



THE WEIGHTED STIELTJES INEQUALITY AND APPLICATIONS

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ABSTRACT. Let $1 < p \leq q < \infty$. Inspired by some results concerning characterization of weighted Hardy type inequalities, where the equivalence of four scales of integral conditions was proved, we use related ideas to find some new equivalent scales of integral conditions related to the Stieltjes transform. By applying our result to weighted inequalities for the Stieltjes transform we obtain four new scales of conditions for characterization of this inequality. We also derive a new characterization for the solvability of a Riccati type equation and show via our new results that this characterization can be done in infinite many ways via our four scales of equivalent conditions.

1. INTRODUCTION

The *Stieltjes transform* S_λ , $\lambda > 0$, is defined for a measurable function h by

$$(1.1) \quad S_\lambda h(x) = \int_0^\infty \frac{h(y) dy}{(x+y)^\lambda}, \quad x > 0.$$

We will consider the *weighted Stieltjes inequality*

$$(1.2) \quad \left(\int_0^\infty (S_\lambda h(x))^q u(x) dx \right)^{\frac{1}{q}} \leq C \left(\int_0^\infty (h(x))^p v(x) dx \right)^{\frac{1}{p}},$$

with $\lambda > 0$, for measurable functions $h \geq 0$, weights u and v and for parameters p, q satisfying $1 < p \leq q < \infty$.

Inequality (1.2) is usually characterized by splitting it into two Hardy type inequalities and, thus, we get two different conditions. Using a "gluing lemma" (see [2, Lemma 2.2]) we can equivalently express these two conditions in one single condition. More precisely, the following proposition was proved:

Date: April 1, 2011.

1991 Mathematics Subject Classification. Primary 26D10, 26D15; Secondary 47B07, 47B38.

Key words and phrases. inequalities, Stieltjes' transform, weights, scales of weight characterizations, Riccati's equation.

The research of the first author was partially supported by the grant no. 201/08/0383 of the Czech Science Foundation and the research of the first a second author by the Institutional Research Plan no. AV0Z10190503 of AS CR

This paper is in final form and no version of it will be submitted for publication elsewhere .

Proposition 1.1. [2, Proposition 4.6] *Let $\lambda \geq 0$, $1 \leq p \leq q \leq \infty$, and suppose that $u(x)$ and $v(x)$ are non-negative extended real-valued functions defined on $(0, \infty)$. Then there exists a constant C independent of h such that the inequality (1.2) holds if and only if*

$$(1.3) \quad K := \sup_{x>0} (S_{\lambda q}(u)(x))^{\frac{1}{q}} \left(S_{\lambda p'}(v^{1-p'})(x) \right)^{\frac{1}{p'}} x^\lambda < \infty.$$

Moreover, the smallest constant C in (1.2) satisfies $C \approx K$.

Here and in the sequel $p' = \frac{p}{p-1}$.

In [3] the equivalence of four scales of integral conditions was proved. These conditions characterize the weighted *Hardy* inequality and contain the usual Muckenhoupt–Bradley and Persson–Stepanov conditions as special cases. The proof was carried out by first proving a related equivalence theorem of independent interest (see Theorem 2.1 below). Inspired by this result we will find some new equivalent scales of integral conditions related to the Stieltjes transform. These conditions can be used to characterize the corresponding weighted inequalities for the Stieltjes transform. We will first prove an equivalence theorem which is of independent interest (see Theorem 2.2) and after that we apply it to get four different scales of conditions for characterization of the mentioned weighted inequalities for the Stieltjes transform (see Theorem 3.1).

Finally, we derive a new characterization for the solvability of an one-dimensional nonlinear second order Riccati type equation (see Theorem 4.1) and show via our new results that this characterization can be done in infinite many ways via our four scales of equivalent conditions (see Theorem 4.3). Our characterizations are necessary and sufficient. To prove the sufficient part which is not difficult, we are using a simple iteration method analogously to that used in [5], but our proof of the necessity part is completely different. It is based on the Equivalence theorem (Theorem 2.2), and we are not using weighted norm inequalities.

Let us mention that the close relation between some boundary value problems and the corresponding integral inequalities is well known and investigated by many authors (see e.g [1], [9], [10], [8, Section 14], [11], [5], [4]).

2. THE NEW EQUIVALENCE THEOREM

We start by formulating the crucial Equivalence theorem from [3]:

Theorem 2.1. [3, Theorem 2.1] *For $-\infty \leq a < b \leq \infty$, α, β and s positive numbers and f, g measurable functions positive a.e. in (a, b) , denote*

$$(2.1) \quad F(x) := \int_x^b f(t)dt, \quad G(x) := \int_a^x g(t)dt$$

and

$$\begin{aligned}
(2.2) \quad B_1(x; \alpha, \beta) &:= F^\alpha(x)G^\beta(x); \\
B_2(x; \alpha, \beta, s) &:= \left(\int_x^b f(t)G^{\frac{\beta-s}{\alpha}}(t)dt \right)^\alpha G^s(x); \\
B_3(x; \alpha, \beta, s) &:= \left(\int_a^x g(t)F^{\frac{\alpha-s}{\beta}}(t)dt \right)^\beta F^s(x); \\
B_4(x; \alpha, \beta, s) &:= \left(\int_a^x f(t)G^{\frac{\beta+s}{\alpha}}(t)dt \right)^\alpha G^{-s}(x); \\
B_5(x; \alpha, \beta, s) &:= \left(\int_x^b g(t)F^{\frac{\alpha+s}{\beta}}(t)dt \right)^\beta F^{-s}(x).
\end{aligned}$$

The numbers $B_1 := \sup_{a < x < b} B_1(x; \alpha, \beta)$ and $B_i(s) = \sup_{a < x < b} B_i(x; \alpha, \beta, s)$ ($i = 2, 3, 4, 5$) are mutually equivalent. The constants in the equivalence relations can depend on α, β and s .

Our main result in this section is the following equivalence theorem:

Theorem 2.2. *Let α, β, λ and s be positive numbers and let ϕ and ψ be measurable functions which are positive a.e. in $(0, \infty)$. Moreover, let*

$$\begin{aligned}
(2.3) \quad A(x; \alpha, \beta, \lambda) &:= \left(S_{\frac{\lambda}{\alpha}} \phi(x) \right)^\alpha \left(S_{\frac{\lambda}{\beta}} \psi(x) \right)^\beta x^\lambda; \\
A_1(x; \alpha, \beta, \lambda, s) &:= \left(S_{\frac{\lambda}{\alpha}} \phi(x) \right)^s \left(S_{\frac{\lambda s}{\beta \alpha}} \left(\left(S_{\frac{\lambda}{\alpha}} \phi(t) \right)^{\frac{\alpha-s}{\beta}} \psi(t) \right) (x) \right)^\beta x^{\frac{\lambda s}{\alpha}}; \\
A_2(x; \alpha, \beta, \lambda, s) &:= \left(S_{\frac{\lambda}{\alpha}} \phi(x) \right)^{-s} \left(S_{\frac{\lambda s}{\beta \alpha}} \left(\left(S_{\frac{\lambda}{\alpha}} \phi(t) \right)^{\frac{\alpha+s}{\beta}} \psi(t) t^{\frac{\lambda s}{\alpha \beta}} \right) (x) \right)^\beta; \\
A_3(x; \alpha, \beta, \lambda, s) &:= \left(S_{\frac{\lambda}{\beta}} \psi(x) \right)^s \left(S_{\frac{\lambda s}{\beta \alpha}} \left(\left(S_{\frac{\lambda}{\beta}} \psi(t) \right)^{\frac{\beta-s}{\alpha}} \phi(t) \right) (x) \right)^\alpha x^{\frac{\lambda s}{\beta}}; \\
A_4(x; \alpha, \beta, \lambda, s) &:= \left(S_{\frac{\lambda}{\beta}} \psi(x) \right)^{-s} \left(S_{\frac{\lambda s}{\beta \alpha}} \left(\left(S_{\frac{\lambda}{\beta}} \psi(t) \right)^{\frac{\beta+s}{\alpha}} \phi(t) t^{\frac{\lambda s}{\beta}} \right) (x) \right)^\alpha.
\end{aligned}$$

The numbers

$$A(\alpha, \beta, \lambda) = \sup_{x > 0} A(x, \alpha, \beta, \lambda)$$

and

$$A_i(\alpha, \beta, \lambda, s) = \sup_{x > 0} A_i(x; \alpha, \beta, \lambda, s), \quad (i = 1, 2, 3, 4)$$

are mutually equivalent. More precisely

$$(2.4) \quad c_i A_i(\alpha, \beta, \lambda, s) \leq A(\alpha, \beta, \lambda) \leq d_i A_i(\alpha, \beta, \lambda, s), \quad i = 1, 2, 3, 4.$$

The positive constants c_i and d_i in the equivalence relations (2.4) can depend on α, β, λ and s .

Proof. We will show that

$$A(\alpha, \beta, \lambda) \approx A_i(\alpha, \beta, \lambda, s)$$

for $i = 1, 2, 3, 4$. In the proof, which is rather technical, we use - among other tools - the fact that the functions $S_{\frac{\lambda}{\alpha}}\phi(x)$ and $S_{\frac{\lambda}{\beta}}\psi(x)$ are decreasing and the functions $x^{\frac{\lambda}{\alpha}}S_{\frac{\lambda}{\alpha}}\phi(x)$ and $x^{\frac{\lambda}{\beta}}S_{\frac{\lambda}{\beta}}\psi(x)$ are increasing, and that

$$\begin{aligned} S_{\frac{\lambda}{\alpha}}\phi(x) &\approx x^{-\frac{\lambda}{\alpha}} \int_0^x \phi(t)dt + \int_x^\infty t^{-\frac{\lambda}{\alpha}}\phi(t)dt \\ &= \frac{\lambda}{\alpha} x^{-\frac{\lambda}{\alpha}} \int_0^x t^{\frac{\lambda}{\alpha}-1} \int_t^\infty y^{-\frac{\lambda}{\alpha}}\phi(y)dydt \\ &= \frac{\lambda}{\alpha} \int_x^\infty t^{-\frac{\lambda}{\alpha}-1} \int_0^t \phi(y)dydt \end{aligned}$$

and

$$\begin{aligned} S_{\frac{\lambda}{\beta}}\psi(x) &\approx x^{-\frac{\lambda}{\beta}} \int_0^x \psi(t)dt + \int_x^\infty t^{-\frac{\lambda}{\beta}}\psi(t)dt \\ &= \frac{\lambda}{\beta} x^{-\frac{\lambda}{\beta}} \int_0^x t^{\frac{\lambda}{\beta}-1} \int_t^\infty y^{-\frac{\lambda}{\beta}}\psi(y)dydt \\ &= \frac{\lambda}{\beta} \int_x^\infty t^{-\frac{\lambda}{\beta}-1} \int_0^t \psi(y)dydt. \end{aligned}$$

The equality in the above formulas follows by using simple calculations and the Fubini theorem; we show only one of them:

$$\begin{aligned} x^{-\frac{\lambda}{\alpha}} \int_0^x \phi(t)dt + \int_x^\infty t^{-\frac{\lambda}{\alpha}}\phi(t)dt &= \\ &= x^{-\frac{\lambda}{\alpha}} \int_0^x t^{-\frac{\lambda}{\alpha}} t^{\frac{\lambda}{\alpha}} \phi(t)dt + x^{-\frac{\lambda}{\alpha}} x^{\frac{\lambda}{\alpha}} \int_x^\infty t^{-\frac{\lambda}{\alpha}} \phi(t)dt \\ &= \frac{\lambda}{\alpha} x^{-\frac{\lambda}{\alpha}} \int_0^x t^{-\frac{\lambda}{\alpha}} \int_0^t y^{\frac{\lambda}{\alpha}-1} dy \phi(t)dt + \frac{\lambda}{\alpha} x^{-\frac{\lambda}{\alpha}} \int_0^x y^{\frac{\lambda}{\alpha}-1} dy \int_x^\infty t^{-\frac{\lambda}{\alpha}} \phi(t)dt \\ &= \frac{\lambda}{\alpha} x^{-\frac{\lambda}{\alpha}} \int_0^x y^{\frac{\lambda}{\alpha}-1} \int_y^x t^{-\frac{\lambda}{\alpha}} \phi(t)dt dy + \frac{\lambda}{\alpha} x^{-\frac{\lambda}{\alpha}} \int_0^x y^{\frac{\lambda}{\alpha}-1} dy \int_x^\infty t^{-\frac{\lambda}{\alpha}} \phi(t)dt \\ &= \frac{\lambda}{\alpha} x^{-\frac{\lambda}{\alpha}} \int_0^x y^{\frac{\lambda}{\alpha}-1} \left(\int_y^x t^{-\frac{\lambda}{\alpha}} \phi(t)dt + \int_x^\infty t^{-\frac{\lambda}{\alpha}} \phi(t)dt \right) dy \\ &= \frac{\lambda}{\alpha} x^{-\frac{\lambda}{\alpha}} \int_0^x y^{\frac{\lambda}{\alpha}-1} \int_y^\infty t^{-\frac{\lambda}{\alpha}} \phi(t)dt dy. \end{aligned}$$

Step 1. $A(\alpha, \beta, \lambda) \approx A_1(\alpha, \beta, \lambda, s)$.

First we note that (using the equivalence relations $(a + b)^p \approx a^p + b^p$ for $p, a, b > 0$ and $\sup(A(x) + B(x)) \approx \sup A(x) + \sup B(x)$)

$$\begin{aligned}
(2.5) \quad A(\alpha, \beta, \lambda) &\approx \sup_{x>0} x^\lambda \left(S_{\frac{\lambda}{\alpha}} \phi(x) \right)^\alpha \left(x^{-\frac{\lambda}{\beta}} \int_0^x \psi(t) dt + \int_x^\infty t^{-\frac{\lambda}{\beta}} \psi(t) dt \right)^\beta \\
&\approx \sup_{x>0} \left(S_{\frac{\lambda}{\alpha}} \phi(x) \right)^\alpha \left(\int_0^x \psi(t) dt \right)^\beta \\
&\quad + \sup_{x>0} x^\lambda \left(S_{\frac{\lambda}{\alpha}} \phi(x) \right)^\alpha \left(\int_x^\infty t^{-\frac{\lambda}{\beta}} \psi(t) dt \right)^\beta \\
&:= I_1 + I_2.
\end{aligned}$$

Now we use the equivalence relation

$$B_1(\alpha, \beta) \approx B_3(\alpha, \beta, s)$$

from Theorem 2.1. Putting $S_{\frac{\lambda}{\alpha}} \phi(x)$ for $F(x)$ and $t^{\frac{\lambda}{\beta}} \psi(t)$ for $g(t)$ we get that

$$(2.6) \quad I_1 \approx \sup_{x>0} \left(S_{\frac{\lambda}{\alpha}} \phi(x) \right)^s \left(\int_0^x \left(S_{\frac{\lambda}{\alpha}} \phi(t) \right)^{\frac{\alpha-s}{\beta}} \psi(t) dt \right)^\beta.$$

Then we use the equivalence relation

$$B_1(\beta, \alpha) \approx B_2(\beta, \alpha, s)$$

from Theorem 2.1. Putting $x^{\frac{\lambda}{\alpha}} S_{\frac{\lambda}{\alpha}} \phi(x)$ for $G(x)$ and $x^{-\frac{\lambda}{\beta}} \psi(x)$ for $f(x)$, we have that

$$(2.7) \quad I_2 \approx \sup_{x>0} x^{\frac{\lambda s}{\alpha}} \left(S_{\frac{\lambda}{\alpha}} \phi(x) \right)^s \left(\int_x^\infty t^{\frac{\lambda s}{\alpha \beta}} \left(S_{\frac{\lambda}{\alpha}} \phi(t) \right)^{\frac{\alpha-s}{\beta}} \psi(t) dt \right)^\beta.$$

Therefore, from (2.5), (2.6) and (2.7), we get that

$$\begin{aligned}
A(\alpha, \beta, \lambda) &\approx \sup_{x>0} \left(S_{\frac{\lambda}{\alpha}} \phi(x) \right)^s \left(\int_0^x \left(S_{\frac{\lambda}{\alpha}} \phi(t) \right)^{\frac{\alpha-s}{\beta}} \psi(t) dt \right)^\beta \\
&\quad + \sup_{x>0} x^{\frac{\lambda s}{\alpha}} \left(S_{\frac{\lambda}{\alpha}} \phi(x) \right)^s \left(\int_x^\infty t^{\frac{\lambda s}{\alpha \beta}} \left(S_{\frac{\lambda}{\alpha}} \phi(t) \right)^{\frac{\alpha-s}{\beta}} \psi(t) dt \right)^\beta \\
&\approx \sup_{x>0} x^{\frac{\lambda s}{\alpha}} \left(S_{\frac{\lambda}{\alpha}} \phi(x) \right)^s \left(x^{-\frac{\lambda s}{\alpha}} \int_0^x \left(S_{\frac{\lambda}{\alpha}} \phi(t) \right)^{\frac{\alpha-s}{\beta}} \psi(t) dt \right. \\
&\quad \left. + \int_x^\infty t^{\frac{\lambda s}{\alpha \beta}} \left(S_{\frac{\lambda}{\alpha}} \phi(t) \right)^{\frac{\alpha-s}{\beta}} \psi(t) dt \right)^\beta \\
&\approx \sup_{x>0} x^{\frac{\lambda s}{\alpha}} \left(S_{\frac{\lambda}{\alpha}} \phi(x) \right)^s \left(S_{\frac{\lambda s}{\beta \alpha}} \left(\left(S_{\frac{\lambda}{\alpha}} \phi(t) \right)^{\frac{\alpha-s}{\beta}} \psi(t) \right) \right)^\beta \\
&\approx A_1(\alpha, \beta, \lambda, s).
\end{aligned}$$

Step 2. $A(\alpha, \beta, \lambda) \approx A_2(\alpha, \beta, \lambda, s)$.

The procedure is similar to that in Step 1.

We use the equivalence relation

$$B_1(\alpha, \beta) \approx B_5(\alpha, \beta, s)$$

from Theorem 2.1. Putting $S_{\frac{\lambda}{\alpha}}\phi(x)$ for $F(x)$ we find that

$$(2.8) \quad I_1 \approx \sup_{x>0} \left(S_{\frac{\lambda}{\alpha}}\phi(x) \right)^{-s} \left(\int_x^\infty \left(S_{\frac{\lambda}{\alpha}}\phi(t) \right)^{\frac{\alpha+s}{\beta}} \psi(t) dt \right)^\beta.$$

Next we use the equivalence relation

$$B_1(\beta, \alpha) \approx B_4(\beta, \alpha, s)$$

from Theorem 2.1. Putting $x^{\frac{\lambda}{\alpha}} S_{\frac{\lambda}{\alpha}}\phi(x)$ for $G(x)$ and $x^{-\frac{\lambda}{\beta}}\psi(x)$ for $f(x)$, we obtain that

$$(2.9) \quad I_2 \approx \sup_{x>0} x^{-\frac{\lambda s}{\alpha}} \left(S_{\frac{\lambda}{\alpha}}\phi(x) \right)^{-s} \left(\int_0^x t^{\frac{\lambda s}{\alpha\beta}} \left(S_{\frac{\lambda}{\alpha}}\phi(t) \right)^{\frac{\alpha+s}{\beta}} \psi(t) dt \right)^\beta.$$

Therefore, from (2.5), (2.8) and (2.9), we conclude that

$$\begin{aligned} A(\alpha, \beta, \lambda) &\approx \sup_{x>0} \left(S_{\frac{\lambda}{\alpha}}\phi(x) \right)^{-s} \left(\int_x^\infty \left(S_{\frac{\lambda}{\alpha}}\phi(t) \right)^{\frac{\alpha+s}{\beta}} \psi(t) dt \right)^\beta \\ &\quad + \sup_{x>0} x^{-\frac{\lambda s}{\alpha}} \left(S_{\frac{\lambda}{\alpha}}\phi(x) \right)^{-s} \left(\int_0^x t^{\frac{\lambda s}{\alpha\beta}} \left(S_{\frac{\lambda}{\alpha}}\phi(t) \right)^{\frac{\alpha+s}{\beta}} \psi(t) dt \right)^\beta \\ &\approx \sup_{x>0} \left(S_{\frac{\lambda}{\alpha}}\phi(x) \right)^{-s} \left(\int_x^\infty \left(S_{\frac{\lambda}{\alpha}}\phi(t) \right)^{\frac{\alpha+s}{\beta}} \psi(t) dt \right. \\ &\quad \left. + x^{-\frac{\lambda s}{\alpha\beta}} \int_0^x t^{\frac{\lambda s}{\alpha\beta}} \left(S_{\frac{\lambda}{\alpha}}\phi(t) \right)^{\frac{\alpha+s}{\beta}} \psi(t) dt \right)^\beta \\ &\approx \sup_{x>0} \left(S_{\frac{\lambda}{\alpha}}\phi(x) \right)^{-s} \left(S_{\frac{\lambda s}{\beta\alpha}} \left(x^{\frac{\lambda s}{\alpha\beta}} \left(S_{\frac{\lambda}{\alpha}}\phi(t) \right)^{\frac{\alpha+s}{\beta}} \psi(t) \right) \right)^\beta \\ &\approx A_2(\alpha, \beta, \lambda, s). \end{aligned}$$

The equivalences $A(\alpha, \beta, \lambda) \approx A_3(\alpha, \beta, \lambda, s)$ and $A(\alpha, \beta, \lambda) \approx A_4(\alpha, \beta, \lambda, s)$ can be proved similarly as those in Step 1 and Step 2, if we use the fact that the expression $A(\alpha, \beta, \lambda)$ is symmetric with respect to ϕ and ψ . Hence we omit the details. \square

Corollary 2.3. *Let λ and p be positive numbers and let ϕ be a measurable function positive a.e. in $(0, \infty)$. Then*

$$(2.10) \quad \begin{aligned} 1 &\approx \sup_{x>0} (S_\lambda\phi(x))^{-p} S_{\lambda p} \left((S_\lambda\phi(t))^{p-1} \phi(t) t^{\lambda(p-1)} \right) (x) \\ &\approx \sup_{x>0} x^{\lambda p} (S_\lambda\phi(x))^p S_{\lambda p} \left((S_\lambda\phi(t))^{-p-1} \phi(t) t^{\lambda p} \right) (x). \end{aligned}$$

Proof. Using the equivalence relations

$$A_2(1, 1, \lambda, 1) \approx A_2(1, 1, \lambda, p) \approx A_1(1, 1, \lambda, p)$$

and putting $(S_\lambda \phi(x))^{-2} x^\lambda \phi(x)$ for $\psi(x)$ we get that

$$\begin{aligned} 1 &= A_2(1, 1, \lambda, 1) \\ &\approx A_2(1, 1, \lambda, p) = \sup_{x>0} (S_\lambda \phi(x))^{-p} (S_{\lambda p} (S_\lambda \phi(t))^{p-1} \phi(t) t^{\lambda(p-1)}) (x) \\ &\approx A_1(1, 1, \lambda, p) = \sup_{x>0} x^{\lambda p} (S_\lambda \phi(x))^p (S_{\lambda p} (S_\lambda \phi(t))^{-p-1} \phi(t) t^{\lambda p}) (x), \end{aligned}$$

and the proof is complete. \square

3. SOME NEW SCALES OF CONDITIONS CHARACTERIZING THE WEIGHTED STIELTJES INEQUALITY

Our main result in this section reads:

Theorem 3.1. *Let $0 < \lambda, s < \infty$, $1 < p \leq q < \infty$, and for the weight functions u and v define*

(3.1)

$$\begin{aligned} K_1(x, s) &:= (S_{\lambda p'}(v^{1-p'})(x))^s \left(S_{\lambda s q p'} \left((S_{\lambda p'}(v^{1-p'})(t))^{\frac{q(1-p's)}{p'}} u(t) \right) (x) \right)^{\frac{1}{q}} x^{\lambda s p'}; \\ K_2(x, s) &:= (S_{\lambda p'}(v^{1-p'})(x))^{-s} \left(S_{\lambda s q p'} \left((S_{\lambda p'}(v^{1-p'})(t))^{\frac{q(1-p's)}{p'}} u(t) t^{\lambda s p' q} \right) (x) \right)^{\frac{1}{q}}; \\ K_3(x, s) &:= (S_{\lambda q}(u)(x))^s \left(S_{\lambda s q p'} \left((S_{\lambda q}(u(t))^{\frac{p'(1-qs)}{q}} (v^{1-p'})(t)) \right) (x) \right)^{\frac{1}{p'}} x^{\lambda s q}; \\ K_4(x, s) &:= (S_{\lambda q}(u)(x))^{-s} \left(S_{\lambda s q p'} \left((S_{\lambda q}(u(t))^{\frac{p'(1+qs)}{q}} (v^{1-p'})(t) t^{\lambda s p' q} \right) (x) \right)^{\frac{1}{p'}}, \end{aligned}$$

and

$$K_i(s) := \sup_{x>0} K_i(x, s), \quad i = 1, 2, 3, 4.$$

Then the Stieltjes inequality (1.2) holds for all measurable functions $f \geq 0$ if and only if any of the quantities $K_i(s)$ $i = 1, 2, 3, 4$, is finite for some s , $0 < s < \infty$. Moreover, for the best constant C in (1.2) we have $C \approx K_i(s)$, $i = 1, 2, 3, 4$. The constants in the equivalence relations can depend on s .

Proof. In (2.3) we put $\alpha = \frac{1}{q}$, $\beta = \frac{1}{p'}$, $\phi(x) = u(x)$ and $\psi(x) = v^{1-p'}(x)$. Then the assertion follows from the fact that

$$K = A\left(\frac{1}{q}, \frac{1}{p'}, \lambda\right) \quad \text{and} \quad K_i(s) = A_i\left(\frac{1}{q}, \frac{1}{p'}, \lambda, s\right), \quad i = 1, 2, 3, 4,$$

are all equivalent, with A and A_i ($i = 1, 2, 3, 4$) defined in (2.3), see Theorem 2.2. Moreover, by Proposition 1.1 the finiteness of K is necessary and sufficient for the inequality (1.2) to hold. Finally, since for the least constant C in (1.2), we have $C \approx K$ it is clear that $C \approx K_i(s)$ and the proof is complete. \square

4. SOME SCALES OF CONDITIONS CHARACTERIZING THE SOLVABILITY OF AN ONE-DIMENSIONAL NONLINEAR SECOND ORDER RICCATI TYPE EQUATION

In this section we consider the solvability problem for the following one-dimensional nonlinear second order Riccati type differential equation on the half line $(0, \infty)$:

$$(4.1) \quad u''(x) + a(x)u^q(x) = -v(x),$$

where $q > 1$, a and v are arbitrary nonnegative functions, and with boundary conditions

$$(4.2) \quad u(0) = u'(\infty) = 0.$$

We are looking for conditions on the function v for which the equation (4.1) accompanied by the boundary conditions (4.2) has a positive weak solution.

By a positive weak solution of the equation (4.1) we understand a nonnegative measurable function u satisfying a.e. on $(0, \infty)$ the equivalent integral equation

$$(4.3) \quad u(x) - \int_0^x ta(t)u^q(t)dt - x \int_x^\infty a(t)u^q(t)dt = \int_0^x tv(t)dt + x \int_x^\infty v(t)dt,$$

which follows from (4.1) by integration.

We define

$$V(x) := \int_0^x tv(t)dt + x \int_x^\infty v(t)dt.$$

Our crucial result in this section reads:

Theorem 4.1. *Let $1 < q < \infty$. Let a and v be nonnegative measurable functions on $(0, \infty)$. (i) If*

$$(4.4) \quad \int_0^x ta(t)V^q(t)dt + x \int_x^\infty a(t)V^q(t)dt \leq q^{-1} \left(\frac{q}{q-1} \right)^{1-q} V(x)$$

for a.e. $x \in (0, \infty)$, then the equation (4.1) with boundary conditions (4.2) has a nonnegative weak solution u , such that

$$V(x) \leq u(x) \leq \frac{q}{q-1} V(x), \quad \text{a.e. on } (0, \infty).$$

(ii) *If the equation (4.1) with boundary conditions (4.2) has a nonnegative weak solution, then there is a positive constant $c > 0$ such that*

$$(4.5) \quad \int_0^x ta(t)V^q(t)dt + x \int_x^\infty a(t)V^q(t)dt \leq cV(x) \quad \text{a.e. on } (0, \infty).$$

Proof. (i) Let us denote

$$H_a g(x) := \int_0^x ta(t)g^q(t)dt + x \int_x^\infty a(t)g^q(t)dt.$$

Using simple iterations we find that

$$u_{n+1} := H_a u_n + V, \quad n = 1, 2, \dots,$$

starting from $u_1 = V$. It follows by induction that if (4.4) holds, then

$$u_n \leq u_{n+1} \quad \text{and} \quad V \leq u_n \leq C_n V,$$

where $C_1 = 1$ and $C_{n+1} = q^{-1} \left(\frac{q}{q-1} \right)^{1-q} C_n + 1$. Since $x_0 = \frac{q}{q-1}$ is the only root of the equation $x = q^{-1} \left(\frac{q}{q-1} \right)^{1-q} x^q + 1$ and $C_1 = 1$, it is easy to see that $\lim_{n \rightarrow \infty} C_n = \frac{q}{q-1}$ and, hence, that exists a solution $u(x) = \lim_{n \rightarrow \infty} u_n(x)$ and such that

$$V(x) \leq u(x) \leq \frac{q}{q-1} V(x).$$

(ii) Let us assume that the equation (4.1) with boundary condition (4.2) has a nonnegative weak solution. Then

$$(4.6) \quad V(x) \leq u(x)$$

and

$$(4.7) \quad H_a u(x) \leq u(x).$$

It is easy to see that

$$(4.8) \quad V(x) \approx x S_1(tv(t))(x)$$

and

$$(4.9) \quad H_a u(x) \approx x S_1(ta(t)u^q(t))(x).$$

Therefore,

$$u(x) \approx x S_1(ta(t)u^q(t))(x) + x S_1(tv(t))(x) = x S_1(ta(t)u^q(t) + tv(t))(x).$$

Let us denote $\phi(t) := ta(t)u^q(t) + tv(t)$ and $\psi(t) := ta(t)$. Then we can rewrite the condition (4.7) to the following form:

$$(4.10) \quad \sup_{x>0} (S_1(\phi)(x))^{-1} S_1((S_1(\phi))^q \psi)(x) \leq C.$$

By Theorem 2.2 we have the following equivalence relation:

$$A_1(q-1, 1, q-1, q-1) \approx A_2(q-1, 1, q-1, 1),$$

which reads

$$\sup_{x>0} x^{q-1} (S_1(\phi)(x))^{q-1} S_{q-1}(\psi)(x)$$

$$\approx \sup_{x>0} (S_1(\phi)(x))^{-1} S_1(\psi(t) (S_1(\phi)(t))^q)(x).$$

We obtain, due to (4.10), that

$$\sup_{x>0} x^{q-1} (S_1(\phi)(x))^{q-1} S_{q-1}(\psi)(x) \leq C.$$

Moreover, by using the estimate (4.6), we get that

$$(4.11) \quad \sup_{x>0} (V(x))^{q-1} S_{q-1}(ta(t))(x) \leq C.$$

Therefore, by using the equivalence (4.8), we get that (4.11) is equivalent with the following condition:

$$(4.12) \quad \sup_{x>0} x^{q-1} (S_1(tv(t))(x))^{q-1} S_{q-1}(ta(t))(x) \leq C.$$

Now we use the equivalence relation

$$A(q-1, q-1, q-1) \approx A_2(q-1, 1, q-1, 1),$$

from Theorem 2.2, and put $\phi(t) = tv(t)$ and $\psi(t) = ta(t)$. Then, from (4.12) we get that

$$\sup_{x>0} (S_1(tv(t))(x))^{-1} S_1(t^{1+q}a(t) (S_1(tv(t))(t))^q)(x) \leq C.$$

Now, using again the equivalence (4.8), we obtain that

$$\sup_{x>0} (V(x))^{-1} x S_1(t^1 a(t) (V(t))^q)(x) \leq C,$$

which is equivalent to the condition (4.5) due of the equivalence (4.9). The proof is complete. \square

Remark 4.2. If we use some of the equivalent relations

$$A(q-1, q-1, q-1) \approx A_i(q-1, 1, q-1, s), \quad i = 1, 2, 3, 4, \quad s > 0,$$

from Theorem 2.2, by same way we can obtain other different but equivalent characterizations of the solvability for the equation (4.1). By Theorem 4.1 we can show that it is also connected to the Stieltjes inequality with weights.

With the information in Remark 4.2 in mind we now formulate the main result of this section:

Theorem 4.3. *Let $1 < q < \infty$. Let a and v be nonnegative measurable functions on $(0, \infty)$. Then the following statements are equivalent:*

(i) *The equation*

$$(4.13) \quad u''(x) + a(x)u^q(x) = -\varepsilon v(x),$$

with boundary condition (4.2) has a nonnegative weak solution u for some $\varepsilon > 0$.

(ii) *There exist positive constants C_1 and s , such that*

$$S_s(t^{s-q+2}a(t) (V(t))^{q-1-s})(x) \leq C_1 (V(x))^{-s}$$

for a.e. x on $(0, \infty)$.

(iii) *There exist positive constants C_2 and s , such that*

$$x^s S_s \left(t^{-q+2} a(t) (V(t))^{q-1+s} \right) (x) \leq C_2 (V(x))^s$$

for a.e. x on $(0, \infty)$.

(iv) *There exist positive constants C_3 and s , such that*

$$S_s \left(tv(t) (S_1(ya(y))(t))^{\frac{q-1-s}{q-1}} \right) (x) \leq C_3 x^{-s} (S_1(ya(y))(x))^{-\frac{s}{q-1}}$$

for a.e. x on $(0, \infty)$.

(v) *There exist positive constants C_4 and s , such that*

$$S_s \left(t^{1+s} v(t) (S_1(ya(y))(t))^{\frac{q-1+s}{q-1}} \right) (x) \leq C_4 (S_1(ya(y))(x))^{\frac{s}{q-1}}$$

for a.e. x on $(0, \infty)$.

(vi) *There exists a positive constant C_5 , such that*

$$\left(\int_0^\infty (S_{q'} f(x))^{q'} xv(x) dx \right)^{\frac{1}{q'}} \leq C_5 \left(\int_0^\infty (f(x))^{q'} (xa(x))^{1-q'} dx \right)^{\frac{1}{q'}}$$

holds for every positive measurable function f on $(0, \infty)$.

Proof. The proof follows by just using Theorem 4.1 combined with our equivalence Theorem 2.2 (see also Remark 4.2). \square

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