{R}_+)$, which implies that $Y(t)\leqslant C$, and using energy estimate we have finally the uniform bound

$$\|\eta_x\|_{L^2(\Omega)} \leqslant C. \tag{79}$$

This allows us to improve Theorem 5.

Theorem 6 *The solution* (v, θ, η) *of the problem (3)(4)(5), for* $\mu = Cte$ *and* $\kappa = Cte$ *satisfies (70) (71) (72) and (73). Moreover, when* $t \to \infty$

$$\|\eta(\cdot, t) - \eta_\infty\|_{C(\Omega)} \to 0, \tag{80}$$

where $\eta_\infty = \frac{1}{M} \int_\Omega \eta^0 \, dx$*.*

Proof: Only the last item has to be checked. After (78) and (79) we have

$$\int_0^\infty \left| \frac{d}{dt} \int_\Omega \left[(\log \eta)_x\right]^2 dx \right| \, dt \leqslant C,$$

implying

$$\int_\Omega \eta_x^2 dx \to 0 \quad \text{when } t \to \infty. \tag{81}$$

Now one observes that there exits a $\xi(t) \in \Omega$ such that $\eta(\xi(t), t) = \frac{1}{M} \int_\Omega \eta^0(x) \, dx \equiv \eta_\infty$. Then one gets

$$\eta(x, t) - \eta(\xi(t), t) = \int_\xi^x \eta_y \, dy,$$

and so

$$|\eta(x, t) - \eta_\infty| \leqslant C \left(\int_\Omega \eta_x^2 \, dx \right)^{1/2},$$

which gives (80) after (81) □

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