CANCELLATION RULE FOR INTERNAL DIRECT PRODUCT DECOMPOSITIONS OF A CONNECTED PARTIALLY ORDERED SET

Ján Jakubík[1], Mária Csontóová[2], Košice

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Abstract. In this note we deal with two-factor internal direct product decompositions of a connected partially ordered set.

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Direct product decompositions of a connected partially ordered set have been investigated by Hashimoto [1].

We apply the notion of internal direct product decomposition of a partially ordered set in the same sense as in [2]; the definition is recalled in Section 1 below.

The following cancellation rule has been proved in [2]:

(A) Let L be a directed partially ordered set and $x_0 \in L$. Let

$$\begin{aligned} \varphi^0 &\colon\ L \to A^0 \times B^0, \\ \psi^0 &\colon\ L \to A^0_1 \times B^0_1 \end{aligned}$$

be internal direct product decompositions of L with the same central element x^0. Suppose that $A^0 = A^0_1$. Then $B^0 = B^0_1$ and $\varphi^0(x) = \psi^0(x)$ for each $x \in L$.

The aim of the present paper is to generalize (A) to the case when L is a connected partially ordered set.

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1. Preliminaries

We recall that a partially ordered set is called connected if for any $x, y \in L$ there are elements $x_0, x_1, x_2, \ldots, x_n$ in L such that

(i) $x = x_0$, $y = x_n$;
(ii) if $i \in \{1, 2, \ldots, n\}$, then the elements x_{i-1} and x_i are comparable.

Let L be a connected partially ordered set. Suppose that we have a direct product decomposition

$$\varphi\colon\ L \to \prod_{i \in I} L_i \tag{1}$$

(i.e., φ is an isomorphism of the partially ordered set L onto the direct product $\prod\limits_{i \in I} L_i$). For $x \in L$ let $\varphi(x) = (\ldots, x_i, \ldots)_{i \in I}$. We denote $x_i = x(L_i)$. Next we put

$$L_i(x) = \{z \in L\colon\ z(L_j) = x(L_j) \quad \text{for each } j \in I \setminus \{i\}\}.$$

Let x^0 be a fixed element of L. For each $i \in I$ we denote $L_i(x^0) = L_i^0$.

For each $x \in L$ and each $i \in I$ there is a unique element y_i in L_i^0 such that $x(L_i) = y_i(L_i)$. Put

$$\varphi^0(x) = (\ldots, y_i, \ldots)_{i \in I}.$$

Then the relation

$$\varphi^0\colon\ L \to \prod_{i \in I} L_i^0 \tag{2}$$

is said to be an internal direct product decomposition of L with the central element x^0.

For each $i \in I$, L_i^0 is isomorphic to L_i.

2. Auxiliary results

In this section we suppose that L is a connected partially ordered set.

Assume that we are given a direct product decompostion

$$\varphi\colon\ L \to A \times B. \tag{1}$$

For $x \in L$ we put $\varphi(x) = (x_A, x_B)$. Sometimes we write $x(A)$ instead of x_A, and similarly for x_B.

Further, for each $x_0 \in L$ we put

$$A(x_0) = \{x \in L\colon\ x(B) = x_0(B)\},$$
$$B(x_0) = \{x \in L\colon\ x(A) = x_0(A)\}.$$

Let $x_1 \in L$, $x_1 \notin A(x_0)$. We put $A(x_0) < A(x_1)$ if there are $x_0^1 \in A(x_0)$ and $x_1^1 \in A(x_1)$ such that $x_0^1 < x_1^1$.

If $x, y, z \in L$ and $z = \sup\{x, y\}$ in L, then we express this fact by writing $z = x \vee y$. The meaning of $v = x \wedge y$ is analogous.

2.1. Lemma. *Let $x_0, x_1 \in L$, $A(x_0) < A(x_1)$, $x_2 \in A(x_1)$. Then there exists x_2^0 in $A(x_0)$ such that*

(i) *$x_2^0 < x_2$;*
(ii) *if $z \in A(x_0)$ and $z < x_2$, then $z \leqslant x_2^0$.*

P r o o f. There exists $x_2^0 \in L(x_0)$ such that

$$\varphi(x_2^0) = (x_2(A), x_0(B)).$$

Then $x_2^0 \in A(x_0)$. We have

$$x_0(B) = x_0^1(B) \leqslant x_1^1(B) = x_2(B),$$

where x_0^1 and x_1^1 are as in the definition of the relation $A(x_0) < A(x_1)$. Thus $x_2^0 \leqslant x_2$. Since $x_2 \notin A(x_0)$, we must have $x_2^0 < x_2$. Therefore (i) is valid.

Let $z \in A(x_0)$ and $z < x_2$. Then $z(B) = x_0(B) = x_2^0(B)$ and $z(A) \leqslant x_2(A)$; hence $z \leqslant x_2^0$. Thus (ii) holds. □

It is obvious that the element x_2^0 is uniquely determined if x_2 and $A(x_0)$ are given and if $A(x_2) > A(x_0)$.

2.2. Lemma. *Let x_0 and x_1 be as in 2.1. Further, let $x_3 \in L$, $x_3 \geqslant x_1$. Then the following conditions are equivalent:*

(i) *$x_3 \in A(x_1)$;*
(ii) *$x_3^0 \vee x_1 = x_3$.*

P r o o f. First we remark that from $x_3 \geqslant x_1$ we infer that $A(x_3) > A(x_0)$, whence in view of 2.1, the element x_3^0 does exist; moreover, we have

$$\varphi(x_3^0) = (x_3(A), x_0(B)).$$

Further, from the relation $A(x_0) < A(x_1)$ we conclude that whenever $t_1 \in A(x_0)$ and $t_2 \in A(x_1)$, then $t_1(B) < t_2(B)$. In particular, $x_0(B) < x_1(B)$. Thus $x_0(B) < x_3(B)$ and $x_3^0(B) < x_3(B)$.

Let (i) be valid. Hence $x_3(B) = x_1(B)$. From $x_3 \geqslant x_1$ we get $x_3(A) \geqslant x_1(A)$. Thus

$$(x_3(A), x_0(B)) \vee (x_1(A), x_1(B)) = (x_3(A), x_3(B)).$$

Therefore (ii) holds.

Conversely, let (ii) be valid. Then

$$x_3^0(B) \vee x_1(B) = x_3(B).$$

We already know that $x_3^0(B) \vee x_1(B) = x_1(B)$. Thus $x_1(B) = x_3(B)$. Hence (i) holds. □

2.3. Corollary. *Let x_0 and x_1 be as in 2.1. Then the set $\{x \in A(x_1)\colon\ x \geqslant x_1\}$ is uniquely determined by $A(x_0)$ and x_1.*

2.4. Lemma. *Let x_0 and x_1 be as in 2.1. Further, let $x_4 \in L$, $x_4 \leqslant x_1$. Then x_4 belongs to $A(x_1)$ if and only if the following conditions are satisfied:*

(i) *$x_4 \vee x_1^0 = x_1$;*
(ii) *$x_4 \notin A(x_0)$;*
(iii) *there exists $t \in A(x_0)$ with $t < x_4$.*

P r o o f. Suppose that x_4 belongs to $A(x_1)$. Then (ii) is obviously valid. In view of 2.1, the condition (iii) is satisfied.

For proving that (i) is valid we have to verify the validity of the relation

$$(x_4(A), x_4(B)) \vee (x_1^0(A), x_1^0(B)) = (x_1(A), x_1(B)). \tag{$*$}$$

We have

$$(x_1^0(A), x_1^0(B)) = (x_1(A), x_0(B)),$$

whence

$$x_4(A) \vee x_1^0(A) = x_4(A) \vee x_1(A) = x_1(A). \tag{$*_1$}$$

Further, in view of (iii), $x_4(B) \geqslant t(B)$. Since $t \in A(x_0)$, we get $t(B) = x_0(B)$. Thus

$$x_4(B) \vee x_1^0(B) = x_4(B) \vee x_0(B) = x_4(B) = x_1(B). \tag{$*_2$}$$

From $(*_1)$ and $(*_2)$ we conclude that $(*)$ is valid.

Conversely, suppose that the conditions (i), (ii) and (iii) are satisfied. From (i) we obtain

$$x_4(B) \vee x_1^0(B) = x_1(B).$$

Further we have $x_1^0(B) = t(B) \leqslant x_4(B)$, whence

$$x_4(B) \vee x_1^0(B) = x_4(B) \vee t(B) = x_4(B).$$

Then $x_4(B) = x_1(B)$, therefore $x_4 \in A(x_1)$. $\square$

2.5. Corollary. *Let x_0 and x_1 be as in* 2.1. *Then the set $\{x \in A(x_1)\colon\ x \leqslant x_1\}$ is uniquely determined by $A(x_0)$ and x_1.*

2.6. Definition. The interval $[u, v]$ of L is said to have the property (α) if

(i) there exist $u^0, v^0 \in A(x_0)$ such that the relations

$$u^0 = \max\{x \in A(x_0)\colon\ x \leqslant u\}, \quad v^0 = \max\{x \in A(x_0)\colon\ x \leqslant v\}$$

are valid;

(ii) $v^0 \vee u = v$.

2.7. Lemma. *Let x_0 and x_1 be as in* 2.1. *Let $z \in L$. The following conditions* (a) *and* (b) *are equivalent:*

(a) *There are elements $z_0, z_1, z_2, \ldots, z_n$ in L such that $z_0 = x_1$, $z_n = z$ and for each $i \in \{1, 2, \ldots, n\}$ we have*
   - (i) *the elements z_{i-1}, z_i are comparable;*
   - (ii) *if $z_{i-1} \leqslant z_i$, then the interval $[z_{i-1}, z_i]$ satisfies the condition (α);*
   - (iii) *if $z_{i-1} \geqslant z_i$, then the interval $[z_i, z_{i-1}]$ satisfies the condition (α).*

(b) $z \in A(x_1)$.

Proof. Assume that (a) is valid. Then in view of 2.2 and 2.4 we obtain $z_1 \in A(x_1)$. Now it suffices to apply induction with respect to n.

Conversely, assume that (b) is valid. Since L is connected, the partially ordered set A is connected as well. It is obvious that the partially ordered sets A and $A(x_1)$ are isomorphic; hence $A(x_1)$ is connected as well. Thus there are elements $z_0, z_1, \ldots, z_n$ in $A(x_1)$ such that $z_0 = x_1$, $z_n = z$ and for each $i \in \{1, 2, \ldots, n\}$ the elements z_{i-1}, z_i are comparable. Then by using 2.1, 2.2 and 2.4 we conclude that (a) is valid. $\square$

2.8. Corollary. *Let x_0 and x_1 be as in* 2.1. *Then the set $A(x_1)$ is uniquely determined by $A(x_0)$ and x_1.*

By a dual argument we obtain

2.9. Corollary. *Let $x_0, x_1 \in L$ be such that $A(x_0) > A(x_1)$. Then the set $A(x_1)$ is uniquely determined by $A(x_0)$ and x_1.*

From 2.8, 2.9 and from the fact that L is connected we conclude

2.10. Lemma. *Let $x_0, x_1 \in L$. Then the set $A(x_1)$ is uniquely determined by $A(x_0)$ and x_1.*

Let $x_0, x_1 \in L$, $x_0 \leqslant x_1$. In view of 2.1 there exists $a(x_0, x_1) \in L$ such that

$$a(x_0, x_1) = \max\{x \in A(x_0) \colon x \leqslant x_1\}.$$

Dually, if $x_0, x_1 \in L$, $x_0 \geqslant x_1$, then there is $b(x_0, x_1) \in L$ with

$$b(x_0, x_1) = \min\{x \in A(x_0) \colon x \geqslant x_1\}.$$

2.11. Lemma. *Let $x_0, x_1 \in L$, $x_0 \leqslant x_1$. Then*

$$x_1 \in B(x_0) \Leftrightarrow a(x_0, x_1) = x_0.$$

Proof. Suppose that $a(x_0, x_1) = x_0$. Hence $x_0(A) = x_1(A)$ and therefore $x_1 \in B(x_0)$.

Conversely, suppose that $x_1 \in B(x_0)$. Then $x_1(A) = x_0(A)$. From $x_0 \leqslant x_1$ we conclude that $x_0(B) \leqslant x_1(B)$.

Let $x \in A(x_0)$, $x \leqslant x_1$. We get $x(A) \leqslant x_1(A)$, whence $x(A) \leqslant x_0(A)$. Further, $x(B) = x_0(B)$. Therefore $x \leqslant x_0$. This yields that $a(x_0, x_1) = x_0$. □

By a dual argument we obtain

2.12. Lemma. *Let $x_0, x_1 \in L$, $x_0 \geqslant x_1$. Then*

$$x_1 \in B(x_0) \Leftrightarrow b(x_0, x_1) = x_0.$$

2.13. Lemma. *Let $x_0, x \in L$. The following conditions are equivalent:*

(a) *There exist elements $z_0, z_1, z_2, \ldots, z_n$ in L such that $z_0 = x_0$, $z_n = x$, for each $i \in \{1, 2, \ldots, n\}$ the elements z_{i-1}, z_i are comparable and $z_i \in B(z_{i-1})$;*

(b) *$x \in B(x_0)$.*

Proof. The implication (a)$\Rightarrow$(b) is obvious. Suppose that (b) is valid. The partially ordered set B is connected, hence so is $B(x_0)$. Thus there exist $z_0, z_1, \ldots, z_n \in B(x_0)$ with the properties as in (a). □

From 2.10–2.13 we obtain

2.14. Lemma. *Let $x_0 \in L$. Then the set $B(x_0)$ is uniquely determined by $A(x_0)$ and x_0.*

In 2.10, A can be replaced by B. Hence 2.14 yields

2.15. Corollary. *Let $x_0, x \in L$. Then the set $B(x)$ is uniquely determined by $A(x_0)$ and x.*

3. Cancellation rule

Suppose that L is a connected partially ordered set and consider direct product decompositions

(1) $\varphi\colon\ L \to A \times B$,

(2) $\varphi_1\colon\ L \to A_1 \times B_1$.

Let $x_0 \in L$. Then from (1) and (2) we can construct internal direct product decompositions

(1′) $\varphi^0\colon\ L \to A^0 \times B^0$,

(2′) $\varphi_1^0\colon\ L \to A_1^0 \times B_1^0$

with the central element x_0.

In view of the definition of the internal direct product decomposition we have

(3) $A^0 = A(x_0), \quad B^0 = B(x_0)$,

(4) $A_1^0 = A_1(x_0), \quad B_1^0 = B_1(x_0)$;

further, if $x \in L$ and $\varphi^0(x) = (x_1, x_2)$, $\varphi_1^0(x) = (x_1', x_2')$, then

(5) $\{x_1\} = A^0 \cap B(x), \quad \{x_2\} = B^0 \cap A(x)$,

(6) $\{x_1'\} = A_1^0 \cap B_1(x), \quad \{x_2'\} = B_1^0 \cap A_1(x)$.

3.1. Theorem. *Let (1′) and (2′) be an internal direct product of a connected partially ordered set L with the central element x_0. Suppose that $A^0 = A_1^0$. Then $B^0 = B_1^0$. Moreover, for each $x \in L$ we have $\varphi^0(x) = \varphi_1^0(x)$.*

P r o o f. The first assertion is a consequence of 2.10, 2.15 and of the relations (3), (4). Then in view of (5) and (6) we infer that $\varphi^0(x) = \varphi_1^0(x)$ for each $x \in L$. □

Let us remark that if $\varphi\colon\ L \to A \times B$ and $\psi\colon\ L \to A_1 \times B_1$ are direct product decompositions of a connected partially ordered set L and if A is isomorphic to A_1, then B need not be isomorphic to B_1.

E x a m p l e. Let N be the set of all positive integers and let X be a linearly ordered set having more than one element. Put

$$L = \prod_{n \in N} X_n,$$

where $X_n = X$ for each $n \in N$. We denote

$$A = \prod_{n > 1} X_n, \quad B = X_1,$$
$$A_1 = \prod_{n > 2} X_n, \quad B_1 = X_1 \times X_2.$$

Then we have direct product decompositions

$$\varphi\colon\ L \to A \times B, \quad \psi \to A_1 \times B_1,$$

A is isomorphic to A_1, but B fails to be isomorphic to B_1.

Further, the notion of the internal direct product decomposition can be used in group theory (where the central element coincides with the neutral element of the corresponding group); cf., e.g. Kurosh [3], p. 104. The result analogous to 3.1 does not hold, in general, for internal direct product decompositions of a group.

E x a m p l e. Let X be the additive group of all reals, $Y = X$, $G = X \times Y$. We put

$$X^0 = \{(x, 0)\colon\ x \in X\},$$
$$Y^0 = \{(0, y)\colon\ y \in Y\},$$
$$Z^0 = \{(x, y) \in G\colon\ x = y\}.$$

Then $Y^0 \neq Z^0$. The group G is the internal direct product of X^0 and Y^0; at the same time, G is the internal direct product of X^0 and Z_0.

We conclude by remarking that the assumption of connectedness of L cannot be omitted in 3.1.

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Authors' addresses: *Ján Jakubík*, Matematický ústav SAV, Grešákova 6, 040 01 Košice, Slovakia, e-mail: `musavke@mail.saske.sk`, *Mária Csontóová*, Katedra matematiky, Stavebná fakulta TU, Vysokoškolská 4, 042 00 Košice, Slovakia.