Admissible rules of Łukasiewicz logic

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Derivable and admissible rules

Consider a propositional logic L, defined by a finitary consequence relation $\vdash_L$ closed under substitution.
A rule

$$\varrho = \frac{\varphi_1, \dots, \varphi_k}{\psi}$$

is

- derivable in L, if $\varphi_1, \dots, \varphi_k \vdash_L \psi$,
- admissible in L, if the set of theorems of L is closed under ϱ: for every substitution σ, if L proves all $\sigma\varphi_i$, then it proves $\sigma\psi$. (We write $\varphi_1, \dots, \varphi_k \mathrel{|\!\!\sim}_L \psi$.)

Typical non-classical logics admit some nonderivable rules.

Properties of admissible rules

Questions about admissibility:

- decidability
- semantic characterization
- description of a basis
- ...

Well-understood for some superintuitionistic and modal logics ($\mathbf{IPC}$, $\mathbf{KC}$, $\mathbf{LC}$; $\mathbf{K4}$, $\mathbf{S4}$, $\mathbf{GL}$, $\mathbf{S4.3}$, $\ldots$).

Almost nothing is known for other nonclassical logics.

Fuzzy logics

Multivalued logics using a linearly ordered algebra of truth values

The three fundamental continuous t-norm logics are:

- Gödel–Dummett logic (**LC**): superintuitionistic; structurally complete (admissible = derivable)
- Product logic (Π): also structurally complete (Cintula & Metcalfe '09)
- Łukasiewicz logic (**Ł**): not structurally complete $\Rightarrow$ nontrivial admissibility problem

Łukasiewicz logic

Connectives: $\to$, $\neg$, $\cdot$, $\oplus$, $\wedge$, $\vee$, $\bot$, $\top$ (not all needed as basic)

Semantics: $[0,1]_{\text{Ł}} = \langle [0,1], \{1\}, \to, \neg, \cdot, \oplus, \min, \max, 0, 1 \rangle$, where

- $x \to y = \min\{1, 1 - x + y\}$
- $\neg x = 1 - x$
- $x \cdot y = \max\{0, x + y - 1\}$
- $x \oplus y = \min\{1, x + y\}$

$[0,1]_{\mathbb{Q}}$ suffices instead of $[0,1]$.
More generally, **Ł** is valid in any MV-algebra.

Calculus: Modus Ponens + finitely many axiom schemata

Algebraization

Ł is algebraizable, its equivalent algebraic semantics is the variety of MV-algebras.

propositional formula = term
rule = quasi-identity
derivable = valid in all MV-algebras
admissible = valid in free MV-algebras

Multiple-conclusion rules

Multiple-conclusion rule: $\Gamma \;/\; \Delta$, where Γ and Δ are finite sets of formulas.

$\Gamma \;/\; \Delta$ is admissible ($\Gamma \mid\!\sim \Delta$) iff for every substitution σ:
if $\vdash \sigma\varphi$ for all $\varphi \in \Gamma$, then $\vdash \sigma\psi$ for some $\psi \in \Delta$.

Example: disjunction property = $\dfrac{p \vee q}{p, q}$

Algebraization: multiple-conclusion rule = clause (disjunction of identities and their negations)

I.o.w., we want to describe the universal theory of free MV-algebras.

McNaughton functions

Free MV-algebra F_n over n generators, n finite:

- The algebra of formulas in n variables modulo Ł-provable equivalence (Lindenbaum–Tarski algebra)
- Explicit description by McNaughton: the algebra of all continuous piecewise linear functions

$$f\colon [0,1]^n \longrightarrow [0,1]$$

  with integer coefficients, with operations defined pointwise (i.e., as a subalgebra of $[0,1]_{Ł}^{[0,1]^n}$)

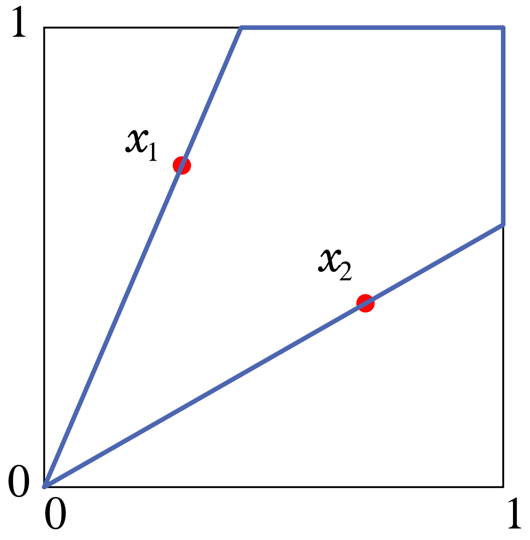
k-tuples of elements of F_n: piecewise linear functions $f\colon [0,1]^n \longrightarrow [0,1]^k$

1-reducibility

Theorem: $\Gamma \mathrel{|\!\!\sim_{\text{Ł}}} \Delta$ iff $F_1 \models \Gamma \,/\, \Delta$

(All free MV-algebras except F_0 have the same universal theory.)

Proof idea: Let $f\colon [0,1]^n \to [0,1]^k$ be a valuation in F_n such that $\Gamma(f) = 1$, $\psi(f) \neq 1$ for all $\psi \in \Delta$. Fix $x_\psi \in [0,1]^n$ such that $\psi(f(x_\psi)) < 1$, and connect them by a suitable piecewise linear curve.



Reparametrization

Recall: valuation to m variables in F_1 = continuous piecewise linear $f\colon [0,1]_{\mathbb{Q}} \to [0,1]^m_{\mathbb{Q}}$ with integer coefficients

Validity of a formula under f only depends on $\mathrm{rng}(f)$
$\Rightarrow$ Question: which piecewise linear curves can be reparametrized to have integer coefficients?

Observation: Let

$$f(t) = a + tb, \quad t \in [t_i, t_{i+1}],$$

where $a, b \in \mathbb{Z}^m$. Then the integer point a lies on the line connecting the points $f(t_i)$, $f(t_{i+1})$. This is independent of parametrization.

Anchoredness

If $X \subseteq \mathbb{Q}^m$, let $A(X)$ be its affine hull (in $\mathbb{Q}^m$)

X is anchored if $A(X) \cap \mathbb{Z}^m \neq \varnothing$

Lemma: X is anchored iff

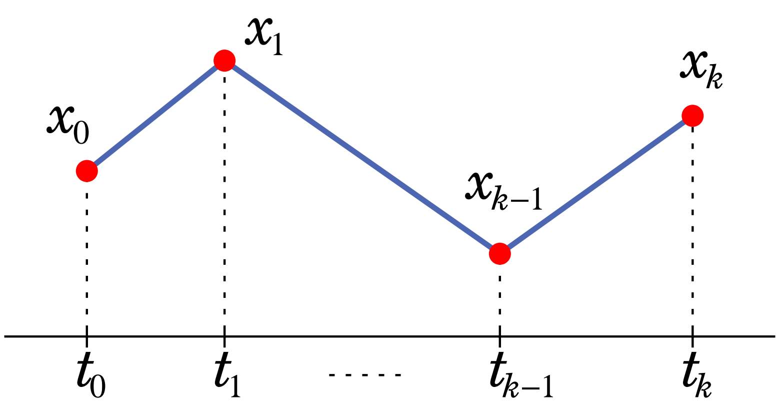
$$\forall u \in \mathbb{Z}^m \, \forall a \in \mathbb{Q} \, [\forall x \in X \, (u^{\mathsf{T}} x = a) \Rightarrow a \in \mathbb{Z}].$$

(Whenever X is contained in a hyperplane defined by an affine function with integer linear coefficients, its constant coefficients must be integer too.)

Lemma: Given $x_0, \ldots, x_k \in \mathbb{Q}^m$, it is decidable whether $\{x_0, \ldots, x_k\}$ is anchored.

Reparametrization (cont'd)

Notation: $L(t_0, x_0; t_1, x_1; \ldots; t_k, x_k) =$



Lemma: If $x_0, \ldots, x_k \in \mathbb{Q}^m$, TFAE:

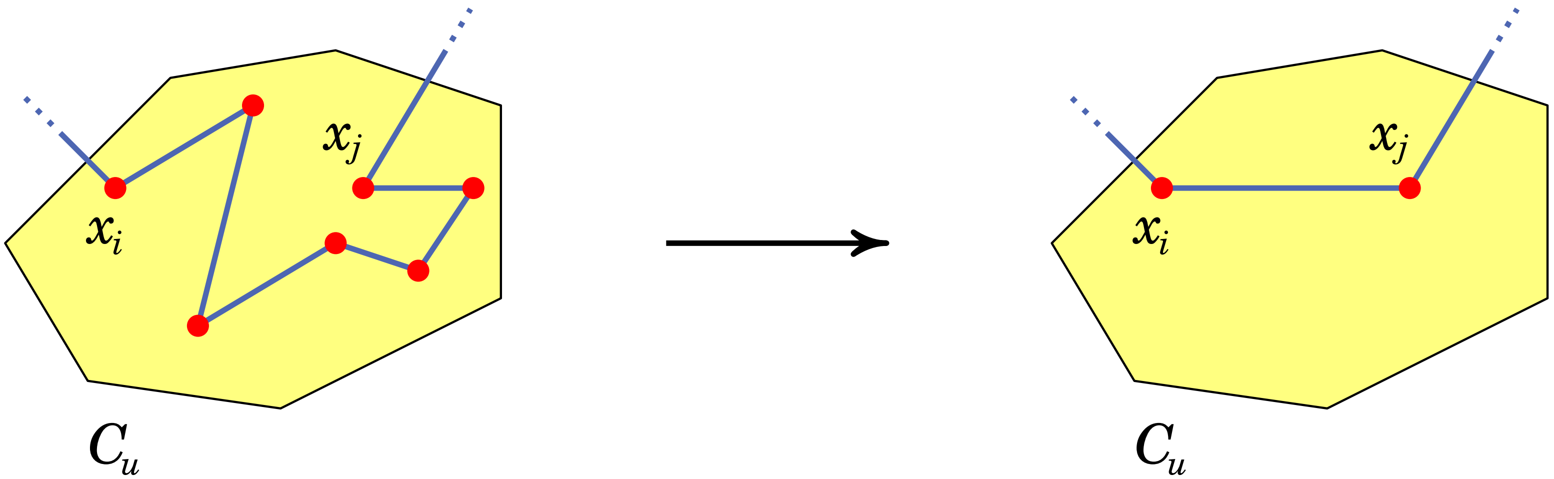
- There exist rationals $t_0 < \cdots < t_k$ such that $L(t_0, x_0; \ldots; t_k, x_k)$ has integer coefficients.
- $\{x_i, x_{i+1}\}$ is anchored for each $i < k$.

Simplification of counterexamples

Goal: Given a counterexample $L(t_0, x_0; \ldots; t_k, x_k)$ for Γ / Δ in F_1, simplify it so that its parameters (e.g., k) are bounded

$\{x \in [0,1]^m_{\mathbb{Q}} \mid \bigwedge \Gamma(x) = 1\}$ is a finite union $\bigcup_{u<r} C_u$ of polytopes.

Idea: If $\mathrm{rng}(L(t_i, x_i; \ldots; t_j, x_j)) \subseteq C_u$, replace $L(t_i, x_i; t_{i+1}, x_{i+1}; \ldots; t_j, x_j)$ with $L(t_i, x_i; t_j, x_j)$

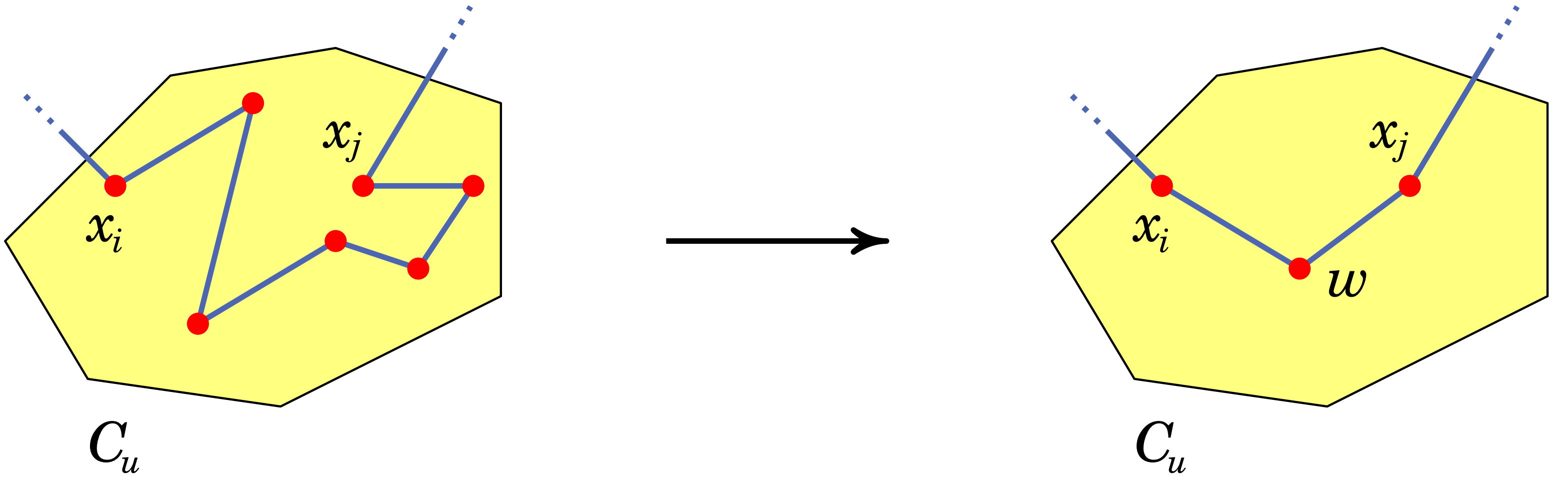


Trouble: $\{x_i, x_j\}$ needn't be anchored: $L(t_i, \frac{1}{2}; t_{i+1}, 0; t_{i+2}, \frac{1}{2})$

Simplification of counterexamples (cont'd)

What cannot be done in one step can be done in two steps:

Lemma: If $C \subseteq \mathbb{Q}^m$ is convex and anchored, and $x, y \in \mathbb{Q}^m$, there exists $w \in C$ such that $\{x, w\}$ and $\{w, y\}$ are anchored.



Main results

Theorem: Admissibility in **Ł** is decidable. Moreover:

- Admissibility in **Ł**, and the universal theory of free MV-algebras, are in $PSPACE$.
- We have explicit bounds on counterexamples for inadmissible rules in F_1.
- Every formula has a finite admissibly saturated approximation in **Ł**.
- We have an explicit basis of **Ł**-admissible rules. There is no finite basis.

Admissibly saturated formulas

A formula φ is admissibly saturated if $\varphi \mathrel{|\!\sim} \Delta \Rightarrow \exists \psi \in \Delta\, \varphi \vdash \psi$.

An admissibly saturated approximation of φ is a finite set Π_φ of a.s. formulas such that $\varphi \mathrel{|\!\sim} \Pi_\varphi$, and $\pi \vdash \varphi$ for each $\pi \in \Pi_\varphi$.

Example: Projective formulas are a.s.

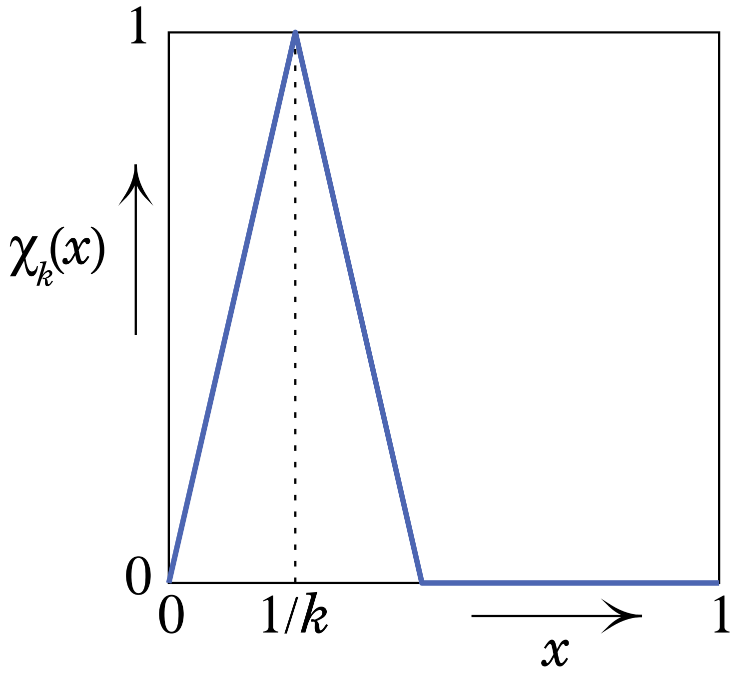
Theorem:

- $\varphi \in F_m$ is a.s. in **Ł** iff $\{x \in [0,1]^m \mid \varphi(x) = 1\}$
  - is connected,
  - hits $\{0,1\}^m$, and
  - is a finite union of anchored polytopes.
- In **Ł**, every formula has an a.s. approximation.

Single-conclusion basis

Theorem: $RCC_3 + \{NA_p \mid p$ is a prime$\}$ is an independent basis of single-conclusion **Ł**-admissible rules.

$$RCC_n = \frac{(q \vee \neg q)^n \to p \quad p \vee \neg p}{p} \qquad NA_k = \frac{p \vee \chi_k(q)}{p}$$



Multiple-conclusion basis

Theorem: $WDP + CC_3 + \{NA_p \mid p$ is a prime$\}$ is an independent basis of multiple-conclusion **Ł**-admissible rules.

$$WDP = \frac{p \vee \neg p}{p, \neg p}$$

$$CC_n = \frac{\neg(q \vee \neg q)^n}{}$$

Thank you for attention!

References

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