Lecture 2: Propagation of light waves and pulses

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- Propagation of plane waves in linear homogeneous isotropic (LHI) media:
  - ☞ refractive index
  - ☞ intensity
  - ☞ absorption
- Propagation of light pulses:
  - ☞ group velocity
  - ☞ group dispersion — chirp
  - ☞ superluminal effects (?)

Wave equation

$$\left.\begin{aligned}\nabla \wedge \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = 0 \\ \nabla \wedge \boldsymbol{H} - \frac{\partial \boldsymbol{D}}{\partial t} = 0\end{aligned}\right\} \Rightarrow \begin{aligned}\nabla^2 \boldsymbol{E} - \varepsilon\mu_0 \frac{\partial^2 \boldsymbol{E}}{\partial t^2} = 0 \\ \nabla^2 \boldsymbol{H} - \varepsilon\mu_0 \frac{\partial^2 \boldsymbol{H}}{\partial t^2} = 0\end{aligned} \quad \text{or} \quad \begin{aligned}\nabla^2 \boldsymbol{E} - \frac{N^2}{c^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2} = 0 \\ \nabla^2 \boldsymbol{H} - \frac{N^2}{c^2}\frac{\partial^2 \boldsymbol{H}}{\partial t^2} = 0\end{aligned}$$

$$\left(\nabla \wedge (\nabla \wedge \boldsymbol{E}) = \nabla\nabla \cdot \boldsymbol{E} - \nabla^2 \boldsymbol{E}\right)$$

well-known identity

$$\varepsilon = \varepsilon_0 \varepsilon_r$$

$$\varepsilon_r = \varepsilon_r' - i\varepsilon_r''$$

dielectric constant

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$$

$$N^2 = \varepsilon\mu_0 c^2 = \varepsilon_r$$

refractive index

$$N = n - i\kappa$$

$$\varepsilon_r' = n^2 - \kappa^2, \qquad \varepsilon_r'' = 2n\kappa,$$

$$n = \sqrt{\frac{1}{2}\left(\sqrt{\varepsilon_r'^2 + \varepsilon_r''^2} + \varepsilon_r'\right)}, \qquad \kappa = \sqrt{\frac{1}{2}\left(\sqrt{\varepsilon_r'^2 + \varepsilon_r''^2} - \varepsilon_r'\right)}$$

Propagation in the vacuum

$$\nabla^2 \boldsymbol{E} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2} = 0$$
$$\nabla^2 \boldsymbol{H} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{H}}{\partial t^2} = 0$$

$$\boldsymbol{E} = \boldsymbol{E}_0 e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})}$$
$$\boldsymbol{H} = \boldsymbol{H}_0 e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})}$$

with $|\boldsymbol{k}| \equiv k = \frac{\omega}{c}$

- Back to Maxwell's equations:

$$\nabla \wedge \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = -i\boldsymbol{k} \wedge \boldsymbol{E} + i\omega\mu_0 \boldsymbol{H} = 0$$
$$\nabla \cdot \boldsymbol{E} = -i\boldsymbol{k} \cdot \boldsymbol{E} = 0$$

$$\frac{1}{\mu_0 \omega}(\boldsymbol{k} \wedge \boldsymbol{E}_0) = \boldsymbol{H}_0$$

$$\boldsymbol{k} \perp \boldsymbol{E}_0 \perp \boldsymbol{H}_0 \perp \boldsymbol{k}$$
$$\boldsymbol{H}_0 = \eta_0^{-1}(\boldsymbol{s} \wedge \boldsymbol{E}_0)$$
$$\boldsymbol{B}_0 = c^{-1}(\boldsymbol{s} \wedge \boldsymbol{E}_0)$$

$$\boldsymbol{s} = \boldsymbol{k}/k$$
$$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \approx 377\Omega$$

Superpositions of plane waves

- In the k-space: 3-dimensional

$$\boldsymbol{E}(t,\boldsymbol{r}) = \iiint \boldsymbol{E}_0(\boldsymbol{k})\, e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})} d\boldsymbol{k}$$

$$\boldsymbol{H}(t,\boldsymbol{r}) = \iiint \boldsymbol{H}_0(\boldsymbol{k})\, e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})} d\boldsymbol{k}$$

- In the ω-space: 1-dimensional (propagation along z: $k_x = k_y = 0$)

$$\boldsymbol{E}(t,z) = \int \boldsymbol{E}_0(\omega)\, e^{i(\omega t - kz)} d\omega$$

$$\boldsymbol{H}(t,z) = \int \boldsymbol{H}_0(\omega)\, e^{i(\omega t - kz)} d\omega$$

- Pulse: central frequency ω_0; central wave vector $k_0 = \omega_0/c$.

$$\boldsymbol{E}(t,z) = \boldsymbol{E}_1(t,z)\; e^{i(\omega_0 t - \boldsymbol{k}_0 z)} = e^{i(\omega_0 t - \boldsymbol{k}_0 z)} \int \boldsymbol{E}_0(\omega)\, e^{i((\omega-\omega_0)t - (k-k_0)z)} d\omega =$$

$$= e^{i(\omega_0 t - \boldsymbol{k}_0 z)} \int \boldsymbol{E}_0(\omega)\, e^{i(\omega-\omega_0)(t - z/c)} d\omega = \boldsymbol{E}_1(t - z/c)\; e^{i(\omega_0 t - \boldsymbol{k}_0 z)}$$

Propagation in the matter

$$\nabla^2 \boldsymbol{E} - \frac{N^2}{c^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2} = 0$$
$$\nabla^2 \boldsymbol{H} - \frac{N^2}{c^2}\frac{\partial^2 \boldsymbol{H}}{\partial t^2} = 0$$

$$\boldsymbol{E} = \boldsymbol{E}_0 e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})}$$
$$\boldsymbol{H} = \boldsymbol{H}_0 e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})}$$

with $|\boldsymbol{k}| \equiv k = \frac{\omega}{c} N$

- Back to Maxwell's equations:

$$\nabla \wedge \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = -i\boldsymbol{k} \wedge \boldsymbol{E} + i\omega\mu_0 \boldsymbol{H} = 0$$
$$\nabla \cdot \boldsymbol{E} = -i\boldsymbol{k} \cdot \boldsymbol{E} = 0$$

$$\frac{1}{\mu_0 \omega}(\boldsymbol{k} \wedge \boldsymbol{E}_0) = \boldsymbol{H}_0$$

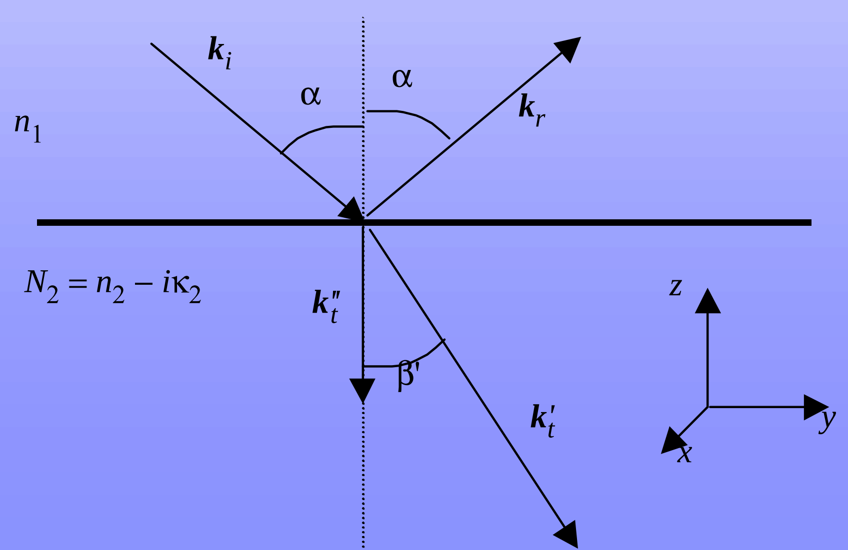
$$\boldsymbol{k} \perp \boldsymbol{E}_0 \perp \boldsymbol{H}_0 \perp \boldsymbol{k}$$
$$\boldsymbol{H}_0 = \frac{N}{\eta_0}(\boldsymbol{s} \wedge \boldsymbol{E}_0)$$
$$\boldsymbol{B}_0 = \frac{N}{c}(\boldsymbol{s} \wedge \boldsymbol{E}_0)$$

$$\boldsymbol{s} = \boldsymbol{k}/k$$
$$\eta = \frac{\eta_0}{N} = \sqrt{\frac{\mu_0}{\varepsilon}}$$

Propagation direction (vector *k*)

$$\boldsymbol{k} = \boldsymbol{k}' - i\boldsymbol{k}''$$

- $\boldsymbol{k}' = n\,\omega/c$, $\boldsymbol{k}'' = \kappa\,\omega/c$
- $\boldsymbol{k}'$ and $\boldsymbol{k}''$ can have different different directions:
  - ☞ $\boldsymbol{k}'\cdot\boldsymbol{r} = const$: constant phase planes
  - ☞ $\boldsymbol{k}''\cdot\boldsymbol{r} = const$: constant amplitude planes
- $\boldsymbol{k}''$: boundary conditions of an absorbing sample



Propagation direction (vector *S*)

- Energy flow (Poynting vector ***S***)

$$\langle \mathbf{S} \rangle = \langle \mathbf{E} \wedge \mathbf{H} \rangle = \tfrac{1}{2}\,\mathrm{Re}\left\{ \mathbf{E} \wedge \mathbf{H}^* \right\}$$

$$\langle \boldsymbol{S} \rangle = \frac{1}{2\omega\mu_0}\mathrm{Re}\left\{ \boldsymbol{E} \wedge \left(\boldsymbol{k}^* \wedge \boldsymbol{E}^* \right) \right\} = \frac{1}{2\omega\mu_0}\mathrm{Re}\left\{ \boldsymbol{k}^* \left(\boldsymbol{E} \cdot \boldsymbol{E}^* \right) - \underbrace{(\boldsymbol{k}^* \cdot \boldsymbol{E})}_{0} \boldsymbol{E}^* \right\} =$$

$$= \frac{1}{2\omega\mu_0}\mathrm{Re}\left\{ \left(\boldsymbol{k}' - i\boldsymbol{k}'' \right) \left(\boldsymbol{E}_0 \cdot \boldsymbol{E}_0^* \right) \, e^{-2\boldsymbol{k}'' \cdot \boldsymbol{r}} \right\} = \frac{\boldsymbol{k}'}{2\omega\mu_0} \left| \boldsymbol{E}_0 \right|^2 \, e^{-2\boldsymbol{k}'' \cdot \boldsymbol{r}}$$

- Propagation along *z* (***k*** // *z*)

$$\left| \boldsymbol{S} \right| = \left(2\omega\mu_0 \right)^{-1} n \frac{\omega}{c} \left| \boldsymbol{E}_0 \right|^2 e^{-2\omega\kappa z / c} = \frac{n}{2\eta_0} \left| \boldsymbol{E}_0 \right|^2 e^{-\alpha z}$$

$$\alpha(\omega) = \frac{2\omega\kappa}{c} = \frac{4\pi\kappa}{\lambda}$$

Optical pulses: dispersion

- Dispersion relation (non-absorbing medium):

$$k = \frac{\omega}{c} n(\omega)$$

- Optical pulse as a linear combination of eigenmodes:

$$\boldsymbol{E}(t,z) = \int \boldsymbol{E}_0(k)\, e^{i(\omega(k)t - kz)} dk = \int \boldsymbol{E}_0(\omega)\, e^{i(\omega t - k(\omega)z)} d\omega$$

- Mean frequency ω_0 and wave vector k_0: $\Delta\omega \ll \omega_0$, $\Delta k \ll k_0$:

$$\omega(k) = \omega_0 + \left(\frac{d\omega}{dk}\right)_0 (k-k_0) + \frac{1}{2}\left(\frac{d^2\omega}{dk^2}\right)_0 (k-k_0)^2 \quad = \quad \omega_0 + v_g (k-k_0) + \frac{\beta}{2}(k-k_0)^2$$

$$k(\omega) = k_0 + \left(\frac{dk}{d\omega}\right)_0 (\omega-\omega_0) + \frac{1}{2}\left(\frac{d^2 k}{d\omega^2}\right)_0 (\omega-\omega_0)^2 \quad = \quad k_0 + v_g^{-1} (\omega-\omega_0) + \frac{\psi}{2}(\omega-\omega_0)^2$$

$\psi = -\beta / v_g^3$ … group velocity dispersion (GVD)

Group velocity (first order effects)

- Propagation of a pulse

$$\boldsymbol{E}(t,z)=\int_{-\infty}^{\infty}\boldsymbol{E}_0(k)\ e^{i(\omega_0 t-k_0 z)}e^{i\left(v_g(k-k_0)t-(k-k_0)z\right)}dk=e^{i(\omega_0 t-k_0 z)}\int_{-\infty}^{\infty}\boldsymbol{E}_0(k)\ e^{i(v_g t-z)(k-k_0)}dk=$$

$$==e^{i(\omega_0 t-k_0 z)}\boldsymbol{E}_1\left(z-v_g t\right)=\underbrace{e^{-ik_0(z-vt)}}_{(A)}\ \underbrace{\boldsymbol{E}_1\left(z-v_g t\right)}_{(B)}$$

- (A): field oscillation with the central frequency

wave front velocity: $$v=\frac{\omega_0}{k_0}=\frac{c}{n(\omega)}$$

- (B): propagation of envelope without deformation

group velocity $$v_g=\left.\frac{d\omega}{dk}\right|_{\omega_0}$$

$$k=\frac{\omega}{c}n(\omega)\qquad /d/dk\qquad\Rightarrow\qquad v_g=\frac{c}{n+\omega\,dn/d\omega}$$

Optical pulses: energy flow

- Poynting vector

$$\left|\left\langle \boldsymbol{S}(z-v_g t)\right\rangle\right| = \frac{n}{2\eta_0}\left|\boldsymbol{E}_1(z-v_g t)\right|^2 e^{-2\omega\kappa z/c}$$

$$c = \frac{1}{\sqrt{\varepsilon_0\mu_0}}, \qquad \eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}$$

- Energy conservation law

$$e^{2\omega\kappa z/c}\frac{\partial\langle U\rangle}{\partial t} = \frac{\omega}{2}\varepsilon''\boldsymbol{E}_1\boldsymbol{E}_1^* + \frac{1}{2}\left(\frac{d(\omega\varepsilon')}{d\omega}\frac{\partial \boldsymbol{E}_1}{\partial t}\cdot\boldsymbol{E}_1^* + \varepsilon'\frac{\partial \boldsymbol{E}_1}{\partial t}\cdot\boldsymbol{E}_1^*\right) =$$

$$= \omega n\kappa\,\varepsilon_0\boldsymbol{E}_1\boldsymbol{E}_1^* + \underbrace{\left(n^2 + \frac{\omega}{2}dn^2/d\omega\right)}_{n(n+\omega dn/d\omega)=nc/v_g}\varepsilon_0\frac{\partial \boldsymbol{E}_1}{\partial t}\cdot\boldsymbol{E}_1^*$$

$$\nabla\cdot\langle\boldsymbol{S}\rangle = \frac{n}{2\eta_0}\frac{\partial}{\partial z}\left|\boldsymbol{E}_1(z-v_g t)\right|^2 e^{-2\omega\kappa z/c} =$$

$$= -\left(\frac{\omega n\kappa}{\eta_0 c}\boldsymbol{E}_1\cdot\boldsymbol{E}_1^* + \frac{n}{\eta_0 v_g}\frac{\partial \boldsymbol{E}_1}{\partial t}\cdot\boldsymbol{E}_1^*\right)e^{-2\omega\kappa z/c} = -\frac{\partial\langle U\rangle}{\partial t}$$

Group velocity dispersion GVD (second order effects)

- Main effect: pulse broadening in time
- This treatment is necessary for
  - ☞ Ultrashort pulse (sub-50 fs) propagation
  - ☞ Ultrashort pulse (ps or sub-ps) generation in lasers (multiple passes through dispersive media)
  - ☞ Short pulse (sub-ns) propagation in fibers (extremely long distances)

$$\omega(k) = \omega_0 + v_g (k - k_0) + \frac{\beta}{2}(k - k_0)^2$$

$$k(\omega) = k_0 + v_g^{-1}(\omega - \omega_0) + \frac{\psi}{2}(\omega - \omega_0)^2$$

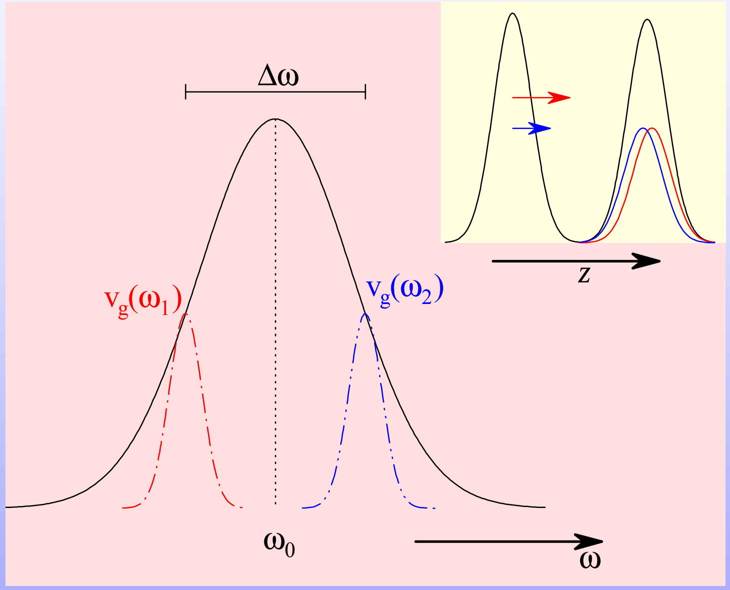
Intuitive treatment

- Spectral components coming from two ends of the spectrum propagate with different velocities

$$\Delta v_g = \frac{dv_g}{d\omega}\Delta\omega$$

- GVD is defined as

$$\psi = \frac{d^2k}{d\omega^2} = \frac{d}{d\omega}\left(\frac{1}{v_g}\right) = -\frac{1}{v_g^2}\frac{dv_g}{d\omega}$$



- Spreading due to a thickness *L* of the dispersive material:

$$\Delta\tau \approx \frac{L}{v_g^2}\left|\Delta v_g\right| = \frac{L}{v_g^2}\left|\frac{dv_g}{d\omega}\right|\Delta\omega = L\,\psi\,\Delta\omega \quad \left(= 4\ln 2\,\frac{\psi L}{\tau_0}\right)$$

for a gaussian pulse shape

Exact treatment

- Superposition of the spectral components

$$\boldsymbol{E}(t,z)=\int_{-\infty}^{\infty}\boldsymbol{E}_0(\omega)\,e^{i(\omega t-k(\omega)z)}\,d\omega=e^{i(\omega_0 t-k_0 z)}\int_{-\infty}^{\infty}\boldsymbol{E}_0(\omega)\,e^{i(\omega-\omega_0)(t-z/v_g)}\;e^{i\beta z/(2v_g^3)(\omega-\omega_0)^2}\,d\omega$$

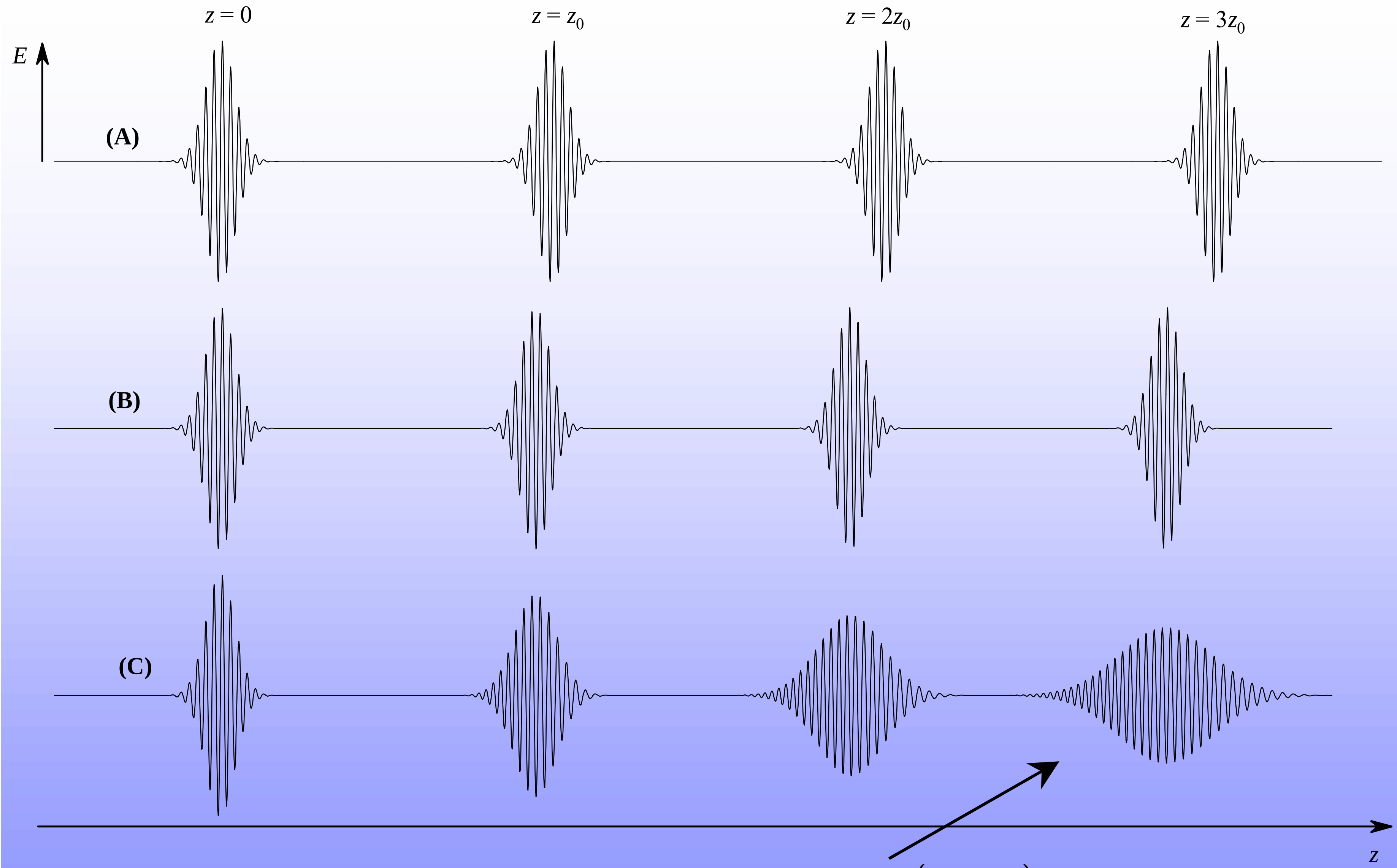
- One gets for a gaussian pulse for a crystal length *L* after all the integrations:

$$\mathbf{E}(t,z)=A\,e^{i(\omega_0 t-k_0 z)}\frac{1}{\sqrt{1+i\delta}}\;e^{-\frac{\alpha\left(t-z/v_g\right)^2(1-i\delta)}{\left(\delta^2+1\right)}}\qquad \text{with}\qquad \delta=2\alpha\psi L=\frac{4\ln 2\,\psi L}{\tau_0^2}$$

- Spreading due to a thickness *L* of the dispersive material:

$$\tau(L)=\tau_0\sqrt{1+\delta^2}=\tau_0\sqrt{1+\left(\frac{4\ln 2\,\psi L}{\tau_0^2}\right)^2}\qquad\left(\Delta\tau\approx\frac{4\ln 2\,\psi L}{\tau_0}\right)$$

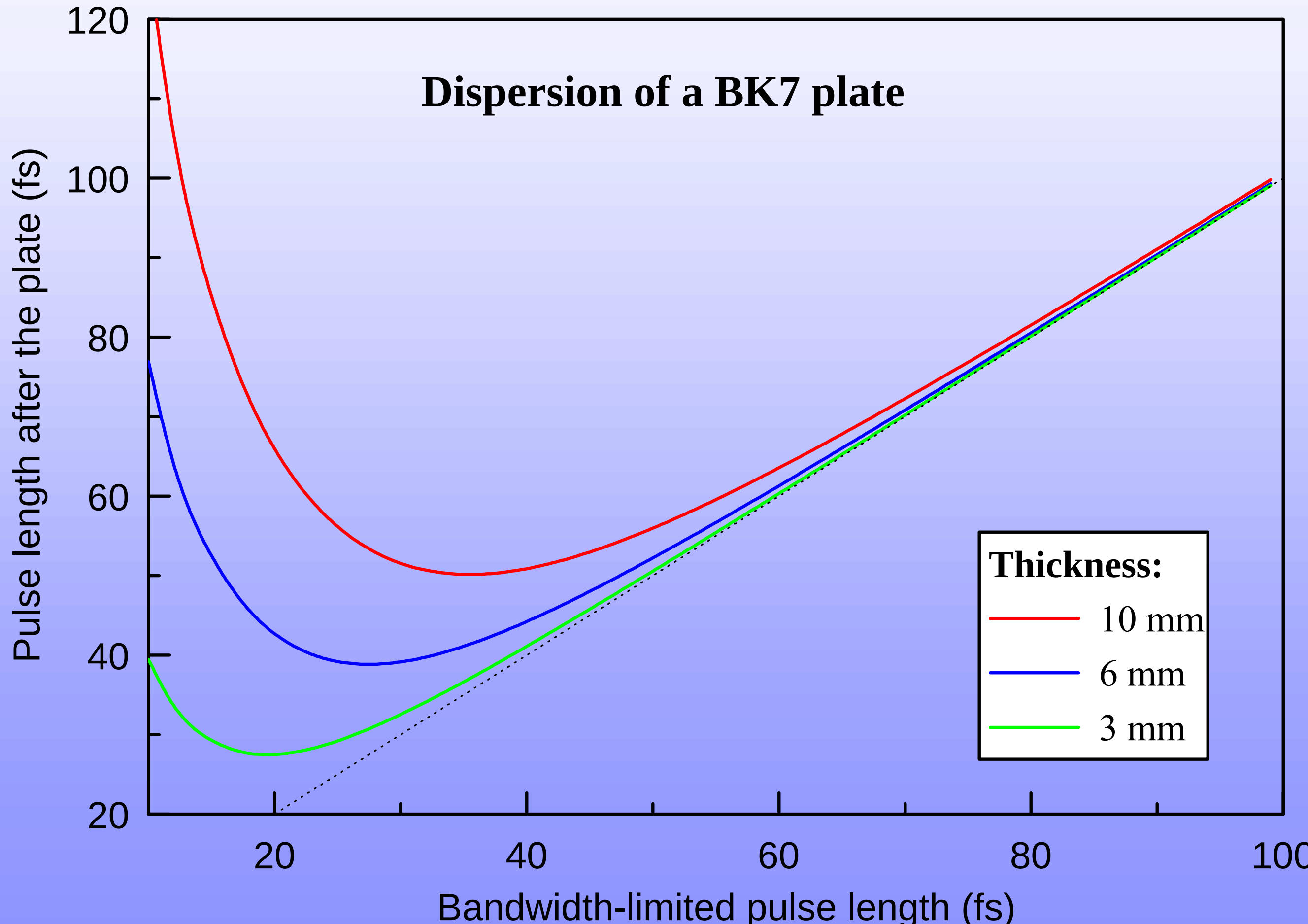
(intuitive treatment)



Mean frequency varies in time: $\omega(z,t) = \frac{\partial \Phi}{\partial t} = \omega_0 + \frac{\left(t - z/v_g\right) 4\alpha^2 \psi L}{1 + \left(2\alpha \psi L\right)^2}$

Chirp (group delay dispersion): $GDD = -\frac{\partial^2 \Phi}{\partial \omega^2} = \psi L$

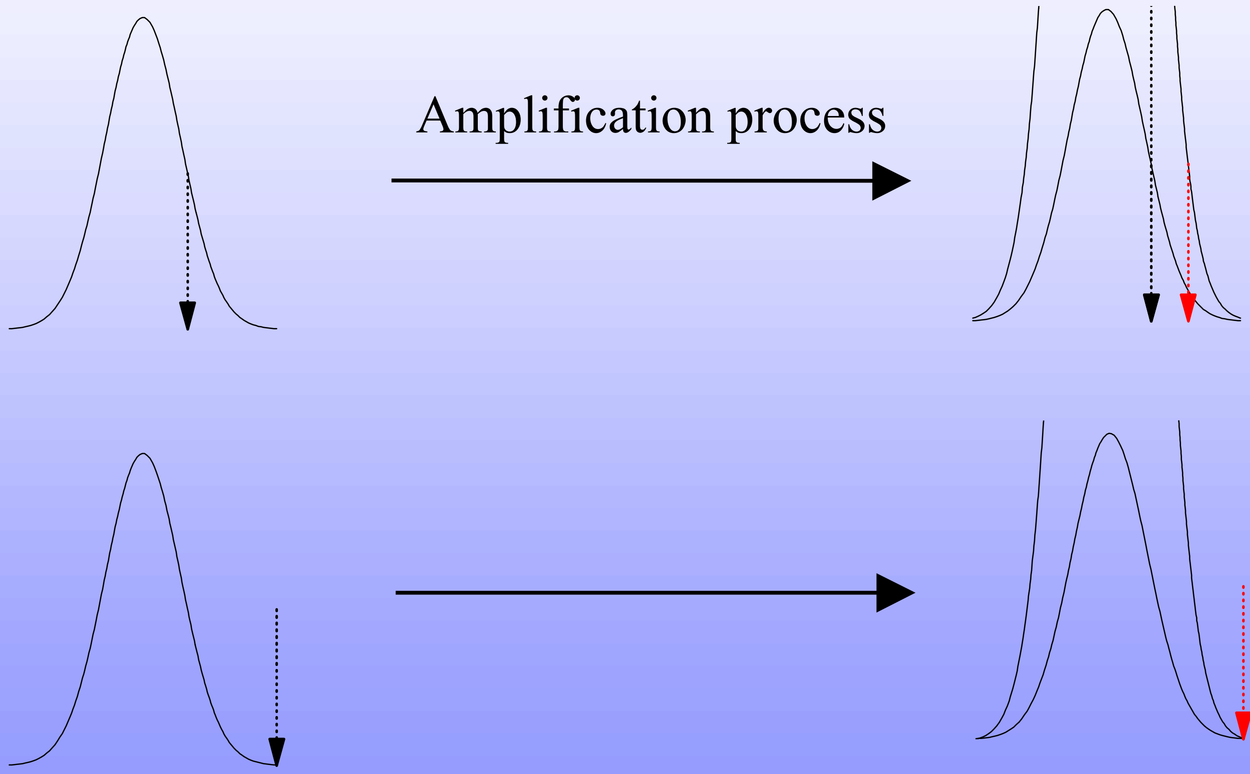
Pulse spreading: example



“Superluminal” effects

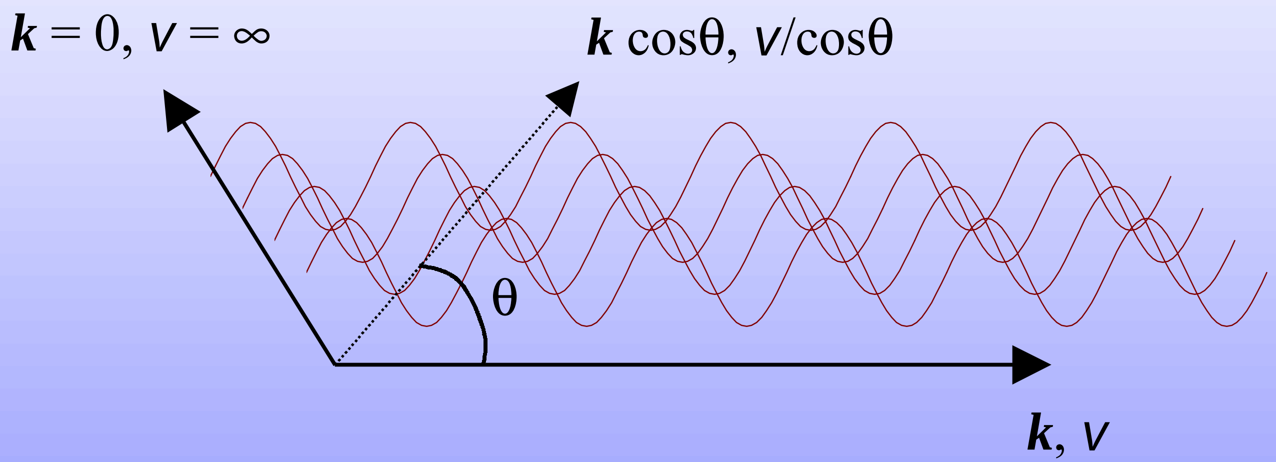
- What can be superluminal?
  - ☞ Phase velocity?
  - ☞ Group velocity?
  - ☞ Velocity of an information?
- Definition of the speed of an information transfer

Information speed



The noise can only slow down the information transfer

Phase velocity



Group velocity

$$v_g = \frac{c}{n + \omega \frac{dn}{d\omega}}$$

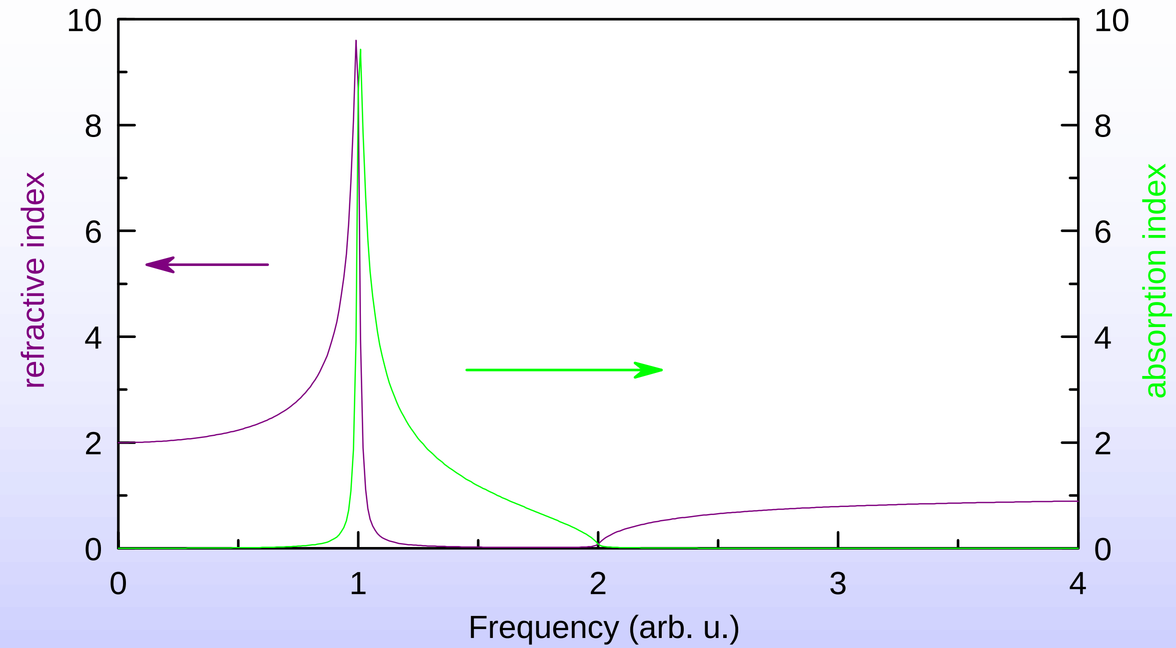
- Group velocity can exceed the vacuum speed of the light:
  - ☞ region of anomalous dispersion
    - dielectric resonance
    - gain medium

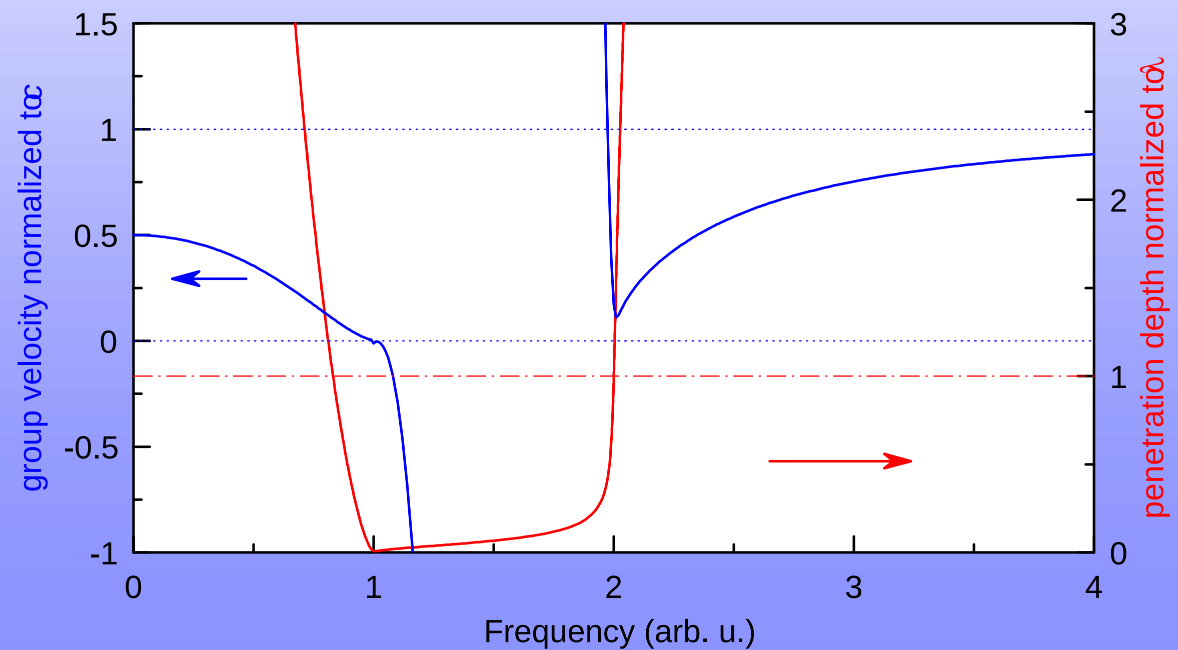
Dielectric resonance

- group velocity

$$v_g = \frac{c}{n + \omega \dfrac{dn}{d\omega}}$$

- group velocity can be superluminal
- penetration depth = fraction of λ





Gain-assisted superluminal light propagation

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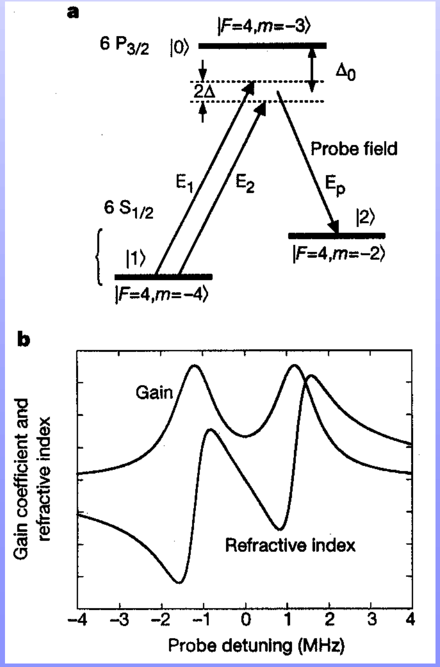
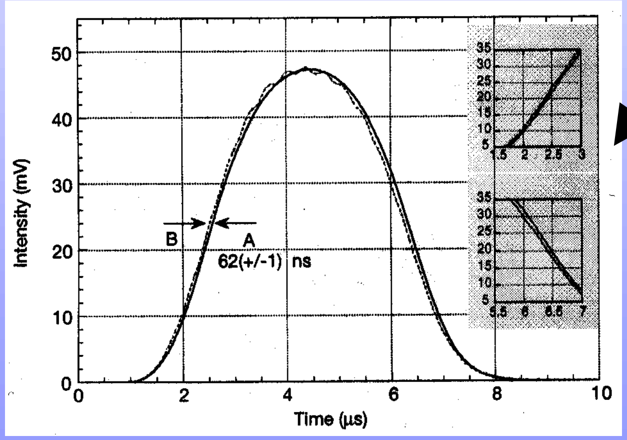
Gain-assisted anomalous dispersion:



Measured pulse advancement corresponds to a negative group velocity



Conclusion:

- pulse maximum and leading / trailing edge of the pulse are advanced
- there is *almost* (!!) no change in the pulse shape
- the very first non-zero signal can *never* (!!) be advanced