1963ApJ...137..135W

FERMI ACCELERATION OF CHARGED PARTICLES

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Received May 16, 1962; revised July 16, 1962

ABSTRACT

Two efficient types of Fermi acceleration by hydromagnetic shocks are presented. One is based on Parker's (1958*a*) work on acceleration between shocks which cross each other. An equivalent situation is that of a single shock crossing a closed magnetic line of force. It is shown that a net gain in particle energy occurs even if the shocks are weak A second type is the acceleration by a shock which moves into stronger magnetic fields. While the particle is accelerated ahead of the shock and is able to penetrate stronger fields, the shock field may become sufficiently stronger that the particle is unable to escape the trap and is accelerated for a long time No such prolonged trapping occurs for the same particle decelerated behind a similar shock moving into weaker fields, so that the net efficiency of acceleration is high. Conditions on acceleration of thermal particles, specifically electrons, are discussed. One condition is that the particle must move faster than the shock.

I. INTRODUCTION

Fermi acceleration of charged particles (Fermi 1949; Spitzer 1956) is based on the constancy of the magnetic moment of a particle encountering varying magnetic fields. If a strong field is moving, it reflects the particle and causes an increase or decrease in particle energy according as the collision is in a head-on or overtaking direction. The major problem with Fermi acceleration is that the particle in the long run experiences nearly as many decelerations as accelerations, making the net rate of acceleration quite small. Parker (1958*a*) reviewed some of the attempts to overcome this difficulty and proposed that particles are accelerated between shocks which cross each other. The following physical concepts emphasized by him are applicable in many circumstances. First, a shock moving into any given region sweeps *ahead* of it all particles that can be trapped, so that, initially, only acceleration occurs. Second, when two shocks approach each other along a line of force, the rate of collisions for particles between the shocks increases, and every particle *must* gain enough energy to cross a shock before the shocks cross each other. Third, the shocks are sharp disturbances of the magnetic fields, and the pitch angles of particles crossing the shocks are reoriented, making further acceleration or deceleration by the next shock possible.

We shall show in Section II that Parker's mechanism is efficient even for shocks of moderate strength, although his result (Parker 1958*a*, eq. [36]) does not indicate this. The computations are based on the equivalent situation of a single shock crossing a closed line of force and intersecting it in two places. In Section III we shall investigate the acceleration by shocks moving through inhomogeneous magnetic fields. We shall demonstrate for a simple case that accelerations ahead of a shock which moves into stronger fields are much more numerous than decelerations behind a similar shock which moves into weaker fields. Generalization and a discussion of its limitations follow. In Section IV we evaluate some of the conditions on particle speed necessary for this acceleration. The discussion centers on acceleration of electrons in the tail of a thermal velocity distribution and the energies which such electrons may achieve.

II. ACCELERATION IN FIELDS WITH UNIFORM STRENGTH

Denote by $\boldsymbol{H}$, $\boldsymbol{v}$, $v_\perp$, θ, and μ the magnetic field, the particle velocity, its component perpendicular to $\boldsymbol{H}$, the pitch angle, and the magnetic moment, respectively, where

$$v_\perp = v \sin\theta \qquad \text{and} \qquad \mu = \frac{v_\perp^2}{H} = \frac{v^2 \sin^2\theta}{H}. \tag{1}$$

The magnetic moment is a constant of the motion if the fractional change in the magnetic field is small during one Larmor period, over one Larmor radius, and over the distance traversed during one Larmor period. In a frame of reference in which the magnetic field is static, the particle energy and v^2 are constant. Since $v_\perp^2 \leq v^2$, the particle is reflected where $v_\perp^2 = v^2$. We denote the field strength at the mirror point by H_{mp},

$$\frac{v_\perp^2}{H} = \frac{v^2}{H_{mp}} \quad \text{and} \quad \frac{H}{H_{mp}} = \sin^2\theta \,. \tag{2}$$

In a frame of reference in which fields stronger than H_{mp} are moving, the reflection of the particle appears as a gain or loss of energy, depending on whether the particle enters the strong fields in a head-on or overtaking direction. Let us consider only particles with speeds much greater than the speed of the reflecting magnetic fields—in our case the speed of the hydromagnetic shock, v_{sh}—and let us neglect all collisional losses of energy. Following Parker (1958*a*), we shall also assume that μ is strictly constant at all times except when the particle crosses a shock, whereupon it is scattered so as to produce an isotropic distribution in pitch angles. (These assumptions will be treated in Sec. IV.) The fractional change in kinetic energy per reflection is of order v_{sh}/v. To maintain this efficiency, it is necessary to find situations in which accelerations statistically outnumber decelerations.

In this section we consider a shock, in which fields are as high as H_{sh}, moving across a curved magnetic line of force along which the undisturbed field strength is a constant, H_0. The condition for reflection of charged particles from either side of the shock is

$$\theta > \theta_c \,, \qquad \sin^2\theta_c = \frac{H_0}{H_{sh}} \,. \tag{3}$$

We are interested specifically in a single shock intersecting one line of force in two points, In Figure 1, *a*, the shock is assumed to form two intersections with a closed line of force for the *whole* time during which the shock sweeps across the line of force. Physically. this need not be the case, either because the line of force is not closed or because the shock front is insufficiently long and intersects the line of force in two places only near the end of its passage across that line of force. Such a situation is represented in Figure 2, *a*. Since it is the simpler case, it is treated first. The two intersections of the shock with the line of force form a single trap ahead of the shock. Figure 2, *b*, shows the variation of field strength along the line of force. As the shock progresses, the trap becomes shorter, the trapped particles are accelerated, and the number of accelerations per second increases. Every particle is able to escape the trap before the length of the trap shrinks to zero and the shock leaves the line of force. Thus the energy of every particle initially trapped with a pitch angle θ is multiplied by a factor $\sin^2\theta/\sin^2\theta_c$. Decelerations of particles approaching the rear of the shock are negligible in number, since the length of the trap behind the shock is assumed infinite in extent and each particle is reflected at most once. Assuming an initially isotropic distribution of pitch angles, the ratio of final to initial energy, averaged over all particles in the region in which the shock forms a trap at any time, is

$$\left\langle\frac{E_f}{E_0}\right\rangle = \int_{\theta_c}^{\pi/2} \frac{\sin^2\theta}{\sin^2\theta_c} \sin\theta \, d\theta + (1 - \cos\theta_c) = 1 + \tfrac{2}{3}\,\frac{\cos^3\theta_c}{\sin^2\theta_c} \,. \tag{4}$$

This represents an appreciable gain in energy.

The case of a closed line of force, as represented in Figure 1, *a*, is important in order to check on the degree to which accelerations ahead of the shock are canceled by subsequent decelerations behind the shock. Figure 1, *b*, shows the strength of the magnetic field measured along the line of force, represented twice. The single shock forms two traps simultaneously. The length of the trap behind the shock increases at the expense of the trap ahead of the shock. It is clear from Figure 1, *b*, that this situation is exactly equivalent to two sets of shocks which cross each other without disturbing each other, the situation treated by Parker (1958*a*). The order of events is the following. When the shock first sweeps onto the line of force, all particles which can be trapped are trapped ahead of the shock and are accelerated. Before the shock leaves the line of force, each such particle must gain enough energy to escape, and its energy is multiplied by the factor $\sin^2\theta/\sin^2\theta_c$, as above. However, the particles which are not initially trapped can cross the shock, and, of these, some have their pitch angles scattered so that they are trapped behind the shock. These are therefore decelerated until the shock leaves the line of force. We compute, first, the case with the maximum possible deceleration. This occurs if *all* particles are captured behind the shock, including those which were first trapped ahead of the shock and then escaped, and if they lose the *maximum* possible

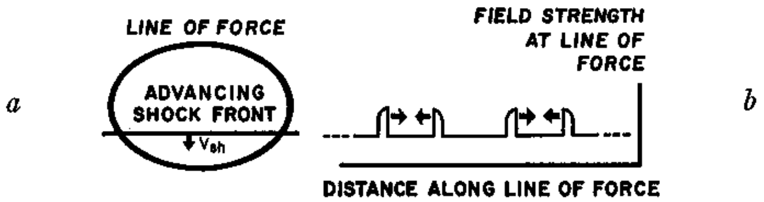

FIG. 1.—*a:* A shock crossing a line of force. *b:* Field strength measured along the line of force (schematic).

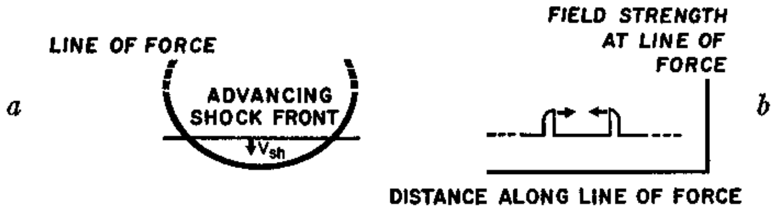

FIG. 2.—Same as Fig. 1

amount of energy, which amounts to a decrease by a factor $\sin^2\theta$, where θ is the pitch angle upon capture. As a result, equation (4) is multiplied by the mean of $\sin^2\theta$ in the range $\theta_c < \theta < \pi/2$. With the assumption of isotropy, this becomes

$$\left\langle\frac{E_f}{E_0}\right\rangle = \left(1+\frac{2}{3}\,\frac{\cos^3\theta_c}{\sin^2\theta_c}\right)\left(1-\frac{1}{3}\cos^2\theta_c\right). \tag{5}$$

A net loss of energy occurs only for $\sin^2\theta_c > 0.8$, that is, $H_{sh}/H_0 < 1.25$. A gain in energy by a factor 1.06 and 1.22 occurs for shock strengths $H_{sh}/H_0 = 1.5$ and 2, respectively, which may not be uncommon in nature.

The extreme case of equation (5) is not likely to be valid when $H_{sh}/H_0 < 1.25$, because $\cos\theta_c < 0.5$, so that less than half of the particles moving backward across the shock are trapped behind the shock and more than half of them recross the shock in the forward direction. For weak shocks, capture is equally probable on both sides of the shock. We therefore consider the following less extreme case, which is more probable for weak shocks. Of the particles which are initially untrapped, half are trapped ahead of the shock; all particles once trapped ahead of the shock are accelerated and escape before the shock leaves the line of force; all the accelerated particles, as well as the other half of the initially untrapped particles, are captured behind the shock and lose the maximum amount of energy. We then find

$$\left\langle\frac{E_f}{E_0}\right\rangle=\left\{\frac{1}{\cos\theta_c}\int_{\theta_c}^{\pi/2}\frac{\sin^3\theta}{\sin^2\theta_c}\,d\theta\,[\cos\theta_c+\tfrac{1}{2}(1-\cos\theta_c)]+\tfrac{1}{2}(1-\cos\theta_c)\right\}\times\frac{1}{\cos\theta_c}\int_{\theta_c}^{\pi/2}\sin^3\theta\,d\theta \tag{6}$$

$$=\frac{2+\sin^2\theta_c}{9\sin^2\theta_c}\,(1+2\sin^2\theta_c+\cos^3\theta_c)\,. \tag{7}$$

This result represents a *net gain* in energy for *all* shock strengths. For $H_{sh}/H_0 = 1.5$ and 2, $\langle E_f/E_0\rangle = 1.12$ and 1.31, respectively, an appreciable gain in energy. For weak shocks with $H_{sh}/H_0 = 1 + \delta$, $\delta \ll 1$, equation (7) reduces to $\langle E_f/E_0\rangle = 1 + \frac{1}{3}\delta^{3/2}$.

Even this second case is probably an underestimate of the actual gain in energy. First, among the particles once accelerated and released, some may later be trapped again ahead of the shock and gain more energy. Second, particles are not likely to lose the maximum amount of energy behind the shock, since this maximum loss implies no motion along the line of force and hence no collisions. Furthermore, particles once accelerated have only little time for subsequent deceleration (Parker 1958*a*). Third, for a weak shock the magnetic moment may be less scattered than has been assumed. A particle once trapped is accelerated as above, but a particle once free is likely to maintain its magnetic moment even when crossing the shock, so that the situation may be more nearly described by equation (4). However, this may be partly offset by an increased average deceleration if the particle is captured, since θ is likely to be closer to θ_c than according to an isotropic distribution.

We conclude, therefore, that shocks traveling through fields of uniform strength and intersecting any line of force in two places lead to an average net increase in particle energy even if the shock strength is small (subject to the conditions on trapping and acceleration discussed in Sec. IV). The same conclusion holds for sequences of shocks crossing each other, thus generalizing Parker's (1958*a*) computations to the case of weak shocks. (His assumption that the shocks cross without affecting each other is plausible only if the shocks are weak.)

III. ACCELERATION IN NON-UNIFORM FIELDS

Consider a surface of low field strength with lines of force connecting it to regions of stronger fields, as shown in Figure 3, *a*. If a shock is moving in this surface, a particle may be trapped on a line of force between the shock and a mirror point in the static field. Figure 3, *b*, shows schematically the field strength along the line of force at various times and shock positions. Four situations can occur, which are in order of decreasing importance: (1) Acceleration ahead of a shock (strictly, its intersection with the line of force) moving into stronger fields. The particle gains energy and is able to penetrate stronger fields, but at the same time the trapping field in the shock may increase sufficiently that the particle remains trapped while gaining energy. (2) Deceleration behind a shock moving into weaker fields. The particle is decelerated and is able to penetrate only weaker fields, but if the acceleration by a similar shock is prolonged as described above, the trapping field decreases sufficiently that the decelerated particle soon escapes across the shock. Thus there exists a *definite asymmetry* in which accelerations far outnumber decelerations, leading to an efficient Fermi acceleration. (3) Acceleration ahead of a shock moving into weaker fields. The particle escapes because it is able to penetrate stronger fields while the trapping field in the shock decreases. (4) Deceleration behind a shock moving into stronger fields. The shock moves into fields strong enough that the particle cannot reach the shock, and deceleration ceases.

A particle is trapped when the field strength inside the shock, H_{sh}, exceeds the field strength at the mirror point of the static field, H_{mp}. These strengths are functions of time. A necessary condition for escape is

$$\frac{d}{dt}\left(\frac{H_{mp}}{H_{sh}}\right) > 0 . \tag{8}$$

We shall show that condition (8) is not satisfied for trapped particles under many circumstances. These particles are trapped more deeply with time and may experience a large change in kinetic energy. The initial trapping occurs either if the pitch angle is scattered when the particle crosses the shock or if H_{sh} becomes larger than H_{mp} while the particle travels between shock and mirror point.

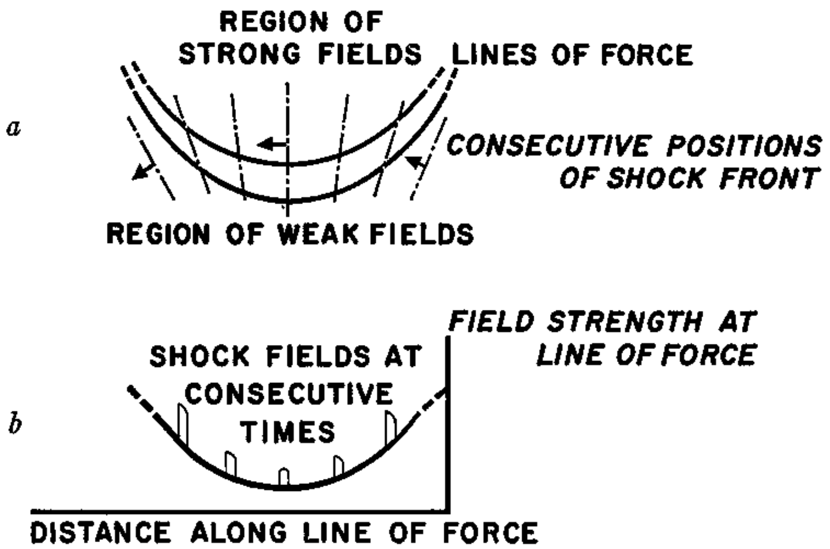

FIG. 3.—Same as Fig. 1

a) A Simple Situation

To derive the essential aspects of this mode of Fermi acceleration, we make several mathematical assumptions which will be relaxed in Section *b*. Let the static field increase linearly with distance,

$$H = H^* \frac{x}{L}, \tag{9}$$

where L is a scale length. Such a field is appropriate near a "neutral plane." In Figure 3, *a*, the x-direction is upward. Let the mirror point due to the static field be located at x_{mp}, so that

$$H_{mp} = H^* \frac{x_{mp}}{L}. \tag{10}$$

The field just behind the shock front is defined by the parameter η through

$$H_{sh} = \eta H = \eta H^* \frac{x_{sh}}{L}, \tag{11}$$

where x_{sh} denotes the position of the shock's point of intersection with the line of force. Let v_{sh} be the speed of this intersection along the line of force, and let the line of force make an angle a with the x-direction. With these definitions, we have

$$\frac{dx_{sh}}{dt}=v_{sh}\cos a\,. \tag{12}$$

The speed v_{sh} is an unspecified function of x which depends on the orientation of the line of force relative to the shock front and on the speed of the shock front (which also may depend on x, yielding refraction of the shock, as indicated in Fig. 3, a).

A second assumption made in this section is that η is constant in time and along the shock front. This may be the case, for instance, in an essentially hydrodynamic shock in which magnetic effects on the shock and dissipation are negligible and v_{sh} is constant. Differentiation of equation (11) now reduces simply to

$$\frac{dH_{sh}}{dt}=\eta\,\frac{H^*}{L}\,v_{sh}\cos a\,. \tag{13}$$

In order to find dH_{mp}/dt, we require the rate of acceleration. Let the particle be trapped at a time $t = 0$ and denote all initial values by sub- or superscripts 0. From equations (1) and (2) we derive

$$H_{mp}=H^0_{mp}\,\frac{v^2}{v_0^2}=\frac{H^*}{L}\,x_{mp}\,,\qquad \frac{dH_{mp}}{dt}=\frac{H^0_{mp}}{v_0^2}\,\frac{dv^2}{dt}=\frac{H^*}{L}\,\frac{dx_{mp}}{dt}\,. \tag{14}$$

Note the constancy of v^2/x_{mp} with time. The rate of acceleration has frequently been derived for fields which are uniform between magnetic traps (for instance, Parker 1958a). In the case of non-uniform fields, the velocity parallel to the line of force, $v_{||}$, changes along the path according to the guiding center approximation,

$$v_{||}=v\left(1-\frac{H}{H_{mp}}\right)^{1/2}. \tag{15}$$

Of special interest is the value of $v_{||}$ just ahead of the shock in the static field,

$$v_{||sh}=v\left(1-\frac{x_{sh}}{x_{mp}}\right)^{1/2}=\left(\frac{v_0^2}{x^0_{mp}}\right)^{1/2}(x_{mp}-x_{sh})^{1/2}, \tag{16}$$

where we have used the constancy of v^2/x_{mp} in the last expression. Note that $v_{||}$ at any one fixed point in space increases with time, since v and H_{mp} increase, but $v_{||sh}$ may increase or decrease, depending on the variation of $x_{mp} - x_{sh}$ with time. A head-on reflection from the shock increases $v_{||}$ at the point just ahead of the shock by $2v_{sh}$. The change in v^2 per reflection from the shock is

$$\Delta v^2=4\,v_{||sh}v_{sh}+4\,v_{sh}^2\,. \tag{17}$$

(The motion of x_{mp} through the static field does not cause any change in energy.)

We evaluate first the rate of acceleration for a particle with $v_{||sh}\gg v_{sh}$, so that the last term in equation (17) may be neglected. The time of flight between reflections from the shock is

$$\Delta t=2\int_{x_{sh}}^{x_{mp}}\frac{dx}{v_{||}\cos a}=\frac{4x_{mp}}{v\cos a}\left(1-\frac{x_{sh}}{x_{mp}}\right)^{1/2}=\frac{4}{\cos a}\left(\frac{x^0_{mp}}{v_0^2}\right)^{1/2}(x_{mp}-x_{sh})^{1/2}, \tag{18}$$

where we have assumed, for simplicity, that a is constant between x_{sh} and x_{mp}. We find, from equations (16)–(18),

$$\frac{\Delta v^2}{\Delta t} = \frac{v_0^2}{x_{mp}^0} v_{sh} \cos a\,, \qquad v_{||sh} \gg v_{sh}\,. \tag{19}$$

This mathematical simplicity is due to the assumed linear variation of H with x. From equations (13) and (14) we derive

$$\frac{dH_{mp}}{dt} = \frac{1}{\eta} \frac{dH_{sh}}{dt}\,, \qquad \frac{dx_{mp}}{dt} = \frac{dx_{sh}}{dt}\,, \tag{20}$$

and

$$\frac{d}{dt}\left(\frac{H_{mp}}{H_{sh}}\right) = \left(1 - \frac{x_{mp}}{x_{sh}}\right) \frac{1}{H_{sh}} \frac{dH_{mp}}{dt}\,. \tag{21}$$

For a particle which is trapped with $v_{||sh} \gg v_{sh}$ and is accelerated ($dH_{mp}/dt > 0$), the right-hand side of equation (21) is negative. Thus this particle is trapped more and more deeply with time, even while it gains energy according to equations (12) and (19),

$$\frac{v^2}{v_0^2} = 1 + \frac{x_{sh} - x_{sh}^0}{x_{mp}^0}\,, \tag{22}$$

where $1/\eta < x_{sh}^0/x_{mp}^0 < 1$, depending on the initial conditions of capture. Thus *acceleration ahead of a shock moving into stronger fields* ($v_{sh} > 0$) *is prolonged.* This conclusion rests on the assumption $v_{||sh} \gg v_{sh}$, which will now be examined.

It follows from equations (16), (18), and (20) that $x_{mp} - x_{sh}$, Δt, and $v_{||sh}$ remain constant in time if a is constant, as was assumed. (This is quite in contrast to the situation in Sec. II, where both trapping fields are discontinuous, the size of the trap must approach zero, and $v_{||sh}$ must increase to cause escape from the trap.) There are three effects when $v_{||sh}$ is comparable with v_{sh}. First, the gain in energy per reflection increases due to the second term in equation (17). This tends to increase $x_{mp} - x_{sh}$ and Δt, which, in turn, tends to reduce the mean rate of acceleration. Second, the shock approaches x_{mp} while the particle travels, reducing Δt. Third, H_{sh} increases during Δt while H_{mp} remains constant, trapping the particle more deeply. The mathematics of the situation are quite involved, but one may compare it with a ball rolling on an inclined plane and bouncing elastically off a barrier which is moving up the plane. One special case follows directly from this analogy. Suppose the ball is initially at rest; upon being hit by the barrier, it gains twice the speed of the barrier; when it reaches its maximum height, the barrier catches up, and the situation repeats itself. Similarly, a particle with initially $v_{||sh} = 0$ has $v_{||sh} = 2v_{sh}$ after acceleration, and it reaches x_{mp} just at the same time as the shock, if this shock travels with constant v_{sh}. Having been trapped once, the particle remains trapped, although it gains energy. We note, however, that a particle with $v_{||sh} = 0$ is not necessarily trapped at all. As we shall see in Section IV, a sufficiently high $v_\perp$ is also required. Except for this special case, little can be deduced as to whether or not a particle with low $v_{||sh}$ remains trapped. Aside from the three effects already mentioned, variations of v_{sh} or a along the line of force may complicate the situation further. Undoubtedly, escape after a few accelerations is possible for a variety of conditions, especially if $v_\perp$ is also not much greater than v_{sh}.

It remains to show that the opposite situation—that of deceleration behind a shock which moves into weaker fields—does not lead to prolonged trapping. If we set $v_{sh} < 0$, it follows that $dH_{mp}/dt < 0$ and the particle is decelerated, but equation (21) in this case is positive. The particle escapes as soon as $H_{mp} = H_{sh}$. When $v_{||sh}$ is not much larger than v_{sh}, a single reflection from the shock practically stops the particle in its motion along the line of force, and no further deceleration occurs.

We conclude that a shock of uniform strength moving in a field whose strength is proportional to x causes only prolonged acceleration, without prolonged deceleration, and represents an efficient means of Fermi acceleration. The gain in energy is limited largely by geometrical conditions. It is possible that the shock drives the particle past the strongest field along the line of force, whereupon the static trap ceases to exist, or it is possible that the shock front has a limited length and the particle is swept past the edge of the shock, as might be the case in Figure 3, *a*. A doubling of the kinetic energy per passing shock may not be infrequent, and larger gains may be possible in fields which encompass a large range of field strengths, such as dipole fields.

In summary, the acceleration of the particle and the changes in the kinetic energies of motion parallel to $\boldsymbol{H}$, $E_{||}$, and perpendicular to $\boldsymbol{H}$, $E_{\perp}$, proceed as follows. A shock sweeps across the line of force, as in Figure 3, *a*. The particle may be trapped several times ahead of and behind the shock, each time escaping across the shock rather rapidly. When the shock's intersection with the line of force moves into stronger fields, the particle experiences a prolonged acceleration. The primary gain in energy from the shock is in the form of $E_{||}$, but this is changed into $E_{\perp}$ as the particle moves with the trap into even higher fields (the magnetic moment remaining constant). Ultimately, the shock front leaves the line of force, and the particle is free to move back and forth on this line of force with mirror points at high H_{mp}. Between these mirror points the particle moves through low field strengths, and most of its energy there is in the form of $E_{||}$. When the next shock arrives, the particle may be scattered several times as it crosses the shock in both directions, and statistically $E_{||}$ is converted back to $E_{\perp}$. This lowers H_{mp}, and the particle is then ready for further prolonged acceleration.

b) Generalization

We no longer assume H proportional to x and η constant. Define

$$H = H^* h\,, \qquad H_{mp} = H^* h_{mp}\,, \qquad \text{and} \qquad H_{sh} = \eta H^* h_{sh}\,, \tag{23}$$

where h is any dimensionless monotonically increasing function of x/L, L is a scale length, and h_{mp} and h_{sh} are the corresponding functions evaluated at x_{mp}/L and x_{sh}/L. The shock strength at x_{sh} is again denoted by η, which may now depend on x_{sh}/L. We define a function $f(\eta)$ through

$$\frac{dH_{sh}}{dt} = \eta \frac{H^*}{L} h'_{sh} f(\eta)\, v_{sh} \cos\alpha\,, \qquad f(\eta) = 1 + \frac{h_{sh}}{h'_{sh}} \frac{\eta'}{\eta}\,, \tag{24}$$

where the prime stands for differentiation with respect to x_{sh}/L. Assuming again that $v_{||sh} \gg v_{sh}$, we derive

$$\frac{d}{dt}\left(\frac{H_{mp}}{H_{sh}}\right) = \frac{1}{H_{sh}} \frac{dH_{mp}}{dt} \{1 - C(x_{sh}, x_{mp}) f(\eta)\}\,, \tag{25}$$

$$C(x_{sh}, x_{mp}) = \frac{h'_{sh}}{2 h_{sh}(1 - h_{sh}/h_{mp})^{1/2}} \int_{x_{sh}}^{x_{mp}} \frac{dx/L}{(1 - h/h_{mp})^{1/2}}\,, \tag{26}$$

in analogy to equation (21). An accelerated particle ($dH_{mp}/dt > 0$) is trapped for a prolonged time if { } is negative. More specifically, it can escape only if { } is positive when $H_{mp} = H_{sh}$.

Consider again the case $f(\eta) = 1$, as in Section *a*. Prolonged trapping ahead of the shock occurs if the coefficient $C(x_{sh}, x_{mp}) > 1$. This inequality actually holds under many conditions. To establish some of these conditions, we multiply the integrand in equation

(26) by $h'/h'_{\max}$, where $h'_{\max}$ is the maximum value attained by h' in the range $x_{sh} \leq x \leq x_{mp}$. The integral can then be evaluated and

$$C(x_{sh}, x_{mp}) > \frac{h'_{sh}}{h'_{max}} \frac{h_{mp}}{h_{sh}}. \tag{27}$$

Since $h_{mp}/h_{sh} > 1$, we have $C(x_{sh}, x_{mp}) > 1$ for any function h for which $h'_{\max} = h'_{sh}$, including all functions with $h'' < 0$. In the case of $h'' > 0$, $h'_{max} = h'_{mp}$, and for the choice $h = (x/L)^n$ we have $h'_{sh}/h_{sh} > h'_{mp}/h_{mp}$, so that again $C(x_{sh}, x_{mp}) > 1$. Although no general proof is apparent, the inequality certainly holds for a greater variety of functions than is indicated here. We conclude that trapping ahead of a shock with uniform shock strength is prolonged in many types of magnetic fields. Note that the inequality $C(x_{sh}, x_{mp}) > 1$ is the more easily satisfied the lower h'', that is, the more gradually h increases with x for a given ratio h_{mp}/h_{sh}.

Considering now only fields yielding $C(x_{sh}, x_{mp}) > 1$, we find that prolonged acceleration nevertheless does not occur if $f(\eta)$ is sufficiently small, that is, if H_{sh} increases so slowly that the accelerated particle may escape. In such a case one obtains, instead, prolonged deceleration of a particle trapped behind a shock moving into weaker fields. Clearly, it is necessary to discuss $f(\eta)$. The shock strength η depends on the speed of the shock and on the energy which is transported by the shock. For the purpose of simple estimates, we assume a highly conducting medium traversed by a weak fast-mode shock (Bazer and Ericson 1959), with the normal to the shock front perpendicular to the undisturbed $\boldsymbol{H}$. The latter assumption makes the ratio of gas densities across the shock equal to the ratio of field strengths. We further assume that the magnetic energy density is either much smaller or much larger than the thermal energy density. The energy carried per cm² of shock front, then, is

$$D = \rho \bar{\eta}^2 V^3 t_0 , \qquad \bar{\eta} = \eta - 1 , \tag{28}$$

where ρ is the gas density, V the speed of the shock, $\bar{\eta} V$ the material speed behind the shock front, and Vt_0 the effective length of the shock (simplified from Österbrock 1961, eq. [53]). Three cases may illustrate the behavior of $f(\eta)$.

i) The magnetic energy density is much smaller than the thermal energy density, the shock is essentially hydrodynamic, and V is the speed of sound. If we assume that D, ρ, V, and t_0 are uniform along the shock, then $\bar{\eta}$ is also uniform, and $f(\eta) = 1$. Acceleration is prolonged, as shown above.

ii) The magnetic energy density is much larger than the thermal energy density, V is the Alfvén speed and proportional to H. In using equation (28) we neglect the refraction of the shock toward regions of low field strength. If we again assume that D, ρ, and t_0 are uniform along the shock, then $\bar{\eta}$ is proportional to $h_{sh}^{-3/2}$ and

$$f(\eta) = 1 - \tfrac{3}{2} \frac{\bar{\eta}}{\eta} = -\tfrac{1}{2} + \frac{3}{2\eta}, \tag{29}$$

independent of the choice of h. At the point of escape, $h_{mp} = \eta h_{sh}$ and $\{ \}$ of equation (25) is a function of η only. It has been evaluated for $h = (x/L)^n$, $n = \frac{1}{2}$, 1, and 2. The result is $\{ \} < 0$ for $n = \frac{1}{2}$, $1 < \eta < \frac{3}{2}$, but $\{ \} > 0$ for $n = \frac{1}{2}$, $\eta > \frac{3}{2}$ and for $n = 1$ or 2, and all $\eta > 1$. Two effects combine to prevent prolonged acceleration when H increases too rapidly with x. First, the uniform D requires a strong decrease of η with increasing x (for $\eta > 3$ $dH_{sh}/dt < 0$, even though $v_{sh} > 0$). Second, the coefficient $C(x_{sh}, x_{mp})$ decreases with increasing n, as pointed out above.

iii) The pure hydromagnetic shock with uniform D assumed in the previous paragraph is not likely to exist very frequently. For instance, an increase of D with x may occur in such a shock which has traveled some distance and which has been dissipated more at low x, where η is larger, than at high x. For $h = x/L$, an accelerated particle does remain trapped if D increases as $(x/L)^m$, $m > 1$. Many other such situations may occur.

To summarize, for a large range in the dependence of H on x, a particle with $v_{||sh} \gg v_{sh}$ accelerated ahead of a shock whose intersection with a line of force moves into stronger fields does not escape if the shock strength is uniform. The same particle may escape if the shock strength decreases sufficiently with increasing x and if, in addition, H increases sufficiently with x. Under that circumstance, one obtains, instead, that a decelerated particle remains trapped behind a shock whose intersection with the line of force moves into weaker fields. A particle with $v_{||sh}$ comparable to v_{sh} may remain trapped in some circumstances, but many other conditions permit escape.

IV. CONDITIONS OF ACCELERATION

Fermi acceleration has generally been considered in connection with cosmic rays, which satisfy $v_{||sh} \gg v_{sh}$. In recent years one has observed or deduced several new kinds of supra-thermal particles, such as the electrons involved in the outer Van Allen belt and its structural variations, solar protons and heavier particles of various energies, and the electrons which emit hard X-rays and/or synchrotron radiation during solar flares. Having demonstrated an efficient method of Fermi acceleration in the previous sections, it is of interest to determine whether these particles may have been accelerated in this fashion. This section is devoted to a more general discussion of the conditions under which thermal charged particles can be Fermi-accelerated.

We show, first, that a significant fraction of thermal particles with speed v are accelerated by shocks of moderate strength only for speeds $v \gtrsim 2v_{sh}$. The slower particles just "bob" in the waves.

To derive this condition on v, consider a shock (or any other magnetic inhomogeneity) moving through a uniform field. The reflection by such a field is based on the limit $v_\perp^2 \leq v^2$. This comparison is meaningful only if v^2 is actually constant during the reflection. We must therefore select a frame of reference in which all magnetic fields are static and test the condition on reflection in this frame. Denote the velocity components and pitch angle in the shock's frame of reference by $v'_{||}$, $v'_\perp$, and θ' and the ratio of field strengths inside and outside the shock by η (independent of the frame of reference). The condition for a head-on collision and reflection becomes

$$\tan \theta' = \frac{v'_\perp}{v'_{||}} = \frac{v_\perp}{v_{||} + v_{sh}} = \frac{\sin \theta}{\cos \theta + v_{sh}/v} > \tan \theta'_c = \frac{1}{(\eta - 1)^{1/2}}. \tag{30}$$

If the pitch angles θ among thermal particles are distributed isotropically, a fraction $\cos \theta_c$ is reflected, where θ_c is the critical pitch angle in the frame in which the shock moves with speed v_{sh}. From equation (30) we derive

$$\text{Fraction accelerated} = -\frac{1}{\eta}\frac{v_{sh}}{v} + \frac{1}{\eta}\left\{(\eta - 1)\left[\eta - \left(\frac{v_{sh}}{v}\right)^2\right]\right\}^{1/2}. \tag{31}$$

Figure 4 shows this fraction as a function of v_{sh}/v for $\eta = 1.5$, 2, and 3. For high v, $v_{sh}/v \to 0$, and the fraction becomes $(1 - 1/\eta)^{1/2}$. No particles are reflected for $v_{sh}/v > (\eta - 1)^{1/2}$. At moderate shock strengths a significant fraction of particles is accelerated for $v \gtrsim 2v_{sh}$.

For a particle reaching the shock from behind, we can again compute $\cos \theta_c$ by using $v'_{||} = v_{||} - v_{sh}$ in equation (30). In addition, we must impose the condition that the particles can actually reach the shock, $v_{||} > v_{sh}$. A fraction $\cos \theta = v_{sh}/v$ cannot reach the shock. Thus the fraction of particles reflected behind the shock is $\cos\theta_c - v_{sh}/v$, which is

$$\text{Fraction decelerated} = -\left(1-\frac{1}{\eta}\right)\frac{v_{sh}}{v}+\frac{1}{\eta}\left\{(\eta-1)\left[\eta-\left(\frac{v_{sh}}{v}\right)^2\right]\right\}^{1/2}. \tag{32}$$

For $\eta \gtrsim 1.5$ this fraction decreases roughly linearly from the high-velocity value $(1 - 1/\eta)^{1/2}$ at $v_{sh}/v = 0$ to 0 at $v_{sh}/v = 1$.

We have here assumed that all magnetic fields are static in the shock's frame of reference. For a shock moving through non-uniform fields, this is not strictly the case, but it should be a good approximation, since the gradient of the magnetic fields is much greater inside the shock than outside. A further condition on the speed of the particles arises if the acceleration occurs in non-uniform fields, as described in Section III.

The reflection and acceleration of a particle by a shock assumes that the magnetic moment is constant and that the guiding center approximation is valid. When, after some acceleration, the Larmor radius or the distance traveled into the shock during one

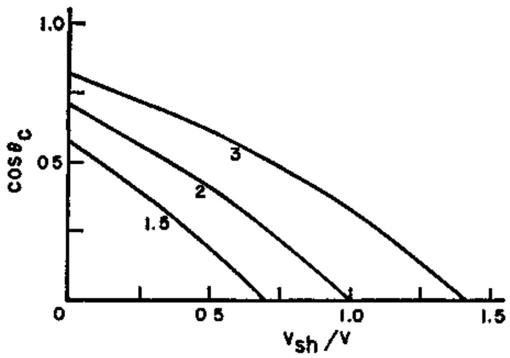

FIG. 4 —The fraction of particles of speed v, which is reflected in a head-on collision with a shock of speed v_{sh}, as a function of v_{sh}/v for shock strengths $\eta = H_{sh}/H_0 = 1.5$, 2, and 3.

Larmor period are comparable with the thickness of the shock, the magnetic moment is not constant, and no further acceleration occurs. This condition limits the energies which can be reached through acceleration by hydromagnetic shocks. The shock thickness is of the order of magnitude of the effective mean-free path of an atom or ion in a direction perpendicular to the shock front. This may be the collisional mean-free path, the Larmor radius, or the stopping distance in electric fields of plasma oscillations set up by the shock itself, whichever is the shortest (Parker 1959).

An example of some interest is the case of shocks moving in an ionized gas of low density, where the shock thickness is of the order of the proton Larmor radius. Larmor radii of electrons and protons become equal when their energies are in the inverse ratio of their masses. With the shock speed comparable to, or greater than, the sound speed, only the thermal electrons are accelerated. Shocks may increase their energies roughly 10^3 fold. For instance, at initial temperatures between 10^5 and 3×10^{6}° K in the solar upper chromosphere and corona, one may expect to obtain kinetic energies between 13 and 390 kev in this manner, corresponding closely to the range of energies required to explain the hard X-rays (Elwert 1961) and the synchrotron radiation (Takakura 1962) associated with solar flares. The details of the acceleration present a separate problem.

Collisions among particles can change both the energy and the pitch angle of any given particle (Spitzer 1956). Since the effect of these collisions is strongly energy-dependent, a minimum injection energy is required for successful acceleration. In the case of shocks passing along any given line of force, one should distinguish three situations.

First, if the collisions knock the particle out of the trap so frequently that the particle is statistically equally often ahead of and behind the shock, then there is no net gain of energy. Second, if the collision frequency is such that the particle is trapped longer ahead of the shock than behind it, on the average, but then loses its energy by collisions before the next shock arrives, the net effect of the shock is to heat all the colliding particles evenly. Third, if the collisions are so infrequent that the particle keeps its energy until the next shock arrives, it is accelerated more by each shock and becomes a supra-thermal particle. In the absence of scattering by shocks, collisions with other particles may also serve to reorient the pitch angle and thereby to make possible repeated Fermi acceleration (Takakura 1962).

V. CONCLUSIONS

We have demonstrated in Sections II and III several situations in which Fermi accelerations far outnumber the decelerations, yielding a high efficiency of acceleration. However, situations also exist in which particles may be efficiently decelerated. In general, only particles with speeds at least several times the shock speed are picked up by the shocks. The gain in energy of an individual particle depends to a large degree on the path of the shock across the line of force; in fields of uniform strength it is limited by the shock strength; in fields of varying strength it is limited by the extent of this variation.

Aside from specific applications, two general situations appear to be of interest for further study. First, dipole fields, such as the geomagnetic and sunspot fields, may contain shocks traversing a great range of field strengths. Particles trapped ahead of such shocks may be very highly accelerated. Second, a natural place for the existence of both shocks and non-uniform fields are regions of pinch instabilities. Such unstable regions may be sources of efficient Fermi acceleration.

Finally, it seems appropriate to take exception to a statement (Parker 1958*b*) to the effect that electrons gain only insignificant energies by Fermi acceleration, since this method increases only the velocity, irrespective of the mass of the particle. This statement would be true if protons and electrons were accelerated starting from the same initial velocity and if they experienced the same number of reflections. However, *thermal* electrons are *favored* in at least some situations over thermal protons because (*a*) they are faster by a factor of 43 and may be picked up and accelerated by shocks, while protons just "bob" in the shocks; (*b*) in a given trap they collide more often with the mirror points because of their greater speed; and (*c*) at a given energy the magnetic moment of an electron is more constant than that of a proton. Furthermore, it is true at all particle energies that magnetic traps such as those discussed in Sections II and III multiply the *energy* of a particle by a factor sufficient to cause escape from the trap, irrespective of the mass of the particle. If electrons and protons are accelerated starting with the same energy, they also attain the same final energy. As a consequence, no prediction can be made for the ratio of energies residing in cosmic-ray heavy particles and electrons created in supernovae until more is known about the details of the acceleration.

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