CONVERGENCE THEOREMS FOR THE PU-INTEGRAL

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Abstract. We give a definition of uniform PU-integrability for a sequence of μ-measurable real functions defined on an abstract metric space and prove that it is not equivalent to the uniform μ-integrability.

Keywords: PU-integral, PU-uniform integrability, μ-uniform integrability

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INTRODUCTION

In [4] we gave the definition of PU-integral on a suitable abstract metric measure space X and proved that this integral is equivalent to the μ-integral. Moreover, we gave an example of a non euclidean space verifying the previous results. In this paper, we give the definition of uniform PU-integrability for a sequence $\{f_n\}_n$ of real functions on X and prove that this concept is not equivalent to the uniform μ-integrability. Then, given a real function f on X, a suitable sequence $\{\bar{f}_n\}_n$ converging to f is defined and some conditions on f are given for $\{\bar{f}_n\}_n$ to be uniform PU-integrable.

PRELIMINARIES

In this paper X denotes a compact metric space, $\mathcal{M}$ a σ-algebra of subsets of X such that each open set is in $\mathcal{M}$, μ a non-atomic, finite, Radon measure on $\mathcal{M}$ such that

(i) each ball $U(x,r)$ centered at x with radius r has a positive measure,

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(ii) for every x in X there is a number $h(x) \in \mathbb{R}$ such that $\mu(U[x,2r]) \leqslant h(x) \times \mu(U[x,r])$ for all $r > 0$ (where $U[x,r]$) is the closed ball),

(iii) $\mu(\partial U(x,r)) = 0$ where $\partial U(x,r)$ is the boundary of $U(x,r)$.

We introduce the following basic concepts.

Definition 1. A partition of unity (PU-partition) in X is, by definition, a finite collection $P = \{(\theta_i, x_i)\}_{i=1}^{p}$ where $x_i \in X$ and θ_i are non negative, μ-measurable and μ-integrable real functions on X such that $\sum\limits_{i=1}^{p} \theta_i(x) = 1$ a.e. in X.

Definition 2. Let δ be a positive function on X. A PU-partition is said to be δ-fine if $S_{\theta_i} = \{x \in X\colon\ \theta_i(x) \neq 0\} \subset U(x_i, \delta(x_i))$, $i = 1, 2, \ldots, p$.

Definition 3. A real function f on X is said to be PU-integrable on X if there exists a real number I with the property that, for every given $\varepsilon > 0$, there is a positive function $\delta\colon\ X \longrightarrow \mathbb{R}$ such that $|\sum\limits_{i=1}^{p} f(x_i) \cdot \int_X \theta_i \mathrm{d}\mu - I \,| < \varepsilon$ for each δ-fine PU-partition $P = \{(\theta_i, x_i)\}_{i=1}^{p}$. The number I is called the PU-integral of f and we write $I =$(PU)$\int_X f$.

Definition 4. A sequence $\{f_n\}_n$ of PU-integrable functions is uniformly PU-integrable on X if for each $\varepsilon > 0$ there exists a positive function δ on X such that

$$\left| \sum_i f_n(x_i) \int_X \theta_i \mathrm{d}\mu - (\mathrm{PU}) \int_X f_n \right| < \varepsilon$$

for all n, whenever $P = \{(\theta_i, x_i)\}_i$ is a δ-fine PU-partition in X.

Definition 5. A sequence $\{f_n\}_n$ of real functions on X is a δ-Cauchy sequence if for each $\varepsilon > 0$ there exist a positive function δ on X and a positive integer $\bar{n}$ such that

$$\left| \sum_i f_n(x_i) \int_X \theta_i \mathrm{d}\mu - \sum_i f_m(x_i) \int_X \theta_i \mathrm{d}\mu \right| < \varepsilon$$

for all $m, n \geqslant \bar{n}$ and for each δ-fine PU-partition $P = \{(\theta_i, x_i)\}_i$.

Definition 6. A sequence $\{f_n\}_n$ of μ-integrable functions is uniformly μ-integrable on X if for each $\varepsilon > 0$ there exists a positive integer k such that

$$\int_{A_k^n} |f_n| \mathrm{d}\mu < \varepsilon$$

for all n, where $A_k^n = \{x \in X\colon\ |f_n(x)| > k\}$.

Definition 7. A real function f has small Riemann tails (sRt) if for each $\varepsilon > 0$ there exist a positive integer $\bar{n}$ and a positive function δ on X such that

$$\left|\sum_i f\chi_{A_n}(x_i)\int_X \theta_i \mathrm{d}\mu\right| < \varepsilon$$

for all $n \geqslant \bar{n}$ whenever $P = \{(\theta_i, x_i)\}_i$ is a δ-fine PU-partition in X, $A_n = \{x \in X\colon |f(x)| > n\}$ and χ_{A_n} is the characteristic function of A_n.

Definition 8. A function f has really small Riemann tails (rsRt) if for each $\varepsilon > 0$ there exist a positive integer n^* and a positive function δ on X such that

$$\left|\sum_i f(x_i)\int_X \theta_i \mathrm{d}\mu\right| < \varepsilon$$

whenever $P = \{(\theta_i, x_i)\}_i$ is an A_{n^*} δ-fine family, e.g. $S_{\theta_i} \subset U(x_i, \delta(x_i))$, $\sum\limits_i \theta_i(x) \leqslant 1$ a.e. in X and $x_i \in A_{n^*}$.

We observe that if f has rsRt then f has sRt but the converse is not usually true.

Part I

Proposition 1. *Let $\{f_n\}$ be a sequence of real functions defined on X such that*

(i) *f_n is* PU*-integrable on X for all n,*

(ii) *$\{f_n(x)\}_n$ converges pointwise to $f(x)$ on X,*

(iii) *$\{f_n\}_n$ is uniformly* PU*-integrable on X,*

then f is PU*-integrable on X and*

$$(\mathrm{PU})\int_X f = \lim_n\, (\mathrm{PU})\int_X f_n.$$

P r o o f. Let $\varepsilon > 0$, there exists a positive function δ on X such that

$$\left|\sum_{i=1}^{p} f_n(x_i')\int_X \theta_i' \mathrm{d}\mu - (\mathrm{PU})\int_X f_n\right| < \frac{\varepsilon}{3}$$

for all n, where $P = \{(\theta_i', x_i')\}_{i=1}^p$ is a fixed δ-fine partition and by (ii), there exists a positive integer n^* such that

$$\left|\sum_{i=1}^{p} f_n(x_i')\int_X \theta_i' \mathrm{d}\mu - \sum_{i=1}^{p} f_m(x_i')\int_X \theta_i' \mathrm{d}\mu\right| < \frac{\varepsilon}{3}$$

for all $m, n \geqslant n^*$.

Consider

$$\begin{aligned}
&\left|(\mathrm{PU})\int_X f_n - (\mathrm{PU})\int_X f_m\right| \\
&\qquad \leqslant \left|(\mathrm{PU})\int_X f_n - \sum_{i=1}^{p} f_n(x_i')\int_X \theta_i' \mathrm{d}\mu\right| \\
&\qquad\quad + \left|\sum_{i=1}^{p} f_n(x_i')\int_X \theta_i' \mathrm{d}\mu - \sum_{i=1}^{p} f_m(x_i')\int_X \theta_i' \mathrm{d}\mu\right| \\
&\qquad\quad + \left|\sum_{i=1}^{p} f_m(x_i')\int_X \theta_i' \mathrm{d}\mu - (\mathrm{PU})\int_X f_m\right| \\
&\qquad < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon
\end{aligned}$$

for all $m, n > n^*$.

So the sequence $\{(\mathrm{PU})\int_X f_n\}_n$ is a Cauchy sequence and let a be its limit. For each $\varepsilon > 0$ there is a positive function δ on X such that

$$\left|\sum_i f_n(x_i)\int_X \theta_i - (\mathrm{PU})\int_X f_n\right| < \frac{\varepsilon}{3}$$

for all n, whenever $P = \{(\theta_i, x_i)\}_i$ is a δ-fine PU-partition, and there is a positive integer $\bar{n}$ such that

$$\left|(\mathrm{PU})\int_X f_n - a\right| < \frac{\varepsilon}{3}$$

and

$$\left|\sum_i f(x_i)\int_X \theta_i \mathrm{d}\mu - \sum_i f_n(x_i)\int_X \theta_i \mathrm{d}\mu\right| < \frac{\varepsilon}{3},$$

for all $n \geqslant \bar{n}$.

Hence

$$\begin{aligned}
&\left|\sum_i f(x_i)\int_X \theta_i \mathrm{d}\mu - a\right| \\
&\qquad \leqslant \left|\sum_i f(x_i)\int_X \theta_i \mathrm{d}\mu - \sum_i f_n(x_i)\int_X \theta_i \mathrm{d}\mu\right| \\
&\qquad\quad + \left|\sum_i f_n(x_i)\int_X \theta_i \mathrm{d}\mu - (\mathrm{PU})\int_X f_n\right| - \left|(\mathrm{PU})\int_X f_n - a\right| < \varepsilon.
\end{aligned}$$

So f is PU-integrable and a is its PU-integral. $\square$

N o t e 1. We observe that this theorem is not equivalent to the generalized Vitali convergence theorem. In fact, if we consider the sequence $\{f_n\}_n$ so defined $f_n(x) = 0$ if $x \in (0,1]$ and $f_n(x) = 2n$ if $x = 0$, it is easy to verify that it is uniformly μ-integrable but it is not uniformly PU-integrable.

Proposition 2. *Let $\{f_n\}_n$ be a sequence of* PU*-integrable functions. Then $\{f_n\}_n$ is a δ-Cauchy sequence iff $\{f_n\}_n$ is uniformly* PU*-integrable and the sequence $\{(\mathrm{PU})\int_X f_n\}_n$ converges.*

P r o o f. If the sequence $\{f_n\}_n$ is uniformly PU-integrable and the sequence $\{(\mathrm{PU})\int_X f_n\}_n$ converges, for $\varepsilon > 0$ there are a positive function δ on X and a positive integer $\bar{n}$ s.t. for each $m, n > \bar{n}$

$$\left|(\mathrm{PU})\int_X f_n - (\mathrm{PU})\int_X f_m\right| < \frac{\varepsilon}{3},$$

and for each δ-fine partition $P = \{(\theta_i, x_i)\}_i$ we have

$$\left|(\mathrm{PU})\int_X f_n - \sum_i f_n(x_i)\int_X \theta_i \mathrm{d}\mu\right| < \frac{\varepsilon}{3}$$

and

$$\left|(\mathrm{PU})\int_X f_m - \sum_i f_m(x_i)\int_X \theta_i \mathrm{d}\mu\right| < \frac{\varepsilon}{3}.$$

Hence

$$\left|\sum_i f_m(x_i)\int_X \theta_i \mathrm{d}\mu - \sum_i f_n(x_i)\int_X \theta_i \mathrm{d}\mu\right| < \varepsilon$$

for all $m, n \geqslant \bar{n}$ and for each δ-fine partition P.

Now, suppose that $\{f_n\}_n$ is a δ-Cauchy sequence.

Let $\varepsilon > 0$, there exist a positive integer $\bar{n}$ and a positive function $\bar{\delta}$ on X s.t. for each $\bar{\delta}$-fine partition $P = \{(\theta_i, x_i)\}_i$ and for $m, n \geqslant \bar{n}$, we have

$$\left|(\mathrm{PU})\int_X f_m - \sum_i f_m(x_i)\int_X \theta_i \,\mathrm{d}\mu\right| < \frac{\varepsilon}{3},$$
$$\left|(\mathrm{PU})\int_X f_n - \sum_i f_n(x_i)\int_X \theta_i \,\mathrm{d}\mu\right| < \frac{\varepsilon}{3}$$

and

$$\left|\sum_i f_m(x_i)\int_X \theta_i \mathrm{d}\mu - \sum_i f_n(x_i)\int_X \theta_i \,\mathrm{d}\mu\right| < \frac{\varepsilon}{3}.$$

For a fixed $\bar{\delta}$-fine partition $P = \{(\theta_i', x_i')\}_i$, consider

$$\begin{aligned}
&\left|(\mathrm{PU})\int_X f_n - (\mathrm{PU})\int_X f_m\right| \\
&\qquad \leqslant \left|(\mathrm{PU})\int_X f_m - \sum_i f_m(x_i')\int_X \theta_i' \mathrm{d}\mu\right| \\
&\qquad\quad + \left|(\mathrm{PU})\int_X f_n - \sum_i f_n(x_i')\int_X \theta_i' \mathrm{d}\mu\right| \\
&\qquad\quad + \left|\sum_i f_m(x_i')\int_X \theta_i' \mathrm{d}\mu - \sum_i f_n(x_i')\int_X \theta_i' \mathrm{d}\mu\right| < \varepsilon
\end{aligned}$$

for all $m,\ n \geqslant \bar{n}$. So it follows that the sequence $\{(\mathrm{PU})\int_X f_n\}_n$ is a Cauchy sequence.

Now, for $\varepsilon > 0$, for each n there is a positive function δ_n on X s.t.

$$(*) \qquad \left|(\mathrm{PU})\int_X f_n - \sum_i f_n(x_i)\int_X \theta_i \mathrm{d}\mu\right| < \varepsilon$$

whenever $P = \{(\theta_i, x_i)\}_i$ is a δ_n-fine partition.

Set δ_0 =min $\{\delta_1, \delta_2, \ldots, \delta_{\bar{n}-1}\}$, then the condition $(*)$ is true for $1 \leqslant n \leqslant (\bar{n}-1)$, whenever P is a δ_0-fine partition. Choose an integer $n_0 \geqslant \bar{n}$ s.t.

$$\left|(\mathrm{PU})\int_X f_n - (\mathrm{PU})\int_X f_m\right| < \frac{\varepsilon}{3}$$

for all $m,\ n \geqslant n_0$. Set $\bar{\delta}_1 = \min\{\bar{\delta}, \delta_{n_0}\}$; for each $n \geqslant n_0$, we have

$$\begin{aligned}
&\left|(\mathrm{PU})\int_X f_n - \sum_i f_n(x_i)\int_X \theta_i \mathrm{d}\mu\right| \\
&\qquad \leqslant \left|\sum_i f_{n_0}(x_i)\int_X \theta_i d\mu - \sum_i f_n(x_i)\int_X \theta_i \mathrm{d}\mu\right| \\
&\qquad\quad + \left|(\mathrm{PU})\int_X f_{n_0} - \sum_i f_{n_0}(x_i)\int_X \theta_i \mathrm{d}\mu\right| \\
&\qquad\quad + \left|(\mathrm{PU})\int_X f_n - (\mathrm{PU})\int_X f_{n_0}\right| < \varepsilon
\end{aligned}$$

whenever $P = \{(\theta_i, x_i)\}_i$ is a $\bar{\delta}_1$-fine partition.

Hence, set $\delta = \min\{\bar{\delta}_1, \delta_0\}$, the relation $(*)$ is true for each n, whenever P is a δ-fine partition. $\square$

Let f be a μ-measurable function on X; if $\{\overline{f}_n\}_n$ is the sequence defined so that

$$\overline{f}_n(x) = \begin{cases} f(x) & \text{if } |f(x)| \leqslant n, \\ 0 & \text{if } |f(x)| > n, \end{cases}$$

then the following propositions hold:

Proposition 3. *The sequence $\{\overline{f}_n\}_n$ is uniformly* PU*-integrable iff f has small Riemann tails.*

Proof. We observe that the functions $\overline{f}_n$ are μ-integrable and by [4] they are PU-integrable. So, if $\{\overline{f}_n\}_n$ is uniformly PU-integrable, by Proposition 1, f is PU-integrable and

$$(\mathrm{PU})\int_X f = \lim_n (\mathrm{PU})\int_X \overline{f}_n.$$

Fixed $\varepsilon > 0$, there exists a positive function δ on X s.t.

$$\left|(\mathrm{PU})\int_X f - \sum_i f(x_i)\int_X \theta_i \mathrm{d}\mu\right| < \frac{\varepsilon}{3}$$

and

$$\left|(\mathrm{PU})\int_X \overline{f}_n - \sum_i \overline{f}_n(x_i)\int_X \theta_i \mathrm{d}\mu\right| < \frac{\varepsilon}{3}$$

for each n, whenever $P = \{(\theta_i, x_i)\}_i$ is a δ-fine PU-partition in X.

Choose $\bar{n}$ s.t.

$$\left|(\mathrm{PU})\int_X \overline{f}_n - (\mathrm{PU})\int_X f\right| < \frac{\varepsilon}{3}$$

for each $n \geqslant \bar{n}$, and let $P_1 = \{(\theta_i', x_i')\}_i$ be a δ-fine PU-partition in X; for $n \geqslant \bar{n}$ consider

$$\begin{aligned}
&\left|\sum_i f\chi_{A_n}(x_i')\int_X \theta_i' \mathrm{d}\mu\right| \\
&\quad = \left|\sum_i f(x_i')\int_X \theta_i' \mathrm{d}\mu - \sum_i \overline{f}_n(x_i')\int_X \theta_i' \mathrm{d}\mu\right| \\
&\quad \leqslant \left|(\mathrm{PU})\int_X f - \sum_i f(x_i')\int_X \theta_i' \mathrm{d}\mu\right| + \left|(\mathrm{PU})\int_X \overline{f}_n - \sum_i \overline{f}_n(x_i')\int_X \theta_i' \mathrm{d}\mu\right| \\
&\qquad + \left|(\mathrm{PU})\int_X \overline{f}_n - (\mathrm{PU})\int_X f\right| < \varepsilon,
\end{aligned}$$

thus f has small Riemann tails.

Now, suppose that f has sRt, then the sequence $\{(\mathrm{PU})\int_X \overline{f}_n\}$ is a Cauchy sequence. In fact, fixed $\varepsilon > 0$, there exists a positive integer $\overline{n}$ s.t. for $m, n \geqslant \overline{n}$ there is a positive function δ on X with the property that if $P = \{(\theta_i, x_i)\}_i$ is a δ-fine PU-partition in X, we have

$$\begin{aligned}
&\left|(\mathrm{PU})\int_X \overline{f}_n - (\mathrm{PU})\int_X \overline{f}_m\right| \\
&\qquad \leqslant \left|(\mathrm{PU})\int_X \overline{f}_n - \sum_i \overline{f}_n(x_i)\int_X \theta_i \mathrm{d}\mu\right| + \left|(\mathrm{PU})\int_X \overline{f}_m - \sum_i \overline{f}_m(x_i)\int_X \theta_i \mathrm{d}\mu\right| \\
&\qquad\quad + \left|\sum_i f(x_i)\int_X \theta_i \mathrm{d}\mu - \sum_i \overline{f}_n(x_i)\int_X \theta_i \mathrm{d}\mu\right| \\
&\qquad\quad + \left|\sum_i f(x_i)\int_X \theta_i \mathrm{d}\mu - \sum_i \overline{f}_m(x_i)\int_X \theta_i \mathrm{d}\mu\right| \\
&\qquad = \left|(\mathrm{PU})\int_X \overline{f}_n - \sum_i \overline{f}_n(x_i)\int_X \theta_i \mathrm{d}\mu\right| + \left|(\mathrm{PU})\int_X \overline{f}_m - \sum_i \overline{f}_m(x_i)\int_X \theta_i \mathrm{d}\mu\right| \\
&\qquad\quad + \left|\sum_i f\chi_{A_n}(x_i)\int_X \theta_i \mathrm{d}\mu\right| + \left|\sum_i f\chi_{A_m}(x_i)\int_X \theta_i \mathrm{d}\mu\right| < \frac{\varepsilon}{4}
\end{aligned}$$

for all $m, n \geqslant \overline{n}$.

Let $\varepsilon > 0$, there exist n_0 and a positive function δ_1 on X s.t.

$$\left|\sum_i f\chi_{A_n}(x_i)\int_X \theta_i \mathrm{d}\mu\right| < \frac{\varepsilon}{4}$$

for each $n \geqslant n_0$, whenever P is a δ_1-fine PU-partition in X.

Choose $n_1 > \max\{\overline{n}, n_0\}$ s.t.

$$\left|(\mathrm{PU})\int_X \overline{f}_n - (\mathrm{PU})\int_X \overline{f}_m\right| < \frac{\varepsilon}{4}$$

for each $m, n \geqslant n_1$, and choose $\delta \leqslant \delta_1$ s.t.

$$\left|(\mathrm{PU})\int_X \overline{f}_n - \sum_i \overline{f}_n(x_i)\int_X \theta_i \mathrm{d}\mu\right| < \frac{\varepsilon}{4}$$

for $1 \leqslant n \leqslant n_1$, whenever $P = \{(\theta_i, x_i)\}_i$ is a δ-fine PU-partition.

Moreover, for each δ-fine PU-partition $P=\{(\theta_i,x_i)\}_i$ and for $n>n_1$ we have

$$
\begin{aligned}
&\left|(\mathrm{PU})\int_X \overline{f}_n-\sum_i \overline{f}_n(x_i)\int_X\theta_i\mathrm{d}\mu\right| \\
&\quad\leqslant\left|\sum_i f(x_i)\int_X\theta_i\mathrm{d}\mu-\sum_i\overline{f}_n(x_i)\int_X\theta_i\mathrm{d}\mu\right| \\
&\qquad+\left|\sum_i f(x_i)\int_X\theta_i\mathrm{d}\mu-\sum_i\overline{f}_{n_1}(x_i)\int_X\theta_i\mathrm{d}\mu\right| \\
&\qquad+\left|(\mathrm{PU})\int_X\overline{f}_{n_1}-\sum_i\overline{f}_{n_1}(x_i)\int_X\theta_i\mathrm{d}\mu\right|+\left|(\mathrm{PU})\int_X\overline{f}_n-(\mathrm{PU})\int_X\overline{f}_{n_1}\right| \\
&\quad=\left|\sum_i f\chi_{A_n}(x_i)\int_X\theta_i\mathrm{d}\mu\right|+\left|\sum_i f\chi_{A_{n_1}}(x_i)\int_X\theta_i\right| \\
&\qquad+\left|(\mathrm{PU})\int_X\overline{f}_{n_1}-\sum_i\overline{f}_{n_1}(x_i)\int_X\theta_i\mathrm{d}\mu\right|+\left|(\mathrm{PU})\int_X\overline{f}_n-(\mathrm{PU})\int_X\overline{f}_{n_1}\right| \\
&\quad<\frac{\varepsilon}{4}+\frac{\varepsilon}{4}+\frac{\varepsilon}{4}+\frac{\varepsilon}{4}=\varepsilon,
\end{aligned}
$$

which proves the uniform PU-convergence of the sequence $\{\overline{f}_n\}_n$. □

Proposition 4. *f has really small Riemann tails iff the sequence $\{\overline{f}_n\}_n$ is uniformly μ-integrable.*

Proof. Set $A_n=\{x\in X\colon |f(x)|>n\}$, we observe that $\overline{|f|_n}=|\overline{f}_n|$ and if the sequence $\{\overline{f}_n\}_n$ is uniformly μ-integrable then so is the sequence $\{\overline{|f|_n}\}_n$.

By the generalized Vitali theorem, it follows that

$$\lim_n\int_X|\overline{f}_n|\mathrm{d}\mu=\int_X|f|\mathrm{d}\mu$$

and

$$\lim_n\int_X|f|\chi_{A_n}\mathrm{d}\mu=\lim_n\int_X(|f|-|\overline{f}_n|)\mathrm{d}\mu=0.$$

Thus, for each $\varepsilon>0$ there exists a positive integer $\bar{n}$ s.t. for each $n\geqslant\bar{n}$ we have

$$\int_X|f|\chi_{A_n}\mathrm{d}\mu<\frac{\varepsilon}{2}$$

and there exists a positive function δ on X s.t.

$$\left|\sum_i|f|\chi_{A_{\bar{n}}}(x_i)\int_X\theta_i\mathrm{d}\mu-\int_X|f|\chi_{A_{\bar{n}}}\mathrm{d}\mu\right|<\frac{\varepsilon}{2}$$

whenever $P = \{(\theta_i, x_i)\}_i$ is a δ-fine PU-partition in X.

We have

$$\begin{aligned}
&\sum_i |f|\chi_{A_{\bar n}}(x_i) \int_X \theta_i \mathrm{d}\mu \\
&\qquad \leqslant \left| \sum_i |f|\chi_{A_{\bar n}} \int_X \theta_i \mathrm{d}\mu - \int_X |f|\chi_{A_{\bar n}} \mathrm{d}\mu \right| + \int_X |f|\chi_{A_{\bar n}} \mathrm{d}\mu \\
&\qquad < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}$$

whenever P is a δ-fine partition.

Suppose that $P_1 = \{(\theta'_i, x'_i)\}_i$ is an $A_{\bar n}$ δ-fine family [see Definition 8], then it can be extended to a δ-fine partition $P = \{(\theta_i, x_i)\}_i$ in X and we have

$$\begin{aligned}
\left| \sum_i f(x'_i) \int_X \theta'_i \mathrm{d}\mu \right| &\leqslant \sum_i |f(x'_i)| \int_X \theta'_i \mathrm{d}\mu \\
&\leqslant \sum_i |f|\chi_{A_{\bar n}}(x_i) \int_X \theta_i \mathrm{d}\mu < \varepsilon.
\end{aligned}$$

Hence f has rsRt.

Now, suppose that f has rsRt, then f has sRt and by the previous Proposition 3 the sequence $\{\overline{f}_n\}_n$ is uniformly PU-integrable; so f is PU-integrable and by the results of [4] f is μ-integrable and so the sequence $\{\overline{f}_n\}_n$ is uniformly μ-integrable. $\square$

N o t e 2. By the results of the two previous propositions, we observe that for the sequence $\{\overline{f}_n\}_n$ the uniform PU-integrability is equivalent to the uniform μ-integrability, but in the general case, they are not equivalent [see Note 1].

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