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BIFURCATION DIRECTION AND EXCHANGE OF STABILITY FOR AN ELLIPTIC UNILATERAL BVP *

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The direction of bifurcation of nontrivial solutions to the elliptic boundary value problem involving unilateral nonlocal boundary conditions is shown in a neighbourhood of bifurcation points of a certain type. Moreover, the stability and instability of bifurcating solutions as well as of the trivial solution is described in the sense of minima of the potential. In particular, an exchange of stability is observed.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ with a Lipschitzian boundary $\partial\Omega$, $1 < N < 5$, let Γ_D and Γ_j, $j = 1, \dots, n$, be pairwise disjoint open (in $\partial\Omega$) subsets of this boundary, $\operatorname{meas}\Gamma_D > 0$. We consider a Crandall-Rabinowitz type bifurcation (a bifurcation from a trivial solution at a simple eigenvalue with exchange of stability, see [4]) for a semilinear elliptic PDE with unilateral nonlocal boundary conditions:

$$\Delta u + pu + au^2 = 0 \quad \text{in } \Omega, \tag{1.1}$$

$$u = 0 \text{ on } \Gamma_D, \quad \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega \setminus \left(\Gamma_D \cup \bigcup_{j=1}^{n} \Gamma_j\right), \tag{1.2}$$

$$\begin{aligned} &\textstyle\int_{\Gamma_j} \psi(u)\,\mathrm{d}\Gamma \le 0, \frac{\partial u}{\partial \nu} = c_j \psi'(u) \le 0 \text{ on } \Gamma_j \text{ with some constant } c_j, \\ &c_j \textstyle\int_{\Gamma_j} \psi(u)\,\mathrm{d}\Gamma = 0,\ j = 1, \dots, n, \end{aligned} \tag{1.3}$$

where $p \in \mathbb{R}$ is the bifurcation parameter, $a \in \mathbb{R}$ is a number, $\psi : \mathbb{R} \to \mathbb{R}$ is a C^3-smooth function and $\frac{\partial u}{\partial \nu}$ is the outer normal derivative. The second condition

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in (1.3) implies, in particular, that for any j, $\psi'(u)$ does not change sign on Γ_j (i.e. $\psi'(u(x))$ is for all $x \in \Gamma_j$ either non-negative or non-positive). The whole condition (1.3) means that a certain ψ-average over any Γ_j cannot exceed the zero value, the flux through any $x \in \Gamma_j$ is proportional to $\psi'(u(x))$ and can go only outwards from the domain Ω. If the ψ-average over Γ_j is strictly negative then there is no flux through Γ_j.

Also the case $a = 0$, i.e. the linearized equation

$$\Delta u + pu = 0 \quad \text{in } \Omega \tag{1.4}$$

with (1.2), (1.3) and also with "linearized" (homogenized) boundary conditions

$$\int_{\Gamma_j} u \, \mathrm{d}\Gamma \le 0, \frac{\partial u}{\partial \nu} = c_j^0 \text{ on } \Gamma_j \text{ with some } c_j^0 \le 0,\ c_j^0 \int_{\Gamma_j} u \, \mathrm{d}\Gamma = 0, j = 1, \dots, n, \tag{1.5}$$

will be of interest for us. The condition (1.5) means that the average over any Γ_j cannot exceed the zero value, the flux through any Γ_j is constant and can go only outwards from the domain Ω. If the average over Γ_j is strictly negative then there is no flux through Γ_j.

By solutions of all boundary value problems mentioned we mean weak solutions, that means solutions of variational inequalities introduced below. As usual in the case of variational inequalities, the linearized problem (1.4), (1.5) is (due to the inequalities in the boundary conditions) nonlinear again but is positively homogeneous. However, let us emphasize that (1.4) with the original boundary conditions (1.2), (1.3) is not even positively homogeneous if ψ' is not constant. In [6] we have proved the existence of a smooth branch of solutions to (1.1)–(1.3) bifurcating from zero at a simple eigenvalue p_0 of (1.4), (1.5). Of course, the u-component of this branch emanates in the direction of the corresponding eigenfunction u_0. Now, our goal is to show that only $\psi''(0)$ together with a and u_0 decide about the p-direction and stability of the bifurcation branch. Let us remark that also in the case $a = 0$ there is generically a direction of bifurcation.

We will show that there are at least three essential differences from the case of classical boundary conditions (see Fig. 2 of Section 3): First, only one half-branch of nontrivial solutions (not two, as for the classical boundary conditions) bifurcates from the branch of trivial solutions. Second, the bifurcating nontrivial solutions can be stable even if the trivial solution is unstable on both sides from the bifurcation point. And third, the bifurcating nontrivial solutions can be unstable even if the trivial solution loses stability at the bifurcation point, i.e. is stable on one side of the bifurcation point. Hence, there is not always exchange of stability if the bifurcation parameter crosses the first bifurcation point (even if there is a loss of stability of the trivial solution). Let us remark that the nonlinear term is quadratic, therefore there is a transcritical bifurcation as well as exchange of stability for the problem with classical boundary conditions (i.e. with $\Gamma_j = \emptyset$ for all $j = 1, \dots, n$).

An analogous result for the particular case of constant ψ' was described already in [5] but to consider the direction of the bifurcation branch it was necessary to have a nontrivial nonlinear perturbation in the equation because (1.3) becomes positively homogeneous in this case. Let us note that in [5], an abstract theory covering the case $\psi(\xi) = \xi$ was studied. The generalization of these abstract results including our present example is the subject of a forthcoming paper [7] based on an equivalence of the variational inequality with the Lagrange equation. The results can be understood as a certain modification of the well-known results for equations (see e.g. [11], Chapter 8.7) to variational inequalities. The method of Lagrange multipliers was used also in [6] to prove the existence of smooth bifurcation families of nontrivial solutions for a class of variational inequalities covering our present BVP.

Let us note that an abstract criterion of stability for variational inequalities without any relation to bifurcation was given in [10]. A loss of stability at the turning points for variational inequalities of a certain type was numerically proved in [3]. Stability and continuation for solutions to obstacle problems were studied in [8].

The main results are formulated in Theorems 2.2 and 2.3. At the end of the text (Section 3) we show possible bifurcation diagrams in the case of one obstacle (obtained with help of numerical computations).

2. Weak Formulation, Main Results

We will assume that the function ψ satisfies

$$\psi(0) = 0, \quad \psi'(0) > 0 \tag{2.1}$$

and the growth conditions

$$\begin{aligned} &|\psi(\xi)| \le c(1+|\xi|^q), \quad \psi'(\xi) \le c(1+|\xi|^{q-1}), \\ &|\psi''(\xi)| \le c(1+|\xi|^{q-2}), \quad |\psi'''(\xi)| \le c(1+|\xi|^{q-3}) \quad \text{for all } \xi \in \mathbb{R} \end{aligned} \tag{2.2}$$

with some $c > 0$ and $q \ge 3$ for $N = 2$, $q = 2\frac{N-1}{N-2}$ for $N = 3$ or $N = 4$. Let us remark that the first assumption in (2.1) ensures that $u = 0$ is a solution for any $p \in \mathbb{R}$ and the second implies that the condition (1.5) is a homogenization of (1.3).

In order to introduce a weak formulation, we consider the Hilbert space

$$H := \{u \in W^{1,2}(\Omega) \, : \, u = 0 \text{ on } \Gamma_D \text{ in the sense of traces}\}$$

with the inner product $\langle u, v\rangle := \int_\Omega \nabla u {\cdot} \nabla v \, \mathrm{d}x$ for any $u, v \in H$. The corresponding norm $\|\cdot\|$ is equivalent to the usual Sobolev norm on H under our assumptions. Let us denote $\mathcal{A} := \{1, \dots, n\}$ and define for any $u \in H$ a set

$$\mathcal{A}(u) := \left\{ \alpha \in \mathcal{A} \, : \, \int_{\Gamma_\alpha} \psi(u) \, \mathrm{d}\Gamma = 0 \right\}.$$

Furthermore, let us introduce a closed set K and a closed convex cone K_0 by

$$\begin{aligned} K &:= \left\{ u \in H \, : \, \textstyle\int_{\Gamma_\alpha} \psi(u) \, \mathrm{d}\Gamma \le 0, \ \alpha \in \mathcal{A} \right\}, \\ K_0 &:= \left\{ u \in H \, : \, \textstyle\int_{\Gamma_\alpha} u \, \mathrm{d}\Gamma \le 0, \ \alpha \in \mathcal{A} \right\} \end{aligned}$$

and functionals $g_\alpha : H \to \mathbb{R}$ by

$$g_\alpha(u) = \int_{\Gamma_\alpha} \psi(u)\, \mathrm{d}\Gamma.$$

Then $\mathcal{A}(u) = \{\alpha \in \mathcal{A} \,:\, g_\alpha(u) = 0\}$ and $v_\alpha := \nabla g_\alpha(0)$, $\alpha \in \mathcal{A}$, satisfy $\langle v_\alpha, u\rangle = \psi'(0)\int_{\Gamma_\alpha} u\, \mathrm{d}\Gamma$ for any $u \in H$. Under the assumption (2.2), standard considerations about Nemyckii operators and the continuity of the embedding of $W^{1,2}(\Omega)$ into $L^q(\partial\Omega)$ (see e.g. Theorem 4.2 in [9]) imply that the functionals g_α are C^3-smooth. Moreover, under the assumption (2.1) the elements $\nabla g_\alpha(u)$ are linearly independent for all $u \in H$. Realizing this and the definition of the local contingent cone to K at u

$$T(K,u) := \{z \in H :\ \text{there exist } w_n \in K, t_n > 0, w_n \to u, t_n(w_n - u) \to z\}$$

(see e.g. [1]) we have $K_0 = T(K,0)$ and

$$T(K,u) = \left\{v \in H \ :\ \int_{\Gamma_\alpha} \psi'(u) v\, \mathrm{d}\Gamma \le 0,\ \alpha \in \mathcal{A}(u)\right\} \text{ for all } u \in K.$$

We define a weak solution to (1.1)–(1.3) as a solution to the problem

$$p \in \mathbb{R}, u \in K : \int_\Omega \nabla u \cdot \nabla\varphi - (pu + au^2)\varphi\, \mathrm{d}x \ge 0 \text{ for all } \varphi \in T(K,u) \tag{2.3}$$

and a weak solution to (1.4), (1.2), (1.5) as a solution to

$$p \in \mathbb{R}, u \in K_0 : \int_\Omega \nabla u \cdot \nabla(\varphi - u) - pu(\varphi - u)\, \mathrm{d}x \ge 0 \ \text{ for all } \varphi \in K_0. \tag{2.4}$$

Finally, we define a functional $\Phi : \mathbb{R} \times H \to \mathbb{R}$ by

$$\Phi(p,u) := \int_\Omega \left(\frac{|\nabla u|^2 - pu^2}{2} - \frac{au^3}{3}\right) \mathrm{d}x.$$

Remark 2.1. Of course, K_0, (2.3) and (2.4) is of the form

$$K_0 = \{u \in H \ :\ \langle v_\alpha, u\rangle \le 0 \text{ for all } \alpha \in \mathcal{A}(0)\}, \tag{2.5}$$

$$p \in \mathbb{R},\ u \in K :\ \langle F(p,u), \psi\rangle \le 0 \text{ for all } \psi \in T(K,u), \tag{2.6}$$

$$p \in \mathbb{R},\ u \in K_0 :\ \left\langle \frac{\partial F}{\partial u}(p,0)u, \varphi - u\right\rangle \le 0 \text{ for all } \varphi \in K_0, \tag{2.7}$$

respectively, with $F : \mathbb{R} \times H \to H$ defined by

$$\langle F(p,u), \varphi\rangle = \int_\Omega -\nabla u \cdot \nabla\varphi + (pu + au^2)\varphi\, \mathrm{d}x \qquad \text{for all } p \in \mathbb{R}, u, \varphi \in H.$$

For N under consideration, the embedding theorems imply that F is well defined and C^∞-smooth. Hence, our problem fits into the abstract framework studied in [7]. Moreover, it is easy to see that the functional Φ satisfies the condition

$$\frac{\partial\Phi}{\partial u}(p,u)v = \langle -F(p,u), v\rangle \text{ for all } p \in \mathbb{R} \text{ and } u, v \in H, \tag{2.8}$$

i.e., $F(p,\cdot)$ is a potential operator for any $p \in \mathbb{R}$.

Remark 2.2. If $u \in H$ is such that $\Delta u \in L^2(\Omega)$, then the normal derivative $\frac{\partial u}{\partial \nu}$ can be defined as a linear bounded functional on the space H by $\left[\frac{\partial u}{\partial \nu}, \varphi\right] = \int_\Omega (\Delta u \cdot \varphi + \nabla u \cdot \nabla \varphi)\, \mathrm{d}x$ for all $\varphi \in H$, where $[\cdot,\cdot]$ is the dual pairing. The non-positivity of $\frac{\partial u}{\partial \nu}$ on Γ_α will be understood in the sense of such functional, i.e. $\left[\frac{\partial u}{\partial \nu}, \varphi\right] \leq 0$ for all $\varphi \in H$, $\varphi \geq 0$ on Γ_α, $\varphi = 0$ on $\partial\Omega \setminus \Gamma_\alpha$. The condition $\frac{\partial u}{\partial \nu} = c_\alpha \psi'(u)$ on Γ_α will mean that the functional $\frac{\partial u}{\partial \nu}$ can be represented on Γ_α by the function $c_\alpha \psi'(u) \in L^{q*}(\partial\Omega)$ (with $\frac{1}{q^*} + \frac{1}{q} = 1$, q from (2.2)), i.e.

$$\left[\frac{\partial u}{\partial \nu}, \varphi\right] = c_\alpha \int_{\Gamma_\alpha} \psi'(u)\varphi \,\mathrm{d}\Gamma \text{ for all } \varphi \in H, \varphi = 0 \text{ on } \partial\Omega \setminus \Gamma_\alpha. \tag{2.9}$$

Observation 2.1. A couple $(p, u) \in \mathbb{R} \times H$ satisfies (2.3) (i.e. it is a weak solution to (1.1)– (1.3)) if and only if u is smooth in Ω, $\Delta u \in L^2(\Omega)$, (1.1) is satisfied in the classical sense and (1.2), (1.3) are satisfied where u on $\partial\Omega$ is understood in the sense of traces and $\frac{\partial u}{\partial \nu}$ is understood in the sense of the functional from Remark 2.2.

Proof is the same as that of Observation 5.3 in [6]. ■

Let us fix a subset $\mathcal{A}_0 := \{\alpha_1, \ldots, \alpha_m\}$ of $\mathcal{A}$. Moreover, let H_0 be a subspace of H given by $H_0 := \left\{u \in H : \int_{\Gamma_\alpha} u \,\mathrm{d}\Gamma = 0,\ \alpha \in \mathcal{A}_0\right\}$. In the main assertions formulated below we will need the boundary value problem (1.4), (1.2),

$$\int_{\Gamma_\alpha} u \,\mathrm{d}\Gamma = 0 \text{ for } \alpha \in \mathcal{A}_0, \tag{2.10}$$

$$\frac{\partial u}{\partial \nu} = c_\alpha \text{ on } \Gamma_\alpha \ \text{ with some } c_\alpha \in \mathbb{R} \text{ for any } \alpha \in \mathcal{A}_0, \tag{2.11}$$

$$\frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma_\alpha \text{ for } \alpha \in \mathcal{A} \setminus \mathcal{A}_0, \tag{2.12}$$

having a weak formulation

$$p \in \mathbb{R},\ u \in H_0 \ : \int_\Omega \nabla u \cdot \nabla \varphi - pu\varphi \,\mathrm{d}x = 0 \ \text{ for all } \varphi \in H_0. \tag{2.13}$$

Let us remark that the problem (2.13) can be written in the abstract framework of the paper [7] as $p \in \mathbb{R}$, $u \in H_0$, $P\frac{\partial F}{\partial u}(p, 0)u = 0$, where P is the orthogonal projection of H onto H_0.

The following theorem gives us the existence of a smooth branch bifurcating from the trivial solutions. The description of the direction and stability of this branch (which is our main goal) will be given in Theorems 2.2 and 2.3 below.

Theorem 2.1. *Let (p_0, u_0) be a weak solution to (1.4), (1.2), (1.5) satisfying (2.10),*

$$\int_{\Gamma_\alpha} u_0 \,\mathrm{d}\Gamma < 0 \textit{ for } \alpha \in \mathcal{A} \setminus \mathcal{A}_0, \tag{2.14}$$

$$\frac{\partial u_0}{\partial \nu} = c_\alpha^0 \textit{ on } \Gamma_\alpha \ \textit{ with some } c_\alpha^0 < 0 \textit{ for any } \alpha \in \mathcal{A}_0. \tag{2.15}$$

Let us assume the following simplicity conditions:

If (p_0, v_0) *is a weak solution to (1.4), (1.2), (2.10)–(2.12) then* $v_0 = cu_0$, $c \in \mathbb{R}$.
If (p_0, v_0) *is a weak solution to (1.4), (1.2), (1.5) then* $v_0 = cu_0$, $c \geq 0$.

Then there exist $\varepsilon > 0$, $s_0 > 0$ *and* C^1*-maps* $\hat{p} : [0, s_0) \to \mathbb{R}$ *and* $\hat{v} : [0, s_0) \to H_0$ *with* $\hat{p}(0) = p_0$, $\hat{v}(0) = 0$ *such that the following holds. The couple* (p, u) *with* $|p - p_0| < \varepsilon$, $\|u\| < \varepsilon$ *and* $\|u\| \neq 0$ *is a weak solution to (1.1)–(1.3) if and only if* $p = \hat{p}(s)$, $u = \hat{u}(s) := s(u_0 + \hat{v}(s))$ *for a certain* $s \in (0, s_0)$. *In this case, moreover,* $u = \hat{u}(s)$ *satisfies (2.12) and*

$$\begin{aligned}
&\textstyle\int_{\Gamma_\alpha} \psi(u)\,\mathrm{d}\Gamma = 0 \text{ for } \alpha \in \mathcal{A}_0,\\
&\tfrac{\partial u}{\partial \nu} = c_\alpha \psi'(u) < 0 \text{ on } \Gamma_\alpha \text{ with some } c_\alpha \in \mathbb{R} \text{ for any } \alpha \in \mathcal{A}_0,\\
&\textstyle\int_{\Gamma_\alpha} \psi(u)\,\mathrm{d}\Gamma < 0 \text{ for } \alpha \in \mathcal{A} \setminus \mathcal{A}_0.
\end{aligned}$$

Proof is the same as that of Theorem 5.4 in [6]. ■

Remark 2.3. If $N = 3$ and a suitable growth estimate for the fourth derivative of ψ is added to (2.2) then the functionals g_α are C^4-smooth and the mappings $\hat{p}$, $\hat{v}$, $\hat{u}$ in Theorem 2.1 are in fact C^2-smooth. If $N = 2$ and $k > 2$ is an arbitrary positive integer then under suitable growth estimates for derivatives of ψ up to the order $k+1$, the functionals g_α are C^{k+1}-smooth and the mappings $\hat{p}$, $\hat{v}$, $\hat{u}$ in Theorem 2.1 are in fact C^{k-1}-smooth.

Theorem 2.2. *Let* (p_0, u_0) *satisfy the assumptions of Theorem 2.1, let* $(\hat{p}(s), \hat{u}(s))$, $s \in [0, s_0)$, *be the bifurcation branch from Theorem 2.1. If*

$$a \int_\Omega u_0^3 \,\mathrm{d}x + 3\psi''(0) \sum_{\alpha \in \mathcal{A}_0} c_\alpha^0 \int_{\Gamma_\alpha} u_0^2 \,\mathrm{d}\Gamma < 0 \tag{2.16}$$

then $\hat{p}(s) > p_0$ *for all* $s \in (0, s_0)$ *and if, moreover,*

$$p_0 \textit{ is the smallest eigenvalue of (1.4), (1.2), (2.10)–(2.12)} \tag{2.17}$$

then $\Phi\left(\hat{p}(s), \cdot\right)$ *attains a strong local minimum on* K *in* $\hat{u}(s)$ *for all* $s \in (0, s_0)$. *If*

$$a \int_\Omega u_0^3 \,\mathrm{d}x + 3\psi''(0) \sum_{\alpha \in \mathcal{A}_0} c_\alpha^0 \int_{\Gamma_\alpha} u_0^2 \,\mathrm{d}\Gamma > 0 \tag{2.18}$$

then $\hat{p}(s) < p_0$ *and* $\Phi\left(\hat{p}(s), \cdot\right)$ *has no local minimum on* K *in* $\hat{u}(s)$ *for all* $s \in (0, s_0)$.

Proof. The direction and stability of bifurcation branches was described in [5], Theorem 9 for a class of variational inequalities in the case when K is a cone with its vertex at the origin in a Hilbert space. The problem (2.3) is included if $\psi(\xi) = \xi$, i.e. K is a cone, and this particular case is described by [5], Theorem 14. These results can be generalized to an abstract class of variational inequalities where K need not be a cone, i.e. inequalitites containing (2.3) with a general ψ. See [7], Theorem 5.5, which is proved by using local equivalence of a variational inequality to an equation with Lagrange multipliers. Theorem 2.2 follows from [7], Theorem 5.5 in the same way as [5], Theorem 14 from [5], Theorem 9.

■

Theorem 2.3. *Let p_0 be the smallest eigenvalue of (1.4), (1.2), (1.5). Then $\Phi(p,\cdot)$ attains a strict local minimum on K at $u = 0$ for any $p \in (0, p_0)$ and $\Phi(p,\cdot)$ has no local minimum on K at $u = 0$ for any $p > p_0$.*

Proof. The assertion follows from [7], Theorem 5.6 in the same way as that of [5], Theorem 15 follows from [5], Theorem 10. ■

Remark 2.4. Contrary to the fact that only one $p_0 \in \mathbb{R}$ can be the smallest eigenvalue of (2.4), the condition (2.17) can be fulfilled for more different values of p_0 corresponding to different subsets $\mathcal{A}_0$ of $\mathcal{A}$, i.e. to different subspaces H_0 of H (maximally 2^n). Hence, Theorem 2.2 can enable us to determine the stability of bifurcating branches in a neighbourhood of several parameters p_0.

Nevertheless, the complete exchange of stability (including the stability of the trivial solution) is ensured only for the unique positive p_0.

3. Example

Let $\Omega \subset \mathbb{R}^2$ be a rectangle $\{x \in (0,1),\ y \in (0,\ell)\}$, $\ell < 1$, $\Gamma_D := \{(x,0);\ x \in (0,1)\} \cup \{(0,y);\ y \in (0,\ell)\}$, $\Gamma_1 := \{(x,\ell);\ x \in (0,1)\}$, $n = 1$, $\mathcal{A} = \{1\}$. If (p,u) satisfies (2.4), i.e. it is a weak solution to (1.4) with

$$u = 0 \text{ on } \Gamma_D, \quad \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega \setminus (\Gamma_D \cup \Gamma_1)\,, \tag{3.1}$$

$$\int_{\Gamma_1} u \,\mathrm{d}\Gamma \le 0,\ \frac{\partial u}{\partial \nu} \le 0,\ \int_{\Gamma_1} u \,\mathrm{d}\Gamma \cdot \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma_1, \tag{3.2}$$

$$\frac{\partial u}{\partial \nu} = c_1^0 \text{ on } \Gamma_1 \text{ with some } c_1^0, \tag{3.3}$$

then one of the following conditions is fulfilled:

$$\int_{\Gamma_1} u \,\mathrm{d}\Gamma < 0, \tag{3.4}$$

$$\int_{\Gamma_1} u \,\mathrm{d}\Gamma = 0. \tag{3.5}$$

If (3.4) holds then we have

$$\frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma_1, \tag{3.6}$$

that means p is an eigenvalue and u is the corresponding eigenfunction with the proper sign of the classical mixed Dirichlet–Neumann boundary value problem (1.4), (3.1), (3.6).

The linearized problem in [5], Example, coincides with the present linearized problem (2.4), only our variational inequality (2.3) is more complicated because of the generality of ψ. In [5] we have proved the following facts about (2.4) (i.e. about (1.4) with (3.1)–(3.3)).

There exist sequences of couples $\left(p^N_{m,n}, u^N_{m,n}\right)$ and $\left(p^P_m, u^P_m\right)$, $m, n = 1, 2, \ldots,$ satisfying (1.4) with (3.1)–(3.3), (3.4) (hence also (3.6)) and (1.4) with (3.1)–(3.3), (3.5), respectively, and $0 < p^N_{1,1} < p^P_1 < p^N_{2,1} \le \cdots$. Moreover,

$$\begin{aligned} &p^N_{m,n} = \left(\left(\tfrac{2m-1}{2}\right)^2 + \left(\tfrac{2n-1}{2\ell}\right)^2\right)\pi^2, \\ &u^N_{m,n}(x,y) = (-1)^m \sin\frac{(2m-1)\pi}{2}x \cdot \sin\frac{(2n-1)\pi}{2\ell}y, \quad m, n = 1, 2, \ldots, \end{aligned}$$

and $u^N_{m,n}$ satisfy (3.4) for all m, n (i.e. they do not satisfy (3.5) for any m, n). In particular,

$$\mathcal{A}(u^N_{m,n}) = \emptyset \text{ for all } m, n \text{ (hence, } \mathcal{A}_0 = \emptyset \text{ in such cases).} \tag{3.7}$$

On the other hand, u^P_m satisfy (3.5) and $\mathcal{A}(u^P_m) = \{1\}$ for all m. Furthermore,

$$\int_\Omega (u^N_{1,1})^3 \,\mathrm{d}x < 0, \quad \int_\Omega (u^P_1)^3 \,\mathrm{d}x > 0 \tag{3.8}$$

(the second integral was computed numerically). Finally, the assumptions of Theorems 2.1, 2.2 and 2.3 are verified for the couples $\left(p^N_{1,1}, u^N_{1,1}\right)$ and $\left(p^P_1, u^P_1\right)$ for the particular case $\psi(\xi) = \xi$ (i.e. for the simplified conditions (2.16) and (2.18)).

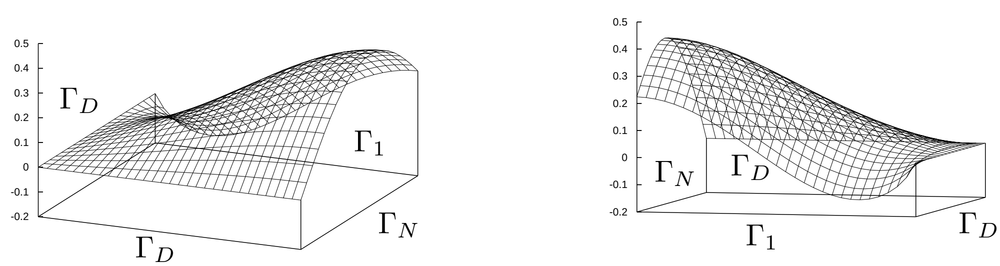

Figure 1. Two different viewpoints on the function u^P_1 corresponding to the parameter $p^P_1 = 16.01$ for $\ell = 0.8$.

In the present paper the further discussion is more complex because of the presence of $\psi''(0)$. Because of (3.7) the terms in (2.16) and (2.18) with $\psi''(0)$ are trivially zero and due to (3.8), Theorem 2.2 gives that if $a > 0$ then the first bifurcating branch (emanating at $p^N_{1,1}$) goes to the right and is stable and if $a < 0$ then it goes to the left and is unstable. If $a = 0$ then we cannot decide about the direction of the first branch.

Due to (3.8), Theorem 2.2 gives that if $a > 0$ and $\psi''(0) < 0$ then the second bifurcating branch (starting at $p_0 = p^P_1 > p^N_{1,1}$) goes to the left and is unstable and if $a < 0$ and $\psi''(0) > 0$ then it goes to the right and is stable.

Without loss of generality we can renorm the eigenfunction $u_0 = u_1^P$ so that the corresponding $c_1^0 = \frac{\partial u_0}{\partial \nu}\Big|_{\Gamma_1} = -1$. Then the numerical computations e.g. for $\ell = 0.8$ give $\int_\Omega (u_1^P)^3 \,\mathrm{d}x = 0.0086287704$ and $\int_{\Gamma_1} (u_1^P)^2 \,\mathrm{d}\Gamma = 0.0172444952$. The left hand side in (2.16) and (2.18) is approximately equal to $3(0.167a - \psi''(0))$ and Theorem 2.2 implies that the second bifurcation branch starting at $p_0 = p_1^P$ goes to the right and is stable or goes to the left and is unstable if and only if $\psi''(0) > 0.167a$ or $\psi''(0) < 0.167a$, respectively. In particular, in the case $a > 0$, the second branch goes to the left and is unstable even in cases when ψ is slightly convex at zero.

We get the following bifurcation diagram in a neigborhood of the two bifurcation points discussed above.

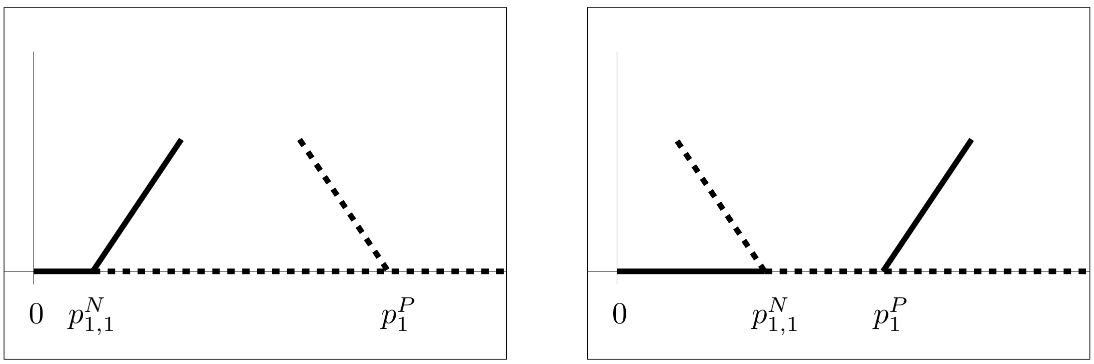

Figure 2. Possible bifurcation diagrams. Left panel $a > 0$ and $\psi''(0) < 0$, right panel $a < 0$ and $\psi''(0) > 0$.

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