ON THE DARBOUX PROBLEM FOR LINEAR HYPERBOLIC FUNCTIONAL-DIFFERENTIAL EQUATIONS

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Abstract. Theorems on the Fredholm alternative and well-posedness of the characteristic initial value problem

$$\frac{\partial^2 u(t,x)}{\partial t\,\partial x} = \ell(u)(t,x) + q(t,x),$$
$$u(t,x_0) = \varphi(t) \quad \text{for } t \in [a,b], \qquad u(t_0,x) = \psi(x) \quad \text{for } x \in [c,d],$$

are established, where $\ell\colon C(\mathcal{D};\mathbb{R}) \to L(\mathcal{D};\mathbb{R})$ is a linear bounded operator, $q \in L(\mathcal{D};\mathbb{R})$, $t_0 \in [a,b]$, $x_0 \in [c,d]$, $\varphi : [a,b] \to \mathbb{R}$, $\psi : [c,d] \to \mathbb{R}$ are absolutely continuous functions, and $\mathcal{D} = [a,b] \times [c,d]$. Some solvability conditions of the problem considered are given as well.

1. Introduction

On the rectangle $\mathcal{D} = [a,b] \times [c,d]$, we consider the linear partial functional-differential equation

$$\frac{\partial^2 u(t,x)}{\partial t\,\partial x} = \ell(u)(t,x) + q(t,x), \tag{1.1}$$

where $\ell\colon C(\mathcal{D};\mathbb{R}) \to L(\mathcal{D};\mathbb{R})$ is a linear bounded operator and $q \in L(\mathcal{D};\mathbb{R})$. As usual, $C(\mathcal{D};\mathbb{R})$ and $L(\mathcal{D};\mathbb{R})$ denote the Banach spaces of continuous and Lebesgue integrable functions, respectively, equipped with the standard norms.

A function $u \in C^*(\mathcal{D};\mathbb{R})$ is said to be a solution to the equation (1.1) if it satisfies the equality (1.1) almost everywhere on the set $\mathcal{D}$.

Various initial and boundary value problems for hyperbolic differential equations and their systems are studied in literature (see, e.g., [4, 8, 9, 11, 14–16, 19, 26, 28, 29] and references therein). We shall consider the so-called characteristic initial value problem (Darboux problem). In this case, the values of the solution u of (1.1) are prescribed on both characteristics $t = t_0$ and $x = x_0$, i.e., the initial conditions are

$$u(t,x_0) = \varphi(t) \quad \text{for } t \in [a,b], \qquad u(t_0,x) = \psi(x) \quad \text{for } x \in [c,d], \tag{1.2}$$

where $t_0 \in [a,b]$, $x_0 \in [c,d]$, and $\varphi\colon [a,b] \to \mathbb{R}$, $\psi\colon [c,d] \to \mathbb{R}$ are absolutely continuous functions such that $\varphi(t_0) = \psi(x_0)$.

A particular case of the problem (1.1), (1.2) (if $t_0 = a$ and $x_0 = c$) is studied in the paper [23]. The aim of this preprint is to generalize the paper mentioned and prove the Fredholm alternative and well-posedness of the problem (1.1), (1.2) (see Sections 4 and 6). Moreover, some conditions are given in Section 5 under which

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the problem (1.1), (1.2) has a unique solution. The results obtained are applied for the equation with deviating arguments

$$\frac{\partial^2 u(t,x)}{\partial t\,\partial x} = p(t,x)u\big(\tau(t,x),\mu(t,x)\big) + q(t,x), \tag{1.1$'$}$$

where $p, q \in L(\mathcal{D};\mathbb{R})$ and $\tau\colon \mathcal{D} \to [a,b]$, $\mu\colon \mathcal{D} \to [c,d]$ are measurable functions.

Let us note that analogous results for the "ordinary" functional-differential equations and their systems are given in [2, 10, 12, 13].

2. Notations and Definitions

The following notation is used throughout the paper.

(1) $\mathbb{N}$ is the set of all natural numbers. $\mathbb{R}$ is the set of all real numbers, $\mathbb{R}_+ = [0, +\infty[$. $\mathrm{Ent}(x)$ denotes the entire part of the number $x \in \mathbb{R}$.
(2) $\mathcal{D} = [a,b] \times [c,d]$, where $-\infty < a < b < +\infty$ and $-\infty < c < d < +\infty$.
(3) The first and the second order partial derivatives of the function $v\colon \mathcal{D} \to \mathbb{R}$ at the point $(t,x) \in \mathcal{D}$ are denoted by $v'_{[1]}(t,x)$ (or $v_t(t,x)$, $\frac{\partial v(t,x)}{\partial t}$), $v'_{[2]}(t,x)$ (or $v_x(t,x)$, $\frac{\partial v(t,x)}{\partial x}$), $v''_{[12]}(t,x)$ (or $v_{tx}(t,x)$, $\frac{\partial^2 v(t,x)}{\partial t\,\partial x}$), and $v''_{[21]}(t,x)$ (or $v_{xt}(t,x)$, $\frac{\partial^2 v(t,x)}{\partial x\,\partial t}$).
(4) $C(\mathcal{D};\mathbb{R})$ is the Banach space of continuous functions $v\colon \mathcal{D} \to \mathbb{R}$ equipped with the norm $\|v\|_C = \max\big\{|v(t,x)| : (t,x) \in \mathcal{D}\big\}$.
(5) $AC([\alpha,\beta];\mathbb{R})$, where $-\infty < \alpha < \beta < +\infty$, is the set of absolutely continuous functions $u\colon [\alpha,\beta] \to \mathbb{R}$.
(6) $C^*(\mathcal{D};\mathbb{R})$ is the set of functions $v\colon \mathcal{D} \to \mathbb{R}$ admitting the representation

$$v(t,x) = e + \int_a^t k(s)\mathrm{d}s + \int_c^x l(\eta)\mathrm{d}\eta + \int_a^t \int_c^x f(s,\eta)\mathrm{d}\eta\mathrm{d}s \quad \text{for } (t,x) \in \mathcal{D},$$

where $e \in \mathbb{R}$, $k \in L([a,b];\mathbb{R})$, $l \in L([c,d];\mathbb{R})$, and $f \in L(\mathcal{D};\mathbb{R})$. Equivalent definitions of the class $C^*(\mathcal{D};\mathbb{R})$ are presented in Proposition 2.1 below.
(7) $L(\mathcal{D};\mathbb{R})$ is the Banach space of Lebesgue integrable functions $p\colon \mathcal{D} \to \mathbb{R}$ equipped with the norm $\|p\|_L = \iint_{\mathcal{D}} |p(t,x)|\mathrm{d}t\mathrm{d}x$.
(8) $\mathcal{L}(\mathcal{D})$ is the set of linear bounded operators $\ell\colon C(\mathcal{D};\mathbb{R}) \to L(\mathcal{D};\mathbb{R})$.
(9) $\mathrm{mes}\,A$ denotes the Lebesgue measure of the set $A \subset R^m$, $m = 1, 2$.
(10) If X, Y are Banach spaces and $T\colon X \to Y$ is a linear bounded operator then $\|T\|$ denotes the norm of the operator T, i. e.,

$$\|T\| = \sup\big\{\|T(z)\|_Y : z \in X,\ \|z\|_X \le 1\big\}.$$

(11) $A \div B$ stands for the symmetric difference of the sets A and B, i. e., $A \div B = (A \setminus B) \cup (B \setminus A)$.

The following proposition dealing with the equivalent characterization of functions absolutely continuous in the sense of Carathéodory plays very important role in our investigation.

Proposition 2.1 ([22, Thm. 2.1])**.** *The following three statements are equivalent:*

(1) *the function $v\colon \mathcal{D} \to \mathbb{R}$ is absolutely continuous on $\mathcal{D}$ in the sense of Carathéodory*[1]*;*
(2) $v \in C^*(\mathcal{D};\mathbb{R})$*;*
(3) *the function $v\colon \mathcal{D} \to \mathbb{R}$ satisfies the conditions:*

[1]This notion is introduced in [3] (see also [22]).

(a) $v(\cdot,x) \in AC([a,b];\mathbb{R})$ *for every* $x \in [c,d]$, $v(a,\cdot) \in AC([c,d];\mathbb{R})$*;*
(b) $v'_{[1]}(t,\cdot) \in AC([c,d];\mathbb{R})$ *for almost every* $t \in [a,b]$*;*
(c) $v''_{[12]} \in L(\mathcal{D};\mathbb{R})$.

Remark 2.1. It is clear that the conditions (3a)–(3c) stated in the previous proposition can be replaced by the symmetric ones, i. e.,

(3) the function $v \colon \mathcal{D} \to \mathbb{R}$ satisfies the conditions:
(A) $v(\cdot,c) \in AC([a,b];\mathbb{R})$, $v(t,\cdot) \in AC([c,d];\mathbb{R})$ for every $t \in [a,b]$;
(B) $v'_{[2]}(\cdot,x) \in AC([a,b];\mathbb{R})$ for almost every $x \in [c,d]$;
(C) $v''_{[21]} \in L(\mathcal{D};\mathbb{R})$.

Moreover, for an arbitrary function $v \in C^*(\mathcal{D};\mathbb{R})$, the equality

$$v''_{[12]}(t,x) = v''_{[21]}(t,x) \quad \text{for a. e. } (t,x) \in \mathcal{D}$$

holds.

Definition 2.1. Let $t_0 \in [a,b]$ and $x_0 \in [c,d]$. An operator $\ell \in \mathcal{L}(\mathcal{D})$ is said to be an (t_0,x_0)–Volterra operator if, for an arbitrary rectangle $\mathcal{D}_0 \subseteq \mathcal{D}$ and every function $v \in C(\mathcal{D};\mathbb{R})$ such that $(t_0,x_0) \in \mathcal{D}_0$ and

$$v(t,x) = 0 \quad \text{for } (t,x) \in \mathcal{D}_0,$$

the relation

$$\ell(v)(t,x) = 0 \quad \text{for a. e. } (t,x) \in \mathcal{D}_0$$

is fulfilled.

Remark 2.2. If the operator ℓ appearing in the equation (1.1) is a (t_0,x_0)–Volterra one, then the problem (1.1), (1.2) can be restricted to an arbitrary rectangle $\mathcal{D}_0 \subseteq \mathcal{D}$ containing the point (t_0,x_0).

Let the operator $\ell \in \mathcal{L}(\mathcal{D})$ be defined by the formula

$$\ell(v)(t,x) = p(t,x)v\big(\tau(t,x),\mu(t,x)\big) \quad \text{for a. e. } (t,x) \in \mathcal{D} \text{ and all } v \in C(\mathcal{D};\mathbb{R}), \tag{2.1}$$

where $p \in L(\mathcal{D};\mathbb{R})$ and $\tau \colon \mathcal{D} \to [a,b]$, $\mu \colon \mathcal{D} \to [c,d]$ are measurable functions. The following statement can be derived from Definition 2.1.

Proposition 2.2. *Let* $t_0 \in [a,b]$ *and* $x_0 \in [c,d]$*. Then the operator* ℓ *defined by the formula* (2.1) *is a* (t_0,x_0)*–Volterra one if and only if the conditions*

$$|p(t,x)|(\tau(t,x)-t)(\tau(t,x)-t_0) \le 0 \quad \text{for a. e. } (t,x) \in \mathcal{D} \tag{2.2}$$

and

$$|p(t,x)|(\mu(t,x)-x)(\mu(t,x)-x_0) \le 0 \quad \text{for a. e. } (t,x) \in \mathcal{D} \tag{2.3}$$

are satisfied.

3. Auxiliary Statements

The following proposition plays a crucial role in the proofs of statements given in Sections 4–6.

Proposition 3.1. *Let* $t_0 \in [a,b]$, $x_0 \in [c,d]$, *and* $\ell \in \mathcal{L}(\mathcal{D})$*. Then the operator* $T \colon C(\mathcal{D};\mathbb{R}) \to C(\mathcal{D};\mathbb{R})$ *defined by the formula*

$$T(v)(t,x) = \int_{t_0}^{t}\int_{x_0}^{x} \ell(v)(s,\eta)\mathrm{d}\eta\mathrm{d}s \quad \text{for } (t,x) \in \mathcal{D},\ v \in C(\mathcal{D};\mathbb{R}) \tag{3.1}$$

is completely continuous.

The statement stated above can be easily proved in the case where the operator ℓ is strongly bounded, i.e., if there exists a function $\eta \in L(\mathcal{D};\mathbb{R}_+)$ such that

$$|\ell(v)(t,x)| \le \eta(t,x)\|v\|_C \quad \text{for a. e. } (t,x) \in \mathcal{D} \text{ and all } v \in C(\mathcal{D};\mathbb{R}). \tag{3.2}$$

H. H. Schaefer proved however that there exists an operator $\ell \in \mathcal{L}(\mathcal{D})$, which is not strongly bounded (see [21]). To prove Proposition 3.1 without the additional requirement (3.2) we need a number of notions and statements from functional analysis. Note here that the proof is analogous to the proof of Proposition 2.9 of [10].

Definition 3.1. Let X be a Banach space, X^* be its dual space.

We say that a sequence $\{x_n\}_{n=1}^{+\infty} \subseteq X$ is weakly convergent if there exists $x \in X$ such that $f(x) = \lim_{n\to+\infty} f(x_n)$ for every $f \in X^*$. The element x is said to be a weak limit of this sequence.

A set $M \subseteq X$ is called weakly relatively compact if every sequence of elements from M contains a subsequence which is weakly convergent in X.

A sequence $\{x_n\}_{n=1}^{+\infty}$ of elements from X is said to be weakly fundamental if the sequence $\{f(x_n)\}_{n=1}^{+\infty}$ is fundamental in $\mathbb{R}$ for every $f \in X^*$.

We say that the space X is weakly complete if every weakly fundamental sequence of elements from X possesses a weak limit in X.

Definition 3.2. Let X and Y be Banach spaces, $T : X \to Y$ be a linear bounded operator. The operator T is said to be weakly completely continuous if it maps a unit ball of X into a weakly relatively compact subset of Y.

Definition 3.3. We say that a set $M \subseteq L(\mathcal{D};\mathbb{R})$ has a property of absolutely continuous integral if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that the relation

$$\left| \iint_E p(t,x)\mathrm{d}t\mathrm{d}x \right| < \varepsilon \quad \text{for every} \quad p \in M$$

holds whenever a measurable set $E \subseteq \mathcal{D}$ is such that $\operatorname{mes} E < \delta$.

The following three lemmas can be found in [6].

Lemma 3.1 (Theorem IV.8.6)**.** *The space $L(\mathcal{D};\mathbb{R})$ is weakly complete.*

Lemma 3.2 (Theorem VI.7.6)**.** *A linear bounded operator mapping the space $C(\mathcal{D};\mathbb{R})$ into a weakly complete Banach space is weakly completely continuous.*

Lemma 3.3 (Theorem IV.8.11)**.** *If a set $M \subseteq L(\mathcal{D};\mathbb{R})$ is weakly relatively compact then it has a property of absolutely continuous integral.*

Proof of Proposition 3.1. Let $M \subseteq C(\mathcal{D};\mathbb{R})$ be a bounded set. We will show that the set $T(M) = \{T(v) : v \in M\}$ is relatively compact in $C(\mathcal{D};\mathbb{R})$. According to Arzelà-Ascoli's lemma, it is sufficient to show that the set $T(M)$ is bounded and equicontinuous.

Boundedness. It is clear that

$$|T(v)(t,x)| \le \int_{t_0}^{t}\int_{x_0}^{x} |\ell(v)(s,\eta)|\mathrm{d}\eta\mathrm{d}s \le \|\ell(v)\|_L \le \|\ell\|\,\|v\|_C$$

for $(t,x) \in \mathcal{D}$ and every $v \in M$. Therefore, the set $T(M)$ is bounded in $C(\mathcal{D};\mathbb{R})$.

Equicontinuity. Let $\varepsilon > 0$ be arbitrary but fixed. Lemmas 3.1 and 3.2 yield that the operator ℓ is weakly completely continuous, that is, the set $\ell(M) = \{\ell(v) :$

$v \in M\}$ is weakly relatively compact subset of $L(\mathcal{D};\mathbb{R})$. Therefore, Lemma 3.3 guarantees that there exists $\delta > 0$ such that the relation

$$\left| \iint_E \ell(v)(t,x)\mathrm{d}t\mathrm{d}x \right| < \frac{\varepsilon}{2} \quad \text{for } v \in M \tag{3.3}$$

holds for every measurable set $E \subseteq \mathcal{D}$ satisfying $\operatorname{mes} E < \max\{b-a, d-c\}\delta$.

On the other hand, for $(t_1,x_1),(t_2,x_2) \in \mathcal{D}$ and $v \in M$, we have

$$\begin{aligned} &|T(v)(t_2,x_2) - T(v)(t_1,x_1)| = \\ &\qquad = \left| \int_{t_0}^{t_2} \int_{x_0}^{x_2} \ell(v)(s,\eta)d\eta ds - \int_{t_0}^{t_1} \int_{x_0}^{x_1} \ell(v)(s,\eta)d\eta ds \right| \leq \\ &\qquad\qquad \leq \left| \iint_{E_1} \ell(v)(s,\eta)dsd\eta \right| + \left| \iint_{E_2} \ell(v)(s,\eta)dsd\eta \right|, \end{aligned}$$

where measurable sets $E_1, E_2 \subseteq \mathcal{D}$ are such that $\operatorname{mes} E_1 \leq (d-c)|t_2 - t_1|$ and $\operatorname{mes} E_2 \leq (b-a)|x_2 - x_1|$. Hence, by virtue of (3.3), we get

$$\begin{aligned} &|T(v)(t_2,x_2) - T(v)(t_1,x_1)| < \varepsilon \\ &\qquad \text{for } (t_1,x_1),(t_2,x_2) \in \mathcal{D},\ |t_2 - t_1| + |x_2 - x_1| < \delta, \text{ and } v \in M, \end{aligned}$$

i.e., the set $T(M)$ is equicontinuous in $C(\mathcal{D};\mathbb{R})$. □

4. Fredholm Property

The main result of this section is the following statement on the Fredholmity of the problem (1.1), (1.2).

Theorem 4.1. *For the unique solvability of the problem* (1.1), (1.2) *it is sufficient and necessary that the homogeneous problem*

$$\frac{\partial^2 u(t,x)}{\partial t\, \partial x} = \ell(u)(t,x), \tag{1.1$_0$}$$

$$u(t,x_0) = 0 \quad \text{for } t \in [a,b], \qquad u(t_0,x) = 0 \quad \text{for } x \in [c,d], \tag{1.2$_0$}$$

has only the trivial solution.

To prove the theorem we need a result stated in [22].

Lemma 4.1 ([22, Proposition 3.5])**.** *Let* $f \in L(\mathcal{D};\mathbb{R})$ *and*

$$u(t,x) = \int_a^t \int_c^x f(s,\eta)\mathrm{d}\eta\mathrm{d}s \quad \text{for } (t,x) \in \mathcal{D}.$$

Then:

(i) *there exists a set* $E \subseteq [a,b]$ *such that* $\operatorname{mes} E = b-a$ *and*

$$u'_{[1]}(t,x) = \int_c^x f(t,\eta)\mathrm{d}\eta \quad \text{for } t \in E \text{ and } x \in [c,d];$$

(ii) *there exists a set* $F \subseteq \mathcal{D}$ *such that* $\operatorname{mes} F = (b-a)(d-c)$ *and*

$$u''_{[12]}(t,x) = f(t,x) \quad \text{for } (t,x) \in F.$$

Proof of Theorem 4.1. Let u be a solution to the problem (1.1), (1.2). It is clear that u is a solution to the equation

$$v = T(v) + f \tag{4.1}$$

in the space $C(\mathcal{D};\mathbb{R})$, where the operator T is given by the relation (3.1) and

$$f(t,x) = -\varphi(t_0) + \varphi(t) + \psi(x) + \int_{t_0}^{t}\int_{x_0}^{x} q(s,\eta)\mathrm{d}\eta\mathrm{d}s \quad \text{for } (t,x) \in \mathcal{D}. \tag{4.2}$$

Conversely, if $v \in C(\mathcal{D};\mathbb{R})$ is a solution to the equation (4.1) with f given by (4.2) then it is easy to verify that $v \in C^*(\mathcal{D};\mathbb{R})$ (see Proposition 2.1) and, by virtue of Lemma 4.1(ii), v is a solution to the problem (1.1), (1.2). Hence, the problem (1.1), (1.2) and the equation (4.1) are equivalent in this sense.

Note also that u is a solution to the homogeneous problem (1.1_0), (1.2_0) if and only if u is a solution to the homogeneous equation

$$v = T(v) \tag{4.3}$$

in the space $C(\mathcal{D};\mathbb{R})$.

According to Proposition 3.1, the operator T is completely continuous. It follows from the Riesz-Schauder theory that the equation (4.1) is uniquely solvable for every $f \in C(\mathcal{D};\mathbb{R})$ if and only if the homogeneous equation (4.3) has only the trivial solution. Therefore, the assertion of the theorem holds. □

Definition 4.1. Let the problem (1.1_0), (1.2_0) have only the trivial solution. An operator $\Omega : L(\mathcal{D};\mathbb{R}) \to C(\mathcal{D};\mathbb{R})$ which assigns to every $q \in L(\mathcal{D};\mathbb{R})$ the solution u of the problem (1.1), (1.2_0) is called the Darboux operator of the problem (1.1_0), (1.2_0).

Remark 4.1. It is clear that the Darboux operator Ω is linear.

If the homogeneous problem (1.1_0), (1.2_0) has a nontrivial solution then, by virtue of Theorem 4.1, there exist functions q, φ, and ψ such that the problem (1.1), (1.2) has either no solution or infinitely many solutions. However, as it follows from the proof of Theorem 4.1, a stronger assertion can be shown in this case.

Proposition 4.1. *Let the problem* (1.1_0)*,* (1.2_0) *have a nontrivial solution. Then, for arbitrary* $\varphi \in \widetilde{C}([a,b],\mathbb{R})$ *and* $\psi \in \widetilde{C}([c,d],\mathbb{R})$ *satisfying* $\varphi(t_0) = \psi(x_0)$*, there exists a function* $q \in L(\mathcal{D};\mathbb{R})$ *such that the problem* (1.1)*,* (1.2) *has no solution.*

Proof. Let u_0 be a nontrivial solution to the problem (1.1_0), (1.2_0), and let $\varphi \in \widetilde{C}([a,b],\mathbb{R})$ and $\psi \in \widetilde{C}([c,d],\mathbb{R})$ be such that $\varphi(t_0) = \psi(x_0)$.

It follows from the proof of Theorem 4.1 that u_0 is also a nontrivial solution to the homogeneous equation (4.3), where the operator T is given by the relation (3.1). Therefore, by the Riesz-Schauder theory, there exists $f \in C(\mathcal{D};\mathbb{R})$ such that the equation (4.1) has no solution.

Then the problem (1.1), (1.2) has no solution for $q \equiv \ell(z)$, where

$$z(t,x) = f(t,x) + \varphi(t_0) - \varphi(t) - \psi(x) \quad \text{for } (t,x) \in \mathcal{D}.$$

Indeed, if the problem indicated has a solution u then the function $u+z$ is a solution to the equation (4.1), which is a contradiction. □

5. Existence and Uniqueness Theorems

In this section, we shall establish some efficient condition guaranteeing the unique solvability of the problems (1.1), (1.2) and (1.1′), (1.2). We will prove, in particular, that the problem (1.1), (1.2) has a unique solution provided that the operator ℓ is a (t_0, x_0)–Volterra one. We first formulate all the results, their proofs are given later.

Introduce the following notation.

Notation 5.1. Let $\ell \in \mathcal{L}(\mathcal{D})$. Define operators $\vartheta_k : C(\mathcal{D};\mathbb{R}) \to C(\mathcal{D};\mathbb{R})$, $k = 0, 1, 2, \dots$, by setting

$$\vartheta_0(v) = v, \quad \vartheta_k(v) = T\big(\vartheta_{k-1}(v)\big) \quad \text{for } v \in C(\mathcal{D};\mathbb{R}),\ k \in \mathbb{N}, \tag{5.1}$$

where the operator T is given by (3.1).

Theorem 5.1. *Let there exist $m \in \mathbb{N}$ and $\alpha \in [0,1[$ such that the inequality*

$$\|\vartheta_m(u)\|_C \le \alpha \|u\|_C \tag{5.2}$$

is satisfied for every solution u of the homogeneous problem (1.1_0), (1.2_0). *Then the problem* (1.1), (1.2) *is uniquely solvable.*

Remark 5.1. The assumption $\alpha \in [0,1[$ in the previous theorem cannot be replaced by the assumption $\alpha \in [0,1]$ (see Example 7.1).

Corollary 5.1. *Let there exist $j \in \mathbb{N}$ such that*

$$\begin{aligned} &\int_a^{t_0}\int_c^{x_0} p_j(s,\eta)\mathrm{d}\eta\mathrm{d}s < 1, \qquad \int_a^{t_0}\int_{x_0}^{d} p_j(s,\eta)\mathrm{d}\eta\mathrm{d}s < 1 \\ &\int_{t_0}^{b}\int_c^{x_0} p_j(s,\eta)\mathrm{d}\eta\mathrm{d}s < 1, \qquad \int_{t_0}^{b}\int_{x_0}^{d} p_j(s,\eta)\mathrm{d}\eta\mathrm{d}s < 1 \end{aligned} \tag{5.3}$$

where $p_1 \equiv |p|$ and

$$p_{k+1}(t,x) = \\ = |p(t,x)| \operatorname{sgn}\big((\tau(t,x) - t_0)(\mu(t,x) - x_0)\big) \int_{t_0}^{\tau(t,x)}\int_{x_0}^{\mu(t,x)} p_k(s,\eta)\mathrm{d}\eta\mathrm{d}s$$
$$\text{for a. e. } (t,x) \in \mathcal{D},\ k \in \mathbb{N}. \tag{5.4}$$

Then the problem (1.1′), (1.2) *is uniquely solvable.*

Remark 5.2. Example 7.1 shows that neither of the strict inequalities (5.3) in Corollary 5.1 can be replaced by the nonstrict one.

Theorem 5.2. *Let ℓ be a (t_0, x_0)–Volterra operator. Then the problem* (1.1), (1.2) *has a unique solution.*

Corollary 5.2. *Let the conditions* (2.2) *and* (2.3) *hold. Then the problem* (1.1′), (1.2) *has a unique solution.*

Proofs. Now we prove statements formulated above.

Proof of Theorem 5.1. According to Theorem 4.1, it is sufficient to show that the homogeneous problem (1.1_0), (1.2_0) has only the trivial solution.

Let u be a solution to the problem (1.1_0), (1.2_0). Then it is clear that

$$u(t,x)=\int_{t_0}^{t}\int_{x_0}^{x}\ell(u)(s,\eta)\mathrm{d}\eta\mathrm{d}s=T(u)(t,x)=\vartheta_1(u)(t,x)\quad\text{for }(t,x)\in\mathcal{D}.$$

Using the last relation, we get

$$u(t,x)=T\big(\vartheta_1(u)\big)(t,x)=\vartheta_2(u)(t,x)\quad\text{for }(t,x)\in\mathcal{D},$$

and thus $u=\vartheta_k(u)$ for every $k\in\mathbb{N}$. Therefore, (5.2) implies

$$\|u\|_C=\|\vartheta_m(u)\|_C\le\alpha\|u\|_C,$$

which guarantees $u\equiv 0$. □

Proof of Corollary 5.1. Let the operator ℓ be defined by the relation (2.1). It is clear that

$$\begin{aligned}&\big|\vartheta_k(v)(t,x)\big|\le\\&\le\operatorname{sgn}\big((t-t_0)(x-x_0)\big)\int_{t_0}^{t}\int_{x_0}^{x}\big|p(s,\eta)\vartheta_{k-1}(v)\big(\tau(s,\eta),\mu(s,\eta)\big)\big|\mathrm{d}\eta\mathrm{d}s\le\\&\le\|v\|_C\operatorname{sgn}\big((t-t_0)(x-x_0)\big)\int_{t_0}^{t}\int_{x_0}^{x}p_k(s,\eta)\mathrm{d}\eta\mathrm{d}s\\&\qquad\text{for }(t,x)\in\mathcal{D},\ k\in\mathbb{N},\ v\in C(\mathcal{D};\mathbb{R}).\end{aligned}$$

Therefore, the assumptions of Theorem 5.1 are satisfied with $m=j$ and

$$\alpha=\max\left\{\operatorname{sgn}\big((t-t_0)(x-x_0)\big)\int_{t_0}^{t}\int_{x_0}^{x}p_j(s,\eta)\mathrm{d}\eta\mathrm{d}s:(t,x)\in\mathcal{D}\right\}.$$

□

To prove Theorem 5.2 we need the following lemma.

Lemma 5.1. *Let $\ell\in\mathcal{L}(\mathcal{D})$ be a (t_0,x_0)–Volterra operator. Then*

$$\lim_{k\to+\infty}\|\vartheta_k\|=0,\tag{5.5}$$

where the operators ϑ_k are defined by the relations (5.1).

Proof. Let $\varepsilon\in\,]0,1[$. According to Proposition 3.1, the operator ϑ_1 is completely continuous. Therefore, by virtue of Arzelà-Ascoli's lemma, there exists $\delta>0$ such that

$$\left|\int_{t_0}^{y_2}\int_{x_0}^{z_2}\ell(w)(s,\eta)d\eta ds-\int_{t_0}^{y_1}\int_{x_0}^{z_1}\ell(w)(s,\eta)d\eta ds\right|\le\varepsilon\,\|w\|_C$$

$$\text{for }(y_1,z_1),(y_2,z_2)\in\mathcal{D},\ |y_2-y_1|+|z_2-z_1|<\delta,\ w\in C(\mathcal{D};\mathbb{R}).\tag{5.6}$$

Let

$$n=\max\left\{\operatorname{Ent}\left(\frac{2(t_0-a))}{\delta}\right),\operatorname{Ent}\left(\frac{2(b-t_0))}{\delta}\right),\right.$$

$$\mathrm{Ent}\left(\frac{2(x_0-c))}{\delta}\right), \mathrm{Ent}\left(\frac{2(d-x_0))}{\delta}\right)\Bigg\} + 1.$$

Choose $y_{n+1} \in [a, t_0]$, $y_{n+2} \in [t_0, b]$ and $z_{n+1} \in [c, x_0]$, $z_{n+2} \in [x_0, d]$ such that $y_{n+2} - y_{n+1} < \delta/2$ and $z_{n+2} - z_{n+1} < \delta/2$, and put

$$y_k = \begin{cases} y_{n+1} - (n+1-k)\,\frac{y_{n+1}-a}{n} & \text{for } k = 1, 2, \dots, n, \\ y_k = y_{n+2} + (k-n-2)\,\frac{b-y_{n+2}}{n} & \text{for } k = n+3, n+4, \dots, 2n+2, \end{cases}$$

$$z_k = \begin{cases} z_{n+1} - (n+1-k)\,\frac{z_{n+1}-a}{n} & \text{for } k = 1, 2, \dots, n, \\ z_k = y_{n+2} + (k-n-2)\,\frac{b-z_{n+2}}{n} & \text{for } k = n+3, n+4, \dots, 2n+2, \end{cases}$$

and

$$\mathcal{D}_k = [y_{n+2-k}, y_{n+1+k}] \times [z_{n+2-k}, z_{n+1+k}] \quad \text{for } k = 1, 2, \dots, n+1.$$

It is clear that, for any $j, r = 1, 2, \dots, 2n+1$, we get

$$|t_2 - t_1| + |x_2 - x_1| < \delta \quad \text{for } (t_1, x_1), (t_2, x_2) \in [y_j, y_{j+1}] \times [z_r, z_{r+1}]. \tag{5.7}$$

Having $w \in C(\mathcal{D}; \mathbb{R})$, we denote

$$\|w\|_i = \|w\|_{C(\mathcal{D}_i;\mathbb{R})} \quad \text{for } i = 1, 2, \dots, n+1.$$

Let $v \in C(\mathcal{D}; \mathbb{R})$ be arbitrary but fixed. We shall show that the relation

$$\|\vartheta_k(v)\|_i \le \alpha_i(k)\varepsilon^k \|v\|_C \quad \text{for } k \in \mathbb{N} \tag{5.8}$$

holds for every $i = 1, 2, \dots, n+1$, where

$$\alpha_i(k) = \alpha_i k^{i-1} \quad \text{for } k \in \mathbb{N},\ i = 1, 2, \dots, n+1 \tag{5.9}$$

and

$$\alpha_1 = 1, \quad \alpha_{i+1} = i + 1 + i\alpha_i \quad \text{for } i = 1, 2, \dots, n. \tag{5.10}$$

By virtue of (5.6) and (5.7), it is easy to verify that, for any $w \in C(\mathcal{D}; \mathbb{R})$ and $i = 1, 2, \dots, n+1$, we have

$$\left| \int_{t_0}^{t} \int_{x_0}^{x} \ell(w)(s, \eta) \mathrm{d}\eta \mathrm{d}s \right| \le i\,\varepsilon\,\|w\|_C \quad \text{for } (t, x) \in \mathcal{D}_i\,. \tag{5.11}$$

We first note that the previous relation immediately implies that

$$\|\vartheta_1(v)\|_i \le i\,\varepsilon\,\|v\|_C \quad \text{for } i = 1, 2, \dots, n+1. \tag{5.12}$$

Furthermore, on account of (5.6), (5.7), and the fact that ℓ is a (t_0, x_0)–Volterra operator, we obtain

$$\big|\vartheta_{k+1}(v)(t, x)\big| = \left| \int_{t_0}^{t} \int_{x_0}^{x} \ell\big(\vartheta_k(v)\big)(s, \eta) \mathrm{d}\eta \mathrm{d}s \right| \le \varepsilon\,\|\vartheta_k(v)\|_1 \quad \text{for } (t, x) \in \mathcal{D}_1,\ k \in \mathbb{N}.$$

Hence, by virtue of (5.12), we get

$$\|\vartheta_k(v)\|_1 \le \varepsilon^k\,\|v\|_C \quad \text{for } k \in \mathbb{N},$$

and thus the relation (5.8) is true for $i = 1$.

Now suppose that the relation (5.8) holds for some $i \in \{1, 2, \dots, n\}$. We shall show that the relation indicated is also true for $i + 1$. With respect to (5.7), we obtain

$$\|\vartheta_{k+1}(v)\|_{i+1} = \max\left\{ \left| \int_{t_0}^{t} \int_{x_0}^{x} \ell\big(\vartheta_k(v)\big)(s, \eta) \mathrm{d}\eta \mathrm{d}s \right| : (t, x) \in \mathcal{D}_{i+1} \right\} =$$

$$= \left| \int_{t_0}^{t_k^*} \int_{x_0}^{x_k^*} \ell\big(\vartheta_k(v)\big)(s,\eta)\mathrm{d}\eta\mathrm{d}s \right| \leq \left| \int_{t_0}^{\hat{t}_k} \int_{x_0}^{\hat{x}_k} \ell\big(\vartheta_k(v)\big)(s,\eta)\mathrm{d}\eta\mathrm{d}s \right| +$$

$$+ \left| \int_{t_0}^{t_k^*} \int_{x_0}^{x_k^*} \ell\big(\vartheta_k(v)\big)(s,\eta)\mathrm{d}\eta\mathrm{d}s - \int_{t_0}^{\hat{t}_k} \int_{x_0}^{\hat{x}_k} \ell\big(\vartheta_k(v)\big)(s,\eta)\mathrm{d}\eta\mathrm{d}s \right| \quad \text{for } k \in \mathbb{N},$$

where $(t_k^*, x_k^*) \in \mathcal{D}_{i+1}$, $(\hat{t}_k, \hat{x}_k) \in \mathcal{D}_i$, and $|t_k^* - \hat{t}_k| + |x_k^* - \hat{x}_k| < \delta$ for $k \in \mathbb{N}$. Therefore, on account of (5.6), (5.11), and the fact that ℓ is a $[t_0, h]$–Volterra operator, we get

$$\|\vartheta_{k+1}(v)\|_{i+1} \leq \varepsilon \, \|\vartheta_k(v)\|_{i+1} + i\,\varepsilon \, \|\vartheta_k(v)\|_i \leq \varepsilon \, \|\vartheta_k(v)\|_{i+1} + i\,\alpha_i(k)\,\varepsilon^{k+1} \, \|v\|_C$$

for $k \in \mathbb{N}$. Consequently,

$$\|\vartheta_{k+1}(v)\|_{i+1} \leq \varepsilon \Big(\varepsilon \, \|\vartheta_{k-1}(v)\|_{i+1} + i\,\alpha_i(k-1)\,\varepsilon^k \, \|v\|_C \Big) + \\ + i\,\alpha_i(k)\,\varepsilon^{k+1} \, \|v\|_C \quad \text{for } k \in \mathbb{N}.$$

Continuing this procedure, on account of (5.12), we obtain

$$\|\vartheta_{k+1}(v)\|_{i+1} \leq \Big(i + 1 + i\,\big(\alpha_i(1) + \cdots + \alpha_i(k)\big) \Big)\,\varepsilon^{k+1} \, \|v\|_C \quad \text{for } k \in \mathbb{N}. \tag{5.13}$$

Using (5.9) and (5.10), it is easy to verify that

$$i + 1 + i\,\big(\alpha_i(1) + \cdots + \alpha_i(k)\big) = i + 1 + i\,\alpha_i\big(1^{i-1} + \cdots + k^{i-1}\big) \leq \\ \leq i + 1 + i\,\alpha_i\,k\,k^{i-1} = i + 1 + i\,\alpha_i\,k^i \leq \\ \leq (i + 1 + i\,\alpha_i)\,k^i = \alpha_{i+1}\,k^i \leq \alpha_{i+1}\,(k+1).$$

Therefore, (5.12) and (5.13) imply

$$\|\vartheta_k(v)\|_{i+1} \leq \alpha_{i+1}(k)\,\varepsilon^k \, \|v\|_C \quad \text{for } k \in \mathbb{N}.$$

Hence, by induction, we have proved that the relation (5.8) is true for every $i = 1, 2, \ldots, n+1$.

Now it is already clear that, for any $k \in \mathbb{N}$, the estimate

$$\|\vartheta_k(v)\|_C = \|\vartheta_k(v)\|_{n+1} \leq \alpha_{n+1}\,k^n\,\varepsilon^k \, \|v\|_C \quad \text{for } v \in C(\mathcal{D}; \mathbb{R})$$

holds, and thus

$$\|\vartheta_k\| \leq \alpha_{n+1}\,k^n\,\varepsilon^k \quad \text{for } k \in \mathbb{N}.$$

Since we suppose $\varepsilon \in]0, 1[$, the last relation yields the validity of the condition (5.5). □

Proof of Theorem 5.2. According to Lemma 5.1, there exists $m_0 \in \mathbb{N}$ such that $\|\vartheta_{m_0}\| < 1$. Moreover, it is clear that

$$\|\vartheta_{m_0}(v)\|_C \leq \|\vartheta_{m_0}\| \, \|v\|_C \quad \text{for } v \in C(\mathcal{D}; \mathbb{R}),$$

because the operator ϑ_{m_0} is bounded. Therefore, the assumptions of Theorem 5.1 are satisfied with $m = m_0$ and $\alpha = \|\vartheta_{m_0}\|$. □

Proof of Corollary 5.2. The assumptions (2.2) and (2.3) guarantee that the operator ℓ given by the relation (2.1) is a (t_0, x_0)–Volterra one. Therefore, the validity of the corollary follows immediately from Theorem 5.2. □

6. Well-posedness

In this part, the well-posedness of the problems (1.1), (1.2) and (1.1′), (1.2) are investigated. We first formulate all the results, their proofs are given later.

For any $k \in \mathbb{N}$, along with the problem (1.1), (1.2) we consider the perturbed problem

$$\frac{\partial^2 u(t,x)}{\partial t\,\partial x} = \ell_k(u)(t,x) + q_k(t,x), \tag{1.1_k}$$

$$u(t,x_k) = \varphi_k(t) \quad \text{for } t \in [a,b], \qquad u(t_k,x) = \psi_k(x) \quad \text{for } x \in [c,d], \tag{1.2_k}$$

where $\ell_k \in \mathcal{L}(\mathcal{D})$, $q_k \in L(\mathcal{D};\mathbb{R})$, $t_k \in [a,b]$, $x_k \in [c,d]$, and $\varphi_k \in AC([a,b];\mathbb{R})$, $\psi_k \in AC([c,d];\mathbb{R})$ are such that $\varphi_k(t_k) = \psi_k(x_k)$.

Introduce the following notation.

Notation 6.1. Let $\Lambda \in \mathcal{L}(\mathcal{D})$, $t^* \in [a,b]$, and $x^* \in [c,d]$. Denote by $M(\Lambda,t^*,x^*)$ the set of all functions $y \in C^*(\mathcal{D};\mathbb{R})$ admitting the representation

$$y(t,x) = \int_{t^*}^{t}\int_{x^*}^{x} \Lambda(z)(s,\eta)\mathrm{d}\eta\mathrm{d}s \quad \text{for } (t,x) \in \mathcal{D},$$

where $z \in C(\mathcal{D};\mathbb{R})$ and $\|z\|_C = 1$.

Theorem 6.1. *Let the problem* (1.1), (1.2) *have a unique solution* u *and*

$$\lim_{k\to+\infty} \lambda_k = 0, \tag{6.1}$$

where

$$\lambda_k = \sup_{\substack{(t,x)\in\mathcal{D}\\ y\in M(\ell_k,t_k,x_k)}} \left\{ \left| \int_{t_k}^{t}\int_{x_k}^{x} \ell_k(y)(s,\eta)\mathrm{d}\eta\mathrm{d}s - \int_{t_0}^{t}\int_{x_0}^{x} \ell(y)(s,\eta)\mathrm{d}\eta\mathrm{d}s \right| \right\} \tag{6.2}$$

for $k \in \mathbb{N}$. *Let, moreover,*

$$\lim_{k\to+\infty} \varrho_k \left[\int_{t_k}^{t}\int_{x_k}^{x} \ell_k(y)(s,\eta)\mathrm{d}\eta\mathrm{d}s - \int_{t_0}^{t}\int_{x_0}^{x} \ell(y)(s,\eta)\mathrm{d}\eta\mathrm{d}s \right] = 0$$

$$\text{uniformly on } \mathcal{D} \text{ for every } y \in C^*(\mathcal{D};\mathbb{R}), \tag{6.3}$$

$$\lim_{k\to+\infty} \varrho_k \left[\int_{t_k}^{t}\int_{x_k}^{x} q_k(s,\eta)\mathrm{d}\eta\mathrm{d}s - \int_{t_0}^{t}\int_{x_0}^{x} q(s,\eta)\mathrm{d}\eta\mathrm{d}s \right] = 0$$

$$\text{uniformly on } \mathcal{D}, \tag{6.4}$$

$$\lim_{k\to+\infty} \varrho_k \|\varphi_k - \varphi\|_C = 0, \qquad \lim_{k\to+\infty} \varrho_k \|\psi_k - \psi\|_C = 0, \tag{6.5}$$

and

$$\lim_{k\to+\infty} \varrho_k \big|\varphi_k(t_k) - \varphi(t_0)\big| = 0, \tag{6.6}$$

where

$$\varrho_k = 1 + \|\ell_k\| \quad \text{for } k \in \mathbb{N}. \tag{6.7}$$

Then there exists $k_0 \in \mathbb{N}$ *such that, for every* $k > k_0$, *the problem* (1.1_k), (1.2_k) *has a unique solution* u_k *and*

$$\lim_{k\to+\infty} \|u_k - u\|_C = 0. \tag{6.8}$$

Remark 6.1. It is clear that the condition (6.6) is equivalent to the condition

$$\lim_{k\to+\infty}\big(1+\|\ell_k\|\big)\big|\psi_k(x_k)-\psi(x_0)\big|=0.$$

Note also that sequences $\{t_k\}$ and $\{x_k\}$ in Theorem 6.1 may not converge to t_0 and x_0, respectively. Indeed, let $\ell_k=\ell=0$[2], $q_k\equiv q\equiv 0$, $a=c=0$, $b=d=1$, $t_0=x_0=1$, $t_k=x_k=1/k$, $\varphi_k\equiv\varphi\equiv\psi_k\equiv\psi\equiv\alpha$, where $\alpha\in AC([0,1];\mathbb{R})$ is such that $\alpha(0)=\alpha(1)$. Then the assumptions of Theorem 6.1 are satisfied whereas $t_k\to 0$ and $x_k\to 0$ when k tends to $+\infty$.

If we suppose that the operators ℓ_k are "uniformly bounded" in the sense of the relation (6.9) then we obtain the following statement.

Corollary 6.1. *Let the problem* (1.1), (1.2) *have a unique solution u, there exist a function $\omega\in L(\mathcal{D};\mathbb{R}_+)$ such that*

$$|\ell_k(y)(t,x)|\le\omega(t,x)\|y\|_C \quad \text{for a. e. } (t,x)\in\mathcal{D} \text{ and all } y\in C(\mathcal{D};\mathbb{R}),\ k\in\mathbb{N}, \tag{6.9}$$

and let

$$\lim_{k\to+\infty}\int_{t_k}^{t}\int_{x_k}^{x}\ell_k(y)(s,\eta)\mathrm{d}\eta\mathrm{d}s=\int_{t_0}^{t}\int_{x_0}^{x}\ell(y)(s,\eta)\mathrm{d}\eta\mathrm{d}s \quad \text{uniformly on } \mathcal{D} \text{ for every } y\in C^*(\mathcal{D};\mathbb{R}). \tag{6.10}$$

Let, moreover,

$$\lim_{k\to+\infty}\int_{t_k}^{t}\int_{x_k}^{x}q_k(s,\eta)\mathrm{d}\eta\mathrm{d}s=\int_{t_0}^{t}\int_{x_0}^{x}q(s,\eta)\mathrm{d}\eta\mathrm{d}s \quad \text{uniformly on } \mathcal{D}, \tag{6.11}$$

$$\lim_{k\to+\infty}\|\varphi_k-\varphi\|_C=0, \qquad \lim_{k\to+\infty}\|\psi_k-\psi\|_C=0, \tag{6.12}$$

and

$$\lim_{k\to+\infty}\varphi(t_k)=\varphi(t_0). \tag{6.13}$$

Then the conclusion of Theorem 6.1 holds.

Remark 6.2. The condition (6.13) is satisfied if and only if

$$\lim_{k\to+\infty}\psi(x_k)=\psi(x_0).$$

Remark 6.3. The assumption (6.9) in the previous corollary is essential and cannot be omitted (see Example 7.2).

Corollary 6.2. *Let the problem* (1.1), (1.2) *have a unique solution u and there exist a function $\omega\in L(\mathcal{D};\mathbb{R}_+)$ such that the relation* (6.9) *holds. Let, moreover, the condition* (6.12) *be satisfied,*

$$\lim_{k\to+\infty}\int_{a}^{t}\int_{c}^{x}\big[\ell_k(y)(s,\eta)-\ell(y)(s,\eta)\big]\mathrm{d}\eta\mathrm{d}s=0 \quad \text{uniformly on } \mathcal{D} \text{ for every } y\in C^*(\mathcal{D};\mathbb{R}), \tag{6.14}$$

$$\lim_{k\to+\infty}\int_{a}^{t}\int_{c}^{x}\big[q_k(y)(s,\eta)-q(s,\eta)\big]\mathrm{d}\eta\mathrm{d}s=0 \quad \text{uniformly on } \mathcal{D}, \tag{6.15}$$

[2] The symbol 0 stands here for the zero operator.

and

$$\lim_{k\to+\infty} t_k = t_0, \qquad \lim_{k\to+\infty} x_k = x_0. \tag{6.16}$$

Then the conclusion of Theorem 6.1 holds.

Corollary 6.2 immediately yields

Corollary 6.3. *Let the homogeneous problem* (1.1_0), (1.2_0) *have only the trivial solution. Then the Darboux operator*[3] *of the problem* (1.1_0), (1.2_0) *is continuous.*

Now we give a statement on the well-posedness of the problem (1.1′), (1.2). For any $k \in \mathbb{N}$, along with the equation (1.1′) we consider the perturbed equation

$$\frac{\partial^2 u(t,x)}{\partial t\,\partial x} = p_k(t,x)u\big(\tau_k(t,x),\mu_k(t,x)\big) + q_k(t,x), \tag{$1.1'_k$}$$

where $p_k, q_k \in L(\mathcal{D};\mathbb{R})$ and $\tau_k\colon \mathcal{D} \to [a,b]$, $\mu_k\colon \mathcal{D} \to [c,d]$ are measurable functions.

Corollary 6.4. *Let the problem* (1.1′), (1.2) *have a unique solution* u, *there exist a function* $\omega \in L(\mathcal{D};\mathbb{R}_+)$ *such that*

$$|p_k(t,x)| \le \omega(t,x) \quad \text{for a. e. } (t,x)\in\mathcal{D},\ k\in\mathbb{N}, \tag{6.17}$$

and let

$$\lim_{k\to+\infty} \int_a^t \int_c^x \big[p_k(s,\eta) - p(s,\eta)\big]\mathrm{d}s\mathrm{d}\eta = 0 \quad \text{uniformly on } \mathcal{D}. \tag{6.18}$$

Let, moreover, the conditions (6.12), (6.15), *and* (6.16) *be satisfied, and*

$$\lim_{k\to+\infty} \operatorname{ess\,sup}\Big\{\big|\tau_k(t,x) - \tau(t,x)\big| : (t,x)\in\mathcal{D}\Big\} = 0, \tag{6.19}$$

$$\lim_{k\to+\infty} \operatorname{ess\,sup}\Big\{\big|\mu_k(t,x) - \mu(t,x)\big| : (t,x)\in\mathcal{D}\Big\} = 0. \tag{6.20}$$

Then there exists $k_0 \in \mathbb{N}$ *such that, for every* $k > k_0$, *the problem* $(1.1'_k)$, (1.2_k) *has a unique solution* u_k *and the relation* (6.8) *holds.*

Remark 6.4. The assumption (6.17) in the previous theorem is essential and cannot be omitted (see Example 7.2).

Finally, we consider the hyperbolic equation without argument deviations

$$u_{tx} = p(t,x)u + q(t,x) \tag{6.21}$$

in which $p, q \in L(\mathcal{D};\mathbb{R})$. For any $k \in \mathbb{N}$, along with the equation indicated we consider the perturbed equation

$$u_{tx} = p_k(t,x)u + q_k(t,x) \tag{6.21_k}$$

where $p_k, q_k \in L(\mathcal{D};\mathbb{R})$.

The following statement can be derived from Theorem 6.1.

Corollary 6.5. *Let the conditions* (6.4)–(6.6) *be satisfied,*

$$\lim_{k\to+\infty} \varrho_k \left[\int_{t_k}^t \int_{x_k}^x p_k(s,\eta)\mathrm{d}\eta\mathrm{d}s - \int_{t_0}^t \int_{x_0}^x p(s,\eta)\mathrm{d}\eta\mathrm{d}s\right] = 0 \quad \text{uniformly on } \mathcal{D}, \tag{6.22}$$

[3]The notion of the Cauchy operator is introduced in Definition 4.1.

and

$$\lim_{k\to+\infty}\varrho_k\int_{t_0}^{t_k}\int_c^d|p(s,\eta)|\mathrm{d}\eta\mathrm{d}s=0,\quad \lim_{k\to+\infty}\varrho_k\int_{x_0}^{x_k}\int_a^b|p(s,\eta)|\mathrm{d}s\mathrm{d}\eta=0,\tag{6.23}$$

where

$$\varrho_k=1+\|p_k\|_L.\tag{6.24}$$

Then the relation (6.8) *holds, where u and u_k are solutions to the problems* (6.21), (1.2) *and* (6.21_k), (1.2_k), *respectively.*

From Corollary 6.5 we get

Corollary 6.6. *Let the conditions* (6.12), (6.15), (6.16), *and* (6.18) *be satisfied, and*

$$\sup\big\{\|p_k\|_L: k\in\mathbb{N}\big\}<+\infty.$$

Then the conclusion of Corollary 6.5 holds.

Corollary 6.6 immediately yields

Corollary 6.7. *Let the conditions* (6.12) *and* (6.16) *be satisfied,*

$$\lim_{k\to+\infty}\|p_k-p\|_L=0,\tag{6.25}$$

and

$$\lim_{k\to+\infty}\|q_k-q\|_L=0.\tag{6.26}$$

Then the conclusion of Corollary 6.5 holds.

6.1. **Proofs.** In order to prove Theorem 6.1, we need the following lemma.

Lemma 6.1. *Let the problem* (1.1_0), (1.2_0) *have only the trivial solution and let the condition* (6.1) *hold, where the numbers λ_k are defined by the formula* (6.2). *Then, for an arbitrary $z\in C^*(\mathcal{D};\mathbb{R})$, there exist $r_0>0$ and $k_0\in\mathbb{N}$ such that*

$$\|y-z\|_C\le r_0(1+\|\ell_k\|)\Big[\|\Delta_k(y)-\Delta_0(z)\|_C+\|\Gamma_k(y,z)\|_C\Big]\quad \text{for } k>k_0,\ y\in C^*(\mathcal{D};\mathbb{R}),\tag{6.27}$$

where

$$\Delta_k(v)(t,x)=-v(t_k,x_k)+v(t,x_k)+v(t_k,x)\quad \text{for } (t,x)\in\mathcal{D},\ v\in C^*(\mathcal{D};\mathbb{R}),\ k\in\mathbb{N}\cup\{0\},\tag{6.28}$$

and

$$\Gamma_k(v,w)(t,x)=\int_{t_k}^t\int_{x_k}^x\big[v''_{[12]}(s,\eta)-\ell_k(v-w)(s,\eta)\big]\mathrm{d}\eta\mathrm{d}s-\\ -\int_{t_0}^t\int_{x_0}^x w''_{[12]}(s,\eta)\mathrm{d}\eta\mathrm{d}s\quad \text{for } (t,x)\in\mathcal{D},\ v,w\in C^*(\mathcal{D};\mathbb{R}),\ k\in\mathbb{N}.\tag{6.29}$$

Proof. Let the operators $T,T_k\colon C(\mathcal{D};\mathbb{R})\to C(\mathcal{D};\mathbb{R})$ be defined by the formulas (3.1) and

$$T_k(v)(t,x)=\int_{t_k}^t\int_{x_k}^x\ell_k(v)(s,\eta)\mathrm{d}\eta\mathrm{d}s\quad \text{for } (t,x)\in\mathcal{D},\ v\in C(\mathcal{D};\mathbb{R}),\ k\in\mathbb{N}.$$

Obviously,

$$\|T_k(y)\|_C \le \|\ell_k(y)\|_L \le \|\ell_k\|\,\|y\|_C \quad \text{for } y \in C(\mathcal{D};\mathbb{R}),\ k \in \mathbb{N}.$$

Therefore, the operators T_k $(k \in \mathbb{N})$ are linear bounded ones, and the relation

$$\|T_k\| \le \|\ell_k\| \quad \text{for } k \in \mathbb{N} \tag{6.30}$$

holds. Moreover, the condition (6.1) with λ_k given by (6.2) can be rewritten in the form

$$\sup\Big\{\|T_k(y) - T(y)\|_C : y \in M(\ell_k, t_k, x_k)\Big\} \to 0 \quad \text{as } k \to +\infty. \tag{6.31}$$

Assume that, on the contrary, the assertion of the lemma is not true. Then there exist $z \in C^*(\mathcal{D};\mathbb{R})$, an increasing sequence $\{k_m\}_{m=1}^{+\infty}$ of natural numbers, and a sequence $\{y_m\}_{m=1}^{+\infty}$ of functions from $C^*(\mathcal{D};\mathbb{R})$ such that, for every $m \in \mathbb{N}$, the relation

$$\|y_m - z\|_C > m(1 + \|\ell_{k_m}\|)\Big[\|\Delta_{k_m}(y_m) - \Delta(z)\|_C + \|\Gamma_{k_m}(y_m, z)\|_C\Big] \tag{6.32}$$

holds. For any $m \in \mathbb{N}$ and $(t,x) \in \mathcal{D}$, we put

$$z_m(t,x) = \frac{y_m(t,x) - z(t,x)}{\|y_m - z\|_C}\,, \tag{6.33}$$

$$v_m(t,x) = \frac{1}{\|y_m - z\|_C}\Big[\Delta_{k_m}(y_m)(t,x) - \Delta(z)(t,x) + \Gamma_{k_m}(y_m, z)(t,x)\Big], \tag{6.34}$$

$$z_{0,m}(t,x) = z_m(t,x) - v_m(t,x), \tag{6.35}$$

$$w_m(t,x) = T_{k_m}(z_{0,m})(t,x) - T(z_{0,m})(t,x) + T_{k_m}(v_m)(t,x). \tag{6.36}$$

Obviously,

$$\|z_m\|_C = 1 \quad \text{for } m \in \mathbb{N}. \tag{6.37}$$

Using (6.28)–(6.29) in the relation (6.34) and, by virtue of the conditions (a)–(c) of Proposition 2.1, we get

$$z_{0,m}(t,x) = T_{k_m}(z_m)(t,x) \quad \text{for } (t,x) \in \mathcal{D},\ m \in \mathbb{N}, \tag{6.38}$$

and thus

$$z_{0,m}(t,x) = T(z_{0,m})(t,x) + w_m(t,x) \quad \text{for } (t,x) \in \mathcal{D},\ m \in \mathbb{N}. \tag{6.39}$$

Moreover, it follows from (6.32) and (6.34) that

$$\|v_m\|_C \le \frac{\|\Delta_{k_m}(y_m) - \Delta(z)\|_C + \|\Gamma_{k_m}(y_m, z)\|_C}{\|y_m - z\|_C} < \frac{1}{m(1 + \|\ell_{k_m}\|)} \tag{6.40}$$

for $m \in \mathbb{N}$. Now the relations (6.30) and (6.40) yield

$$\|T_{k_m}(v_m)\|_C \le \|T_{k_m}\|\,\|v_m\|_C \le \frac{\|\ell_{k_m}\|}{m(1 + \|\ell_{k_m}\|)} < \frac{1}{m} \quad \text{for } m \in \mathbb{N}. \tag{6.41}$$

Note that the expression (6.38) and the condition (6.37) guarantee the validity of the inclusion $z_{0,m} \in M(\ell_{k_m}, t_{k_m}, x_{k_m})$ for $m \in \mathbb{N}$, and thus, in view of (6.31), we obtain

$$\lim_{m \to +\infty} \|T_{k_m}(z_{0,m}) - T(z_{0,m})\|_C = 0. \tag{6.42}$$

According to (6.41) and (6.42), it follows from (6.36) that

$$\lim_{m \to +\infty} \|w_m\|_C = 0, \tag{6.43}$$

and, by virtue of (6.37) and (6.40), the equality (6.35) implies $\|z_{0,m}\|_C < 2$ for $m \in \mathbb{N}$. Since the sequence $\{\|z_{0,m}\|_C\}_{m=1}^{+\infty}$ is bounded and the operator T is completely continuous (see Proposition 3.1), there exists a subsequence of $\{T(z_{0,m})\}_{m=1}^{+\infty}$ which is convergent. We can assume without loss of generality that the sequence $\{T(z_{0,m})\}_{m=1}^{+\infty}$ is convergent, i. e., there exists $z_0 \in C(\mathcal{D};\mathbb{R})$ such that

$$\lim_{m\to+\infty} \|T(z_{0,m}) - z_0\|_C = 0.$$

Then it is clear that

$$\lim_{m\to+\infty} \|z_{0,m} - z_0\|_C = 0, \tag{6.44}$$

because the functions $z_{0,m}$ admit the representation (6.39) and the relation (6.43) holds. However, the estimate (6.40) is true for v_m and thus, the equality (6.35) yields

$$\lim_{m\to+\infty} \|z_m - z_0\|_C = 0,$$

which, together with (6.37), guarantees $\|z_0\|_C = 1$. Since the operator T is continuous and the conditions (6.43) and (6.44) are fulfilled, the relation (6.39) yields $z_0 = T(z_0)$. Consequently, it is easy to verify that $z_0 \in C^*(\mathcal{D};\mathbb{R})$ (see Proposition 2.1) and, by virtue of Lemma 4.1(ii), z_0 is a nontrivial solution to the homogeneous problem (1.1_0), (1.2_0), which is a contradiction. □

Proof of Theorem 6.1. Since the problem (1.1), (1.2) has a unique solution, the homogeneous problem (1.1_0), (1.2_0) has only the trivial solution. Therefore, the assumptions of Lemma 6.1 are satisfied, and thus there exist $r_0 > 0$ and $k_0 \in \mathbb{N}$ such that

$$\|y\|_C \le r_0(1 + \|\ell_k\|)\Big[\|\Delta_k(y)\|_C + \|\Gamma_k(y,0)\|_C\Big] \quad \text{for } k > k_0,\ y \in C^*(\mathcal{D};\mathbb{R}) \tag{6.45}$$

and

$$\|y - u\|_C \le r_0(1 + \|\ell_k\|)\Big[\|\Delta_k(y) - \Delta_0(u)\|_C + \|\Gamma_k(y,u)\|_C\Big] \quad \text{for } k > k_0,\ y \in C^*(\mathcal{D};\mathbb{R}), \tag{6.46}$$

where the operators Δ_k and Γ_k are given by the formulas (6.28) and (6.29), respectively.

If, for some $k \in \mathbb{N}$, u_0 is a solution to the problem

$$\begin{gathered} \frac{\partial^2 u(t,x)}{\partial t\,\partial x} = \ell_k(u)(t,x), \\ u(t,x_k) = 0 \quad \text{for } t \in [a,b], \qquad u(t_k,x) = 0 \quad \text{for } x \in [c,d] \end{gathered} \tag{6.47}$$

then $\Delta_k(u_0) \equiv 0$ and $\Gamma_k(u_0,0) \equiv 0$. Therefore, the relation (6.45) guarantees that, for every $k > k_0$, the homogeneous problem (6.47) has only the trivial solution. Hence, for every $k > k_0$, the problem (1.1_k), (1.2_k) has a unique solution u_k (see Theorem 4.1). Then we get

$$\begin{aligned} \Delta_k(u_k)(t,x) &= -\varphi_k(t_k) + \varphi_k(t) + \psi_k(x) \quad && \text{for } (t,x) \in \mathcal{D},\ k > k_0, \\ \Delta_0(u)(t,x) &= -\varphi(t_0) + \varphi(t) + \psi(x) && \text{for } (t,x) \in \mathcal{D}, \end{aligned}$$

and

$$\Gamma_k(u,u_k)(t,x) = \int_{t_k}^t \int_{x_k}^x \ell_k(u)(s,\eta)\mathrm{d}\eta\mathrm{d}s - \int_{t_0}^t \int_{x_0}^x \ell(u)(s,\eta)\mathrm{d}\eta\mathrm{d}s +$$

$$+\int_{t_k}^{t}\int_{x_k}^{x} q_k(s,\eta)\mathrm{d}\eta\mathrm{d}s - \int_{t_0}^{t}\int_{x_0}^{x} q(s,\eta)\mathrm{d}\eta\mathrm{d}s \quad \text{for } (t,x)\in\mathcal{D},\ k>k_0.$$

Using the relations (6.3)–(6.6), we get

$$\lim_{k\to+\infty}(1+\|\ell_k\|)\Big[\|\Delta_k(u_k)-\Delta(u)\|_C+\|\Gamma_k(u_k,u)\|_C\Big]=0. \tag{6.48}$$

On the other hand, it follows from the inequality (6.46) that

$$\|u_k-u\|_C\le r_0(1+\|\ell_k\|)\Big[\|\Delta_k(u_k)-\Delta(u)\|_C+\|\Gamma_k(u_k,u)\|_C\Big] \quad \text{for } k>k_0 \tag{6.49}$$

and thus, by virtue of the relation (6.48), the condition (6.8) holds. □

Proof of Corollary 6.1. We shall show that the assumptions of Theorem 6.1 are satisfied. Indeed, the relation (6.9) yields $\|\ell_k\|\le\|\omega\|_L$ for $k\in\mathbb{N}$. Therefore, it is clear that, by virtue of the relations (6.10)–(6.13), the assumptions (6.3)–(6.6) of Theorem 6.1 are fulfilled. It remains to show that the condition (6.1) holds, where the numbers λ_k are given by the formula (6.2).

Assume that, on the contrary, the condition (6.1) does not hold. Then there exist $\varepsilon_0>0$, an increasing sequence $\{k_m\}_{m=1}^{+\infty}$ of natural numbers, and a sequence $\{y_m\}_{m=1}^{+\infty}$ such that

$$y_m\in M\big(\ell_{k_m},t_{k_m},x_{k_m}\big) \quad \text{for } m\in\mathbb{N} \tag{6.50}$$

and

$$\max_{(t,x)\in\mathcal{D}}\left\{\left|\int_{t_{k_m}}^{t}\int_{x_{k_m}}^{x}\ell_{k_m}(y_m)(s,\eta)\mathrm{d}\eta\mathrm{d}s-\int_{t_0}^{t}\int_{t_0}^{x}\ell(y_m)(s,\eta)\mathrm{d}\eta\mathrm{d}s\right|\right\}\ge\varepsilon_0 \quad \text{for } m\in\mathbb{N}. \tag{6.51}$$

In view of (6.50) and Notation 6.1, we get

$$y_m(t,x)=\int_{t_{k_m}}^{t}\int_{x_{k_m}}^{x}\ell_{k_m}(z_m)(s,\eta)\mathrm{d}\eta\mathrm{d}s \quad \text{for } (t,x)\in\mathcal{D},\ m\in\mathbb{N},$$

where $z_m\in C(\mathcal{D};\mathbb{R})$ and $\|z_m\|_C=1$ for $m\in\mathbb{N}$. Since we suppose that the operators ℓ_k are uniformly bounded in the sense of condition (6.9), we obtain $\|y_m\|_C\le\|\omega\|_L$ for $m\in\mathbb{N}$, and thus the sequence $\{y_m\}_{m=1}^{+\infty}$ is bounded in the space $C(\mathcal{D};\mathbb{R})$. We will show that the sequence indicated is also equicontinuous. Let $\varepsilon>0$ be arbitrary but fixed. Since the function ω is integrable on $\mathcal{D}$, there exists $\delta>0$ such that the relation

$$\iint_E\omega(t,x)\mathrm{d}t\mathrm{d}x<\frac{\varepsilon}{2} \tag{6.52}$$

holds for every measurable set $E\subseteq\mathcal{D}$ satisfying $\operatorname{mes}E<\max\{b-a,d-c\}\delta$. Using the condition (6.9), for any $(t_1,x_1),(t_2,x_2)\in\mathcal{D}$ and $m\in\mathbb{N}$, we get

$$\left|\int_{t_{k_m}}^{t_2}\int_{x_{k_m}}^{x_2}\ell_{k_m}(z_m)(s,\eta)\mathrm{d}\eta\mathrm{d}s-\int_{t_{k_m}}^{t_1}\int_{x_{k_m}}^{x_1}\ell_{k_m}(z_m)(s,\eta)\mathrm{d}\eta\mathrm{d}s\right|\le$$
$$\le\sum_{k=1}^{2}\iint_{E_k}\omega(s,\eta)\mathrm{d}s\mathrm{d}\eta,$$

where the measurable sets $E_1,E_2\subseteq\mathcal{D}$ are such that $\operatorname{mes}E_1=(d-c)|t_2-t_1|$ and $\operatorname{mes}E_2=(b-a)|x_2-x_1|$. Therefore, by virtue of (6.52), we have

$$|y_m(t_2,x_2) - y_m(t_1,x_1)| < \varepsilon$$
$$\text{for } (t_1,x_1),(t_2,x_2) \in \mathcal{D},\ |t_2 - t_1| + |x_2 - x_1| < \delta,\ m \in \mathbb{N}.$$

Consequently, the sequence $\{y_m\}_{m=1}^{+\infty}$ is equicontinuous in the space $C(\mathcal{D};\mathbb{R})$. Therefore, according to the Arzelà-Ascoli lemma, we can assume without loss of generality that the sequence indicated is convergent. Hence, there exists $p_0 \in \mathbb{N}$ such that

$$\|y_m - y_{p_0}\|_C < \frac{\varepsilon_0}{2(\|\omega\|_L + \|\ell\| + 1)} \quad \text{for } m \geq p_0. \tag{6.53}$$

Since $y_{p_0} \in C^*(\mathcal{D};\mathbb{R})$ and the relation (6.10) holds, there exists $p_1 \in \mathbb{N}$ such that

$$\max_{(t,x)\in\mathcal{D}} \left\{ \left| \int_{t_k}^{t}\int_{x_k}^{x} \ell_k(y_{p_0})(s,\eta)\mathrm{d}\eta\mathrm{d}s - \int_{t_0}^{t}\int_{x_0}^{x} \ell(y_{p_0})(s,\eta)\mathrm{d}\eta\mathrm{d}s \right| \right\} < \frac{\varepsilon_0}{2} \quad \text{for } k \geq p_1. \tag{6.54}$$

Now we choose a number $M \in \mathbb{N}$ satisfying $M \geq p_0$ and $k_M \geq p_1$. It is clear that

$$\left| \int_{t_{k_M}}^{t}\int_{x_{k_M}}^{x} \ell_{k_M}(y_M)(s,\eta)\mathrm{d}\eta\mathrm{d}s - \int_{t_0}^{t}\int_{x_0}^{x} \ell(y_M)(s,\eta)\mathrm{d}\eta\mathrm{d}s \right| \leq$$
$$\leq \left| \int_{t_{k_M}}^{t}\int_{x_{k_M}}^{x} \ell_{k_M}(y_M - y_{p_0})(s,\eta)\mathrm{d}\eta\mathrm{d}s \right| + \left| \int_{t_0}^{t}\int_{x_0}^{x} \ell(y_{p_0} - y_M)(s,\eta)\mathrm{d}\eta\mathrm{d}s \right| +$$
$$+ \left| \int_{t_{k_M}}^{t}\int_{x_{k_M}}^{x} \ell_{k_M}(y_{p_0})(s,\eta)\mathrm{d}\eta\mathrm{d}s - \int_{t_0}^{t}\int_{x_0}^{x} \ell(y_{p_0})(s,\eta)\mathrm{d}\eta\mathrm{d}s \right| \quad \text{for } (t,x) \in \mathcal{D}.$$

Therefore, by virtue of the conditions (6.9), (6.53), and (6.54), the last relation yields

$$\max_{(t,x)\in\mathcal{D}} \left\{ \left| \int_{t_{k_M}}^{t}\int_{x_{k_M}}^{x} \ell_{k_M}(y_M)(s,\eta)\mathrm{d}\eta\mathrm{d}s - \int_{t_0}^{t}\int_{x_0}^{x} \ell(y_M)(s,\eta)\mathrm{d}\eta\mathrm{d}s \right| \right\} \leq$$
$$\leq \|\omega\|_L \|y_M - y_{p_0}\|_C + \frac{\varepsilon_0}{2} + \|\ell\| \|y_{p_0} - y_M\|_C < \varepsilon_0, \tag{6.55}$$

which contradicts the condition (6.51).

The contradiction obtained proves the validity of the condition (6.1), and thus all the assumptions of Theorem 6.1 are satisfied. □

To prove Corollary 6.2 we need the following lemma.

Lemma 6.2. *Let the condition* (6.16) *and* $\{\sigma_k\}_{k=1}^{+\infty}$ *be a sequence of functions from* $L(\mathcal{D};\mathbb{R})$ *such that*

$$\lim_{k\to+\infty} \int_a^t\int_c^x \big[\sigma_k(s,\eta) - \sigma(s,\eta)\big]\mathrm{d}\eta\mathrm{d}s = 0 \quad \textit{uniformly on } \mathcal{D}, \tag{6.56}$$

where $\sigma \in L(\mathcal{D};\mathbb{R})$. *Then*

$$\lim_{k\to+\infty} \int_{t_k}^{t}\int_{x_k}^{x} \sigma_k(s,\eta)\mathrm{d}\eta\mathrm{d}s = \int_{t_0}^{t}\int_{x_0}^{x} \sigma(s,\eta)\mathrm{d}\eta\mathrm{d}s \quad \textit{uniformly on } \mathcal{D}. \tag{6.57}$$

Proof. It is easy to verify that

$$\int_{t_k}^{t}\int_{x_k}^{x} \sigma_k(s,\eta)\mathrm{d}\eta\mathrm{d}s - \int_{t_0}^{t}\int_{x_0}^{x} \sigma(s,\eta)\mathrm{d}\eta\mathrm{d}s =$$

$$= \int_a^{t_k}\int_c^{x_k}\big[\sigma_k(s,\eta)-\sigma(s,\eta)\big]\mathrm{d}\eta\mathrm{d}s + \int_a^{t}\int_c^{x}\big[\sigma_k(s,\eta)-\sigma(s,\eta)\big]\mathrm{d}\eta\mathrm{d}s +$$
$$+\left(\int_a^{t_k}\int_c^{x_k}\sigma(s,\eta)\mathrm{d}\eta\mathrm{d}s - \int_a^{t_0}\int_c^{x_0}\sigma(s,\eta)\mathrm{d}\eta\mathrm{d}s\right) + \int_{t_k}^{t_0}\int_c^{x}\sigma(s,\eta)\mathrm{d}\eta\mathrm{d}s -$$
$$-\int_a^{t_k}\int_c^{x}\big[\sigma_k(s,\eta)-\sigma(s,\eta)\big]\mathrm{d}\eta\mathrm{d}s + \int_a^{t}\int_{x_k}^{x_0}\sigma(s,\eta)\mathrm{d}\eta\mathrm{d}s -$$
$$-\int_a^{t}\int_c^{x_k}\big[\sigma_k(s,\eta)-\sigma(s,\eta)\big]\mathrm{d}\eta\mathrm{d}s \quad \text{for } (t,x)\in\mathcal{D}.$$

Therefore, using the assumptions (6.16) and (6.56), we get the validity of the condition (6.57). □

Proof of Corollary 6.2. We shall show that the assumptions of Corollary 6.1 are satisfied. Indeed, according to Lemma 6.2, the assumptions (6.14)–(6.16) guarantee the validity of the conditions (6.10) and (6.11). On the other hand, the condition (6.13) is obviously satisfied, because the function φ is continuous and $t_k \to t_0$ when k tends to $+\infty$. □

To prove Corollary 6.4 we need the following statement, which is a two-dimensional analogy of the well-known Krasnoselskii-Krein's lemma.

Lemma 6.3. *Let $p, p_k \in L(\mathcal{D};\mathbb{R})$ and let $\alpha, \alpha_k\colon \mathcal{D}\to\mathbb{R}$ be measurable and essentially bounded functions for ($k\in\mathbb{N}$). Assume that the relations* (6.17) *and* (6.18) *are satisfied, and*

$$\lim_{k\to+\infty} \operatorname{ess\,sup}\Big\{|\alpha_k(t,x)-\alpha(t,x)| : (t,x)\in\mathcal{D}\Big\} = 0. \tag{6.58}$$

Then

$$\lim_{k\to+\infty}\int_a^t\int_c^x\big[p_k(s,\eta)\alpha_k(s,\eta)-p(s,\eta)\alpha(s,\eta)\big]\mathrm{d}\eta\mathrm{d}s = 0 \quad \textit{uniformly on } \mathcal{D}. \tag{6.59}$$

Proof. Without loss of generality we can assume that

$$|p(t,x)| \le \omega(t,x) \quad \text{for a. e. } (t,x)\in\mathcal{D}. \tag{6.60}$$

Let $\varepsilon>0$ be arbitrary but fixed. According to (6.58), there exists $k_0\in\mathbb{N}$ such that

$$\iint_{\mathcal{D}} \omega(t,x)|\alpha_k(t,x)-\alpha(t,x)|\mathrm{d}t\mathrm{d}x < \frac{\varepsilon}{4} \quad \text{for } k\ge k_0. \tag{6.61}$$

Since the function α is measurable and essentially bounded, there exists a function $w\in C(\mathcal{D};\mathbb{R})$, which has continuous derivatives up to the second order and such that

$$\iint_{\mathcal{D}} \omega(t,x)|\alpha(t,x)-w(t,x)|\mathrm{d}t\mathrm{d}x < \frac{\varepsilon}{4}. \tag{6.62}$$

For any $k\in\mathbb{N}$, we put

$$f_k(t,x) = \int_a^t\int_c^x\big[p_k(s,\eta)-p(s,\eta)\big]\mathrm{d}\eta\mathrm{d}s \quad \text{for } (t,x)\in\mathcal{D}.$$

Clearly, the condition (6.18) can be rewritten in the form

$$\lim_{k\to+\infty}\|f_k\|_C = 0. \tag{6.63}$$

It can be verified by direct computation that

$$\int_a^t\int_c^x\big[p_k(s,\eta)-p(s,\eta)\big]w(s,\eta)\mathrm{d}\eta\mathrm{d}s=f_k(t,x)w(t,x)-$$
$$-\int_a^t f_k(s,x)w'_{[1]}(s,x)\mathrm{d}s-\int_c^x f_k(t,\eta)w'_{[2]}(t,\eta)\mathrm{d}\eta+$$
$$+\int_a^t\int_c^x f_k(s,\eta)w''_{[12]}(s,\eta)\mathrm{d}\eta\mathrm{d}s\quad\text{for }(t,x)\in\mathcal{D},\ k\in\mathbb{N}.$$

Consequently, using (6.63), we get

$$\lim_{k\to+\infty}\int_a^t\int_c^x\big[p_k(s,\eta)-p(s,\eta)\big]w(s,\eta)\mathrm{d}\eta\mathrm{d}s=0\quad\text{uniformly on }\mathcal{D}.$$

Hence, there exists $k_1\geq k_0$ such that

$$\left|\int_a^t\int_c^x\big[p_k(s,\eta)-p(s,\eta)\big]w(s,\eta)\mathrm{d}\eta\mathrm{d}s\right|<\frac{\varepsilon}{4}\quad\text{for }(t,x)\in\mathcal{D},\ k\geq k_1.\tag{6.64}$$

On the other hand, it is clear that, for any $(t,x)\in\mathcal{D}$ and $k\in\mathbb{N}$,

$$\int_a^t\int_c^x\big[p_k(s,\eta)\alpha_k(s,\eta)-p(s,\eta)\alpha(s,\eta)\big]\mathrm{d}\eta\mathrm{d}s=$$
$$=\int_a^t\int_c^x p_k(s,\eta)\big[\alpha_k(s,\eta)-\alpha(s,\eta)\big]\mathrm{d}\eta\mathrm{d}s+$$
$$+\int_a^t\int_c^x\big[p_k(s,\eta)-p(s,\eta)\big]w(s,\eta)\mathrm{d}\eta\mathrm{d}s+$$
$$+\int_a^t\int_c^x\big[p_k(s,\eta)-p(s,\eta)\big]\big[\alpha(s,\eta)-w(s,\eta)\big]\mathrm{d}\eta\mathrm{d}s.$$

Therefore, in view of (6.17), (6.60)–(6.62), and (6.64), we get

$$\left|\int_a^t\int_c^x\big[p_k(s,\eta)\alpha_k(s,\eta)-p(s,\eta)\alpha(s,\eta)\big]\mathrm{d}\eta\mathrm{d}s\right|\leq$$
$$\leq\iint_{\mathcal{D}}\omega(s,\eta)|\alpha_k(s,\eta)-\alpha(s,\eta)|\mathrm{d}s\mathrm{d}\eta+$$
$$+\left|\int_a^t\int_c^x\big[p_k(s,\eta)-p(s,\eta)\big]w(s,\eta)\mathrm{d}\eta\mathrm{d}s\right|+$$
$$+2\iint_{\mathcal{D}}\omega(s,\eta)|\alpha(s,\eta)-w(s,\eta)|\mathrm{d}s\mathrm{d}\eta<$$
$$<\frac{\varepsilon}{4}+\frac{\varepsilon}{4}+2\,\frac{\varepsilon}{4}=\varepsilon\quad\text{for }(t,x)\in\mathcal{D},\ k\geq k_1\,,$$

that is, the relation (6.59) is true. □

Proof of Corollary 6.4. Let the operator ℓ be defined by the formula (2.1). Put

$$\ell_k(v)(t,x)=p_k(t,x)v\big(\tau_k(t,x),\mu_k(t,x)\big)$$
$$\text{for a. e. }(t,x)\in\mathcal{D}\text{ and all }v\in C(\mathcal{D};\mathbb{R}),\ k\in\mathbb{N}.\tag{6.65}$$

We will show that the condition (6.14) is satisfied. Indeed, let $y \in C^*(\mathcal{D};\mathbb{R})$ be arbitrary but fixed. It is clear that the conditions (6.19) and (6.20) guarantee the validity of the relation (6.58), where

$$\alpha_k(t,x) = y\big(\tau_k(t,x),\mu_k(t,x)\big), \quad \alpha(t,x) = y\big(\tau(t,x),\mu(t,x)\big)$$

for a. e. $(t,x) \in \mathcal{D}$ and all $k \in \mathbb{N}$. Therefore, it follows from Lemma 6.3 that the condition (6.59) holds, i. e., the condition (6.14) is fulfilled. Moreover, by virtue of the relation (6.17), the condition (6.9) is satisfied.

Consequently, the assertion of the corollary follows from Corollary 6.2. □

Proof of Corollary 6.5. Notice that, according to Corollary 5.2, the problems (6.21)–(1.2) and (6.21_k)– (1.2_k) have unique solutions u and u_k, respectively.

Let the operators ℓ and ℓ_k be defined by the formulas

$$\ell(v)(t,x) = p(t,x)v(t,x) \quad \text{for a. e. } (t,x) \in \mathcal{D} \text{ and all } v \in C(\mathcal{D};\mathbb{R}), \tag{6.66}$$

and

$$\ell_k(v)(t,x) = p_k(t,x)v(t,x) \quad \text{for a. e. } (t,x) \in \mathcal{D}, \ \text{all } v \in C(\mathcal{D};\mathbb{R}), \ k \in \mathbb{N}, \tag{6.67}$$

respectively. Obviously,

$$\|\ell_k\| = \|p_k\|_L \quad \text{for } k \in \mathbb{N}. \tag{6.68}$$

Therefore, it is clear that the assumptions (6.4)–(6.6) of Theorem 6.1 are satisfied. In order to apply Theorem 6.1, it remains to show that the condition (6.1) and (6.3) are fulfilled.

It is easy to see that

$$\left|\int_{t_k}^{t}\int_{x_k}^{x}\big[p_k(s,\eta)-p(s,\eta)\big]\mathrm{d}\eta\mathrm{d}s\right| \le$$
$$\le \left|\int_{t_k}^{t}\int_{x_k}^{x}p_k(s,\eta)\mathrm{d}\eta\mathrm{d}s - \int_{t_0}^{t}\int_{x_0}^{x}p(s,\eta)\mathrm{d}\eta\mathrm{d}s\right| +$$
$$+\left|\int_{t_0}^{t_k}\int_{c}^{d}|p(s,\eta)|\mathrm{d}\eta\mathrm{d}s\right| + \left|\int_{x_0}^{x_k}\int_{a}^{b}|p(s,\eta)|\mathrm{d}s\mathrm{d}\eta\right| \quad \text{for } (t,x) \in \mathcal{D}, \ k \in \mathbb{N}.$$

Therefore, the conditions (6.22) and (6.23) guarantee that

$$\lim_{k\to+\infty} \varrho_k \|f_k\|_C = 0, \tag{6.69}$$

where

$$f_k(t,x) = \int_{t_k}^{t}\int_{x_k}^{x}\big[p_k(s,\eta)-p(s,\eta)\big]\mathrm{d}\eta\mathrm{d}s \quad \text{for } (t,x) \in \mathcal{D}, \ k \in \mathbb{N}. \tag{6.70}$$

We first note that, for an arbitrary $y \in C(\mathcal{D};\mathbb{R})$, we have

$$\left|\int_{t_k}^{t}\int_{x_k}^{x}\ell_k(y)(s,\eta)\mathrm{d}\eta\mathrm{d}s - \int_{t_0}^{t}\int_{x_0}^{x}\ell(y)(s,\eta)\mathrm{d}\eta\mathrm{d}s\right| \le$$
$$\le \left|\int_{t_k}^{t}\int_{x_k}^{x}\big[p_k(s,\eta)-p(s,\eta)\big]y(s,\eta)\mathrm{d}\eta\mathrm{d}s\right| +$$
$$+\left|\int_{t_0}^{t_k}\int_{c}^{d}|p(s,\eta)y(s,\eta)|\mathrm{d}\eta\mathrm{d}s\right| +$$

$$+\left|\int_{x_0}^{x_k}\int_a^b |p(s,\eta)y(s,\eta)|\mathrm{d}s\mathrm{d}\eta\right| \quad \text{for } (t,x)\in\mathcal{D},\ k\in\mathbb{N}. \tag{6.71}$$

Moreover, for an arbitrary $y \in C^*(\mathcal{D};\mathbb{R})$, we can verify by direct computation that

$$\int_{t_k}^{t}\int_{x_k}^{x}\big[p_k(s,\eta)-p(s,\eta)\big]y(s,\eta)\mathrm{d}\eta\mathrm{d}s = f_k(t,x)y(t,x)-$$
$$-\int_{t_k}^{t} f_k(s,x)y'_{[1]}(s,x)\mathrm{d}s - \int_{x_k}^{x} f_k(t,\eta)y'_{[2]}(t,\eta)\mathrm{d}\eta +$$
$$+\int_{t_k}^{t}\int_{x_k}^{x} f_k(s,\eta)y''_{[12]}(s,\eta)\mathrm{d}\eta\mathrm{d}s \quad \text{for } (t,x)\in\mathcal{D},\ k\in\mathbb{N}. \tag{6.72}$$

Let $k \in \mathbb{N}$ and $y \in M(\ell_k, t_k, x_k)$ be arbitrary but fixed. Then, by virtue of Notation 6.1 and Lemma 4.1, we get

$$|y(t,x)| = \left|\int_{t_k}^{t}\int_{x_k}^{x} p_k(s,\eta)z(s,\eta)\mathrm{d}\eta\mathrm{d}s\right| \le \varrho_k \quad \text{for } (t,x)\in\mathcal{D}, \tag{6.73}$$

$$|y'_{[1]}(t,x)| = \left|\int_{x_k}^{x} p_k(t,\eta)z(t,\eta)\mathrm{d}\eta\right| \le \int_c^d |p_k(t,\eta)|\mathrm{d}\eta$$
$$\text{for a.e. } t\in[a,b] \text{ and all } x\in[c,d], \tag{6.74}$$

$$|y'_{[2]}(t,x)| = \left|\int_{t_k}^{t} p_k(s,x)z(s,x)\mathrm{d}s\right| \le \int_a^b |p_k(s,x)|\mathrm{d}s$$
$$\text{for all } t\in[a,b] \text{ and a.e. } x\in[c,d], \tag{6.75}$$

and

$$|y''_{[12]}(t,x)| = |p_k(t,x)z(t,x)| \le |p_k(t,x)| \quad \text{for a.e. } (t,x)\in\mathcal{D}. \tag{6.76}$$

Using relations (6.73)–(6.76), it follows from the inequalities (6.71) and (6.72) that

$$\left|\int_{t_k}^{t}\int_{x_k}^{x}\ell_k(y)(s,\eta)\mathrm{d}\eta\mathrm{d}s - \int_{t_0}^{t}\int_{x_0}^{x}\ell(y)(s,\eta)\mathrm{d}\eta\mathrm{d}s\right| \le$$
$$\le 4\varrho_k\|f_k\|_C + \varrho_k\left|\int_{t_0}^{t_k}\int_c^d |p(s,\eta)|\mathrm{d}\eta\mathrm{d}s\right| +$$
$$+\varrho_k\left|\int_{x_0}^{x_k}\int_a^b |p(s,\eta)|\mathrm{d}s\mathrm{d}\eta\right| \quad \text{for } (t,x)\in\mathcal{D},\ k\in\mathbb{N}.$$

Therefore, according to the relations (6.23) and (6.69), the condition (6.1) holds, where the numbers λ_k are given by the formula (6.2).

Now let $y \in C^*(\mathcal{D};\mathbb{R})$ be arbitrary but fixed. Put

$$\varrho_0 = \|y\|_C + \max\left\{\int_a^b |y'_{[1]}(s,x)|\mathrm{d}s : x\in[c,d]\right\} +$$
$$+\max\left\{\int_c^d |y'_{[2]}(t,\eta)|\mathrm{d}\eta : t\in[a,b]\right\} + \|y''_{[12]}\|_L. \tag{6.77}$$

Then the inequalities (6.71) and (6.72) imply that

$$\left|\int_{t_k}^{t}\int_{x_k}^{x}\ell_k(y)(s,\eta)\mathrm{d}\eta\mathrm{d}s-\int_{t_0}^{t}\int_{x_0}^{x}\ell(y)(s,\eta)\mathrm{d}\eta\mathrm{d}s\right|\le$$

$$\le\varrho_0\left(\|f_k\|_C+\left|\int_{t_0}^{t_k}\int_{c}^{d}|p(s,\eta)|\mathrm{d}\eta\mathrm{d}s\right|+\right.$$

$$\left.+\left|\int_{x_0}^{x_k}\int_{a}^{b}|p(s,\eta)|\mathrm{d}s\mathrm{d}\eta\right|\right)\quad\text{for } (t,x)\in\mathcal{D},\ k\in\mathbb{N}.$$

According to the relations (6.23) and (6.69), the last inequality yields the validity of the condition (6.3).

Consequently, the assertion of the corollary follows from Theorem 6.1. □

Proof of Corollary 6.6. We will show that all the assumptions of Corollary 6.5 are satisfied. Indeed, in view of the relations (6.12) and (6.16), the assumptions (6.5), (6.6), and (6.23) are satisfied. Moreover, by virtue of the relations (6.15), (6.16), and (6.18), Lemma 6.2 guarantees the validity of the conditions (6.4) and (6.22). □

7. Counter-examples

Example 7.1. Let $p\in L(\mathcal{D};\mathbb{R}_+)$ be such that

$$\int_{t_0}^{b}\int_{x_0}^{d}p(s,\eta)\mathrm{d}\eta\mathrm{d}s=1$$

and let the operator ℓ be defined by the relation

$$\ell(v)(t,x)=p(t,x)v(b,d)\quad\text{for a.e. } (t,x)\in\mathcal{D}\text{ and all } v\in C(\mathcal{D};\mathbb{R}).$$

Then the condition (5.2) with $\alpha=1$ is satisfied for every $m\in\mathbb{N}$ and $v\in C(\mathcal{D};\mathbb{R})$. Moreover,

$$\int_{t_0}^{b}\int_{x_0}^{d}p_j(s,\eta)\mathrm{d}\eta\mathrm{d}s=1\quad\text{for every}\quad j\in\mathbb{N},$$

where the function p_j is given by the formula (5.4).

On the other hand, the problem (1.1_0), (1.2_0) has a nontrivial solution

$$u(t,x)=\int_{a}^{t}\int_{c}^{x}p(s,\eta)d\eta ds\quad\text{for } (t,x)\in\mathcal{D}.$$

This example shows that the assumption $\alpha\in[0,1[$ in Theorem 5.1 cannot be replaced by the assumption $\alpha\in[0,1]$, and the strict inequality

$$\int_{t_0}^{b}\int_{x_0}^{d}p_j(s,\eta)\mathrm{d}\eta\mathrm{d}s<1$$

in Corollary 5.1 cannot be replaced by the nonstrict one. The optimality of the other strict inequalities in (5.3) can be justified analogously.

Example 7.2. Let

$$g_k(t)=k\cos(k^2t),\quad h_k(t)=k\sin(k^2t)\quad\text{for } t\ge0,\ k\in\mathbb{N},\tag{7.1}$$

and

$$y_k(t) = -k \int_0^t \exp\left(\frac{\sin(k^2 t)}{k} - \frac{\sin(k^2 s)}{k}\right) \sin(k^2 s) ds \quad \text{for } t \ge 0, \ k \in \mathbb{N}. \tag{7.2}$$

It is not difficult to verify that, for every $k \in \mathbb{N}$,

$$y_k'(t) = g_k(t) y_k(t) + h_k(t) \quad \text{for } t \ge 0 \tag{7.3}$$

and

$$|y_k(t)| \le 1 + e + te^2 \quad \text{for } t \ge 0, \tag{7.4}$$

because

$$\begin{aligned} y_k(t) = \frac{1}{k} \cos(k^2 t) - \frac{1}{k} \exp\left(\frac{\sin(k^2 t)}{k}\right) + \\ + \frac{1}{2} \int_0^t \exp\left(\frac{\sin(k^2 t)}{k} - \frac{\sin(k^2 s)}{k}\right) ds + \\ + \frac{1}{2} \int_0^t \exp\left(\frac{\sin(k^2 t)}{k} - \frac{\sin(k^2 s)}{k}\right) \cos(2k^2 s) ds \quad \text{for } t \ge 0. \end{aligned}$$

Moreover,

$$\lim_{k \to +\infty} y_k(t) = \frac{t}{2} \quad \text{for } t \ge 0. \tag{7.5}$$

Now, let $p \equiv 0$, $q \equiv 0$, $t_0 = a$, $x_0 = c$, $\varphi \equiv 0$, $\psi \equiv 0$, and

$$\tau(t, x) = t, \quad \mu(t, x) = x \quad \text{for } (t, x) \in \mathcal{D}.$$

For any $k \in \mathbb{N}$, we put $t_k = a$, $x_k = c$, $\varphi_k \equiv 0$, $\psi_k \equiv 0$,

$$p_k(t, x) = g_k(t - a) g_k(x - c) \quad \text{for } (t, x) \in \mathcal{D},$$

$$\begin{aligned} q_k(t, x) = h_k(t - a) y_k'(x - c) + y_k'(t - a) h_k(x - c) - \\ - h_k(t - a) h_k(x - c) \quad \text{for } (t, x) \in \mathcal{D}, \end{aligned}$$

and

$$\tau_k(t, x) = t, \quad \mu_k(t, x) = x \quad \text{for } (t, x) \in \mathcal{D}.$$

According to (7.1), (7.3), and (7.4), it is clear that the assumptions of Theorem 6.1 are satisfied except of (6.9). Let $\ell, \ell_k \in \mathcal{L}(\mathcal{D})$ be operators defined by (2.1) and (6.65), respectively. Then, it is not difficult to verify that the assumptions of Corollary 6.1 are fulfilled except of (6.17).

On the other hand,

$$u(t, x) = 0 \quad \text{for } (t, x) \in \mathcal{D}$$

and

$$u_k(t, x) = y_k(t - a) y_k(x - c) \quad \text{for } (t, x) \in \mathcal{D}, \ k \in \mathbb{N}$$

are solutions to the problems $(1.1')$, (1.2) and $(1.1_k')$, (1.2_k), respectively, as well as the problems (1.1), (1.2) and (1.1_k), (1.2_k), respectively. However, in view of (7.5), we get

$$\begin{aligned} \lim_{k \to +\infty} \Big(u_k(t, x) - u(t, x)\Big) = \lim_{k \to +\infty} y_k(t - a) y_k(x - c) = \\ = \frac{(t - a)(x - c)}{4} \quad \text{for } (t, x) \in \mathcal{D}, \end{aligned}$$

that is, the relation (6.8) is not true.

This example shows that the assumption (6.17) in Corollary 6.1 and the assumption (6.9) in Theorem 6.1 are essential and they cannot be omitted.

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