133 (2008) MATHEMATICA BOHEMICA No. 4, 435–445

# GENERAL CONDITIONS GUARANTEEING THE SOLVABILITY OF THE CAUCHY PROBLEM FOR FUNCTIONAL DIFFERENTIAL EQUATIONS

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(Received November 8, 2007)

*Abstract.* New general unique solvability conditions of the Cauchy problem for systems of general linear functional differential equations are established. The class of equations considered covers, in particular, linear equations with transformed argument, integro-differential equations, neutral type equations and their systems of an arbitrary order.

*Keywords*: functional differential equation, Cauchy problem, initial value problem, differential inequality

*MSC 2000*: 34K06, 34K10

## 1. Introduction and problem formulation

The purpose of this paper, which has been motivated by the recent work [3], is to establish new general conditions sufficient for the unique solvability of the Cauchy problem for systems of linear functional differential equations. It is rather interesting to point out that, as we show below, fairly general results on the solvability of the initial value problem can be obtained by using an abstract approach based upon order-theoretical considerations. In this way we extend and strengthen several results that have been established in the papers [8], [3] directly by the techniques of calculus.

The proof of the main results obtained in this paper is based on the application of [7, Theorem 49.4], which ensures the unique solvability of an abstract equation with an operator satisfying Lipschitz-type conditions with respect to a suitable cone.

The research was supported in part by AS CR, Institutional Research Plan No. AV0Z10190503, GA CR, Grant No. 201/06/0254 (A. Rontó) and DFFD, Grant No. 0107U003322, and National Scholarship Programme of the Slovak Republic (N. Dilna).

Here we deal with the initial value problem for systems of linear functional differential equations of the general form [2], [1]. More precisely, we consider the system of functional differential equations

$$u_k'(t) = (l_k u)(t) + q_k(t), \qquad t \in [a,b],\ k = 1,2,\dots,n, \tag{1}$$

subjected to the initial condition

$$u_k(a) = c_k, \qquad k = 1,2,\dots,n, \tag{2}$$

where $-\infty < a < b < +\infty$, $n \in \mathbb{N}$, $l_k\colon D([a,b],\mathbb{R}^n) \to L_1([a,b],\mathbb{R})$, $k = 1,2,\dots,n$, are linear operators, $\{q_k\,;\ k = 1,2,\dots,n\} \subset L_1([a,b],\mathbb{R})$ are given functions, and $\{c_k\,;\ k = 1,2,\dots,n\} \subset \mathbb{R}$. Here $D([a,b],\mathbb{R}^n)$ and $L_1([a,b],\mathbb{R})$ are, respectively, the Banach spaces of absolutely continuous and Lebesgue integrable vector functions on the interval $[a,b]$ (see Sec. 2 for the notation). It should be noted that, in contrast to the case considered in [3], [8], setting (1) covers, in particular, neutral type systems because the right-hand side member there may contain various terms with derivatives.

The solution of the initial value problem (1), (2) is understood in the sense of the following standard definition (see, e.g., [2], [1]).

**Definition 1.** We say that a vector function $u = (u_k)_{k=1}^n\colon [a,b] \to \mathbb{R}^n$ is a solution of the problem (1), (2) if it satisfies system (1) almost everywhere on the interval $[a,b]$ and possesses property (2) at the point $a$.

We shall use in the sequel a natural notion of positivity of a linear operator.

**Definition 2.** A linear operator $l = (l_k)_{k=1}^n\colon D([a,b],\mathbb{R}^n) \to L_1([a,b],\mathbb{R}^n)$ is said to be positive if

$$\operatorname*{vrai\,min}_{t\in[a,b]} (l_k u)(t) \geqslant 0, \qquad k = 1,2,\dots,n,$$

for any $u = (u_k)_{k=1}^n$ from $D^+([a,b],\mathbb{R}^n)$.

The following definition is motivated by a notion used, in particular, in [3], [8].

**Definition 3.** A linear operator $l = (l_k)_{k=1}^n\colon D([a,b],\mathbb{R}^n) \to L_1([a,b],\mathbb{R}^n)$ is said to belong to the set $\mathcal{S}_a([a,b],\mathbb{R}^n)$ if the semi-homogeneous problem (1), (3),

$$u_k(a) = 0, \qquad k = 1,2,\dots,n, \tag{3}$$

has a unique solution $u = (u_k)_{k=1}^n$ for any $\{q_k\,;\ k = 1,2,\dots,n\} \subset L_1([a,b],\mathbb{R})$ and, moreover, the solution of (1), (3) possesses the property

$$\min_{t\in[a,b]} u_k(t) \geqslant 0, \qquad k = 1,2,\dots,n, \tag{4}$$

whenever the functions $q_k$, $k = 1,2,\dots,n$, appearing in (1) are non-negative almost everywhere on $[a,b]$.

R e m a r k 1. It follows from Lemma 1 (1), Sec. 4.2, that $\mathcal{S}_a([a,b],\mathbb{R}^1)$ strictly contains the set $\mathcal{S}_{ab}(a)$ described by [3, Definition 2.1].

In the cases where the operator $l$ appearing in (1) satisfies the inclusion

$$l \in \mathcal{S}_a([a,b],\mathbb{R}^n), \tag{5}$$

one sometimes says (see, e.g., [4]) that the theorem on integration of a differential inequality is true for problem (1), (3). Our aim here is to establish the global solvability conditions for problem (1), (2) assuming that certain operators associated with the problem possess the property indicated.

## 2. Notation

The following notation is used throughout the paper.

1. $\mathbb{R} := (-\infty,\infty)$, $\mathbb{N} := \{1,2,3,\ldots\}$.
2. $\|x\| := \max\limits_{1\leqslant k\leqslant n} |x_k|$ for $x = (x_k)_{k=1}^n \in \mathbb{R}^n$.
3. $D([a,b],\mathbb{R}^n)$ is the Banach space of absolutely continuous functions $[a,b] \to \mathbb{R}^n$ equipped with the norm
$$D([a,b],\mathbb{R}^n) \ni u \longmapsto \|u(a)\| + \int_a^b \|u'(s)\|\,\mathrm{d}s.$$
4. The set $D^+([a,b],\mathbb{R}^n)$ is defined by the formula
$$D^+([a,b],\mathbb{R}^n) := \Big\{u = (u_k)_{k=1}^n \in D([a,b],\mathbb{R}^n);\ \min_{\xi\in[a,b]} u_k(\xi) \geqslant 0 \\ \text{for all } k = 1,2,\ldots,n\Big\}. \tag{6}$$
5. The set $D^{++}([a,b],\mathbb{R}^n)$ is introduced by the formula
$$D^{++}([a,b],\mathbb{R}^n) := \Big\{u = (u_k)_{k=1}^n \in D([a,b],\mathbb{R}^n);\ \min_{\xi\in[a,b]} u_k(\xi) \geqslant 0 \\ \text{and } \operatorname*{vrai\,min}_{\xi\in[a,b]} u_k'(\xi) \geqslant 0 \text{ for all } k = 1,2,\ldots,n\Big\}. \tag{7}$$
6. $D_0([a,b],\mathbb{R}^n)$ (or $D_0^+([a,b],\mathbb{R}^n)$, $D_0^{++}([a,b],\mathbb{R}^n)$, respectively) is the set of all $u = (u_k)_{k=1}^n$ from $D([a,b],\mathbb{R}^n)$ (or $D^+([a,b],\mathbb{R}^n)$, $D^{++}([a,b],\mathbb{R}^n)$, respectively) for which $u_k(a) = 0$, $k = 1,2,\ldots,n$.
7. $L_1([a,b],\mathbb{R}^n)$ is the Banach space of all Lebesgue integrable vector-functions $u\colon [a,b] \to \mathbb{R}^n$ with the standard norm
$$L_1([a,b],\mathbb{R}^n) \ni u \longmapsto \int_a^b \|u(s)\|\,\mathrm{d}s.$$

## 3. Conditions for unique sovability

The following theorem, which strengthens [8, Theorem 3.3], gives general criteria for the unique solvability of the initial value problem (1), (2) for arbitrary perturbation terms.

**Theorem 1.** *Let there exist positive linear operators* $p_i = (p_{ik})_{k=1}^n \colon D([a,b], \mathbb{R}^n) \to L_1([a,b], \mathbb{R}^n)$, $i = 0, 1$, *satisfying the inclusions*

$$p_1 \in \mathcal{S}_a([a,b], \mathbb{R}^n), \ \ p_0 + p_1 \in \mathcal{S}_a([a,b], \mathbb{R}^n), \tag{8}$$

*and such that the inequalities*

$$|(l_k u)(t) - (p_{1k} u)(t)| \leqslant (p_{0k} u)(t), \qquad t \in [a,b], \ \ k = 1, 2, \ldots, n, \tag{9}$$

*hold for an arbitrary non-negative absolutely continuous vector function* $u \colon [a,b] \to \mathbb{R}^n$ *with property* (3).

*Then the initial value problem* (1), (2) *has a unique solution for arbitrary* $\{q_k ;\ k = 1, 2, \ldots, n\} \subset L_1([a,b], \mathbb{R})$ *and* $\{c_k ;\ k = 1, 2, \ldots, n\} \subset \mathbb{R}$. *In particular, the corresponding homogeneous problem* (3) *for the system*

$$u_k'(t) = (l_k u)(t), \qquad t \in [a,b], \ \ k = 1, 2, \ldots, n, \tag{10}$$

*has only the trivial solution.*

The method used in the proof of Theorem 1 allows one to obtain a more general version of [8, Corollary 3.1]. Theorem 2 given below also generalises [3, Theorem 2.2].

**Theorem 2.** *Let there exist positive linear operators* $l_i \colon D([a,b], \mathbb{R}^n) \to L_1([a,b], \mathbb{R}^n)$, $i = 0, 1$, *satisfying the inclusions*

$$l_0 \in \mathcal{S}_a([a,b], \mathbb{R}^n), \ \ -\tfrac{1}{2} l_1 \in \mathcal{S}_a([a,b], \mathbb{R}^n), \tag{11}$$

*and such that the inequalities*

$$|(l_k u)(t) + (l_{1k} u)(t)| \leqslant (l_{0k} u)(t), \qquad t \in [a,b], \ \ k = 1, 2, \ldots, n, \tag{12}$$

*hold for an arbitrary non-negative absolutely continuous function* $u \colon [a,b] \to \mathbb{R}^n$ *with property* (3).

*Then the initial value problem* (1), (2) *has a unique solution for arbitrary* $\{q_k ;\ k = 1, 2, \ldots, n\} \subset L_1([a,b], \mathbb{R})$ *and* $\{c_k ;\ k = 1, 2, \ldots, n\} \subset \mathbb{R}$.

The role played by the mappings $l_i\colon\ D([a,b],\mathbb{R}^n) \to L_1([a,b],\mathbb{R}^n)$, $i=0,1$, in condition (12) is very close to that of the positive and negative parts of a mapping on a Banach lattice. The next result deals with the case where $l$ admits decomposition in the form

$$l = l_0 - l_1, \tag{13}$$

where $l_0$ and $l_1$ could be referred to as the positive and negative parts of $l$.

**Theorem 3.** *Let us assume that the operator $l$ admits representation* (13) *where $l_i\colon\ D([a,b],\mathbb{R}^n) \to L_1([a,b],\mathbb{R}^n)$, $i=0,1$, are certain positive linear operators such that the inclusions*

$$l_0 \in \mathcal{S}_a([a,b],\mathbb{R}^n),\ \tfrac{1}{2}(l_0 - l_1) \in \mathcal{S}_a([a,b],\mathbb{R}^n), \tag{14}$$

*are satisfied.*

*Then the initial value problem* (1), (2) *has a unique solution for arbitrary $\{q_k;\ k=1,2,\dots,n\} \subset L_1([a,b],\mathbb{R})$ and $\{c_k;\ k=1,2,\dots,n\} \subset \mathbb{R}$.*

The proofs of the results presented above as well as some subsidiary statements are given in the subsequent sections.

Remark 2. Conditions (14) and (11) appearing in the theorems presented are unimprovable in a certain sense. More precisely, condition (11) cannot be replaced by its weaker versions

$$(1-\varepsilon)\, l_0 \in \mathcal{S}_a([a,b],\mathbb{R}^n),\ -\tfrac{1}{2} l_1 \in \mathcal{S}_a([a,b],\mathbb{R}^n),$$

and

$$l_0 \in \mathcal{S}_a([a,b],\mathbb{R}^n),\ -(2+\varepsilon)^{-1} l_1 \in \mathcal{S}_a([a,b],\mathbb{R}^n),$$

no matter how small the constant $\varepsilon \in (0,1)$ may be. In a similar way, condition (14) can be weakened to neither of the two conditions

$$\begin{gathered}(1-\varepsilon)\, l_0 \in \mathcal{S}_a([a,b],\mathbb{R}^n),\ \tfrac{1}{2}(l_0 - l_1) \in \mathcal{S}_a([a,b],\mathbb{R}^n),\\ l_0 \in \mathcal{S}_a([a,b],\mathbb{R}^n),\ (2+\varepsilon)^{-1}(l_0 - l_1) \in \mathcal{S}_a([a,b],\mathbb{R}^n),\end{gathered}$$

for any small positive $\varepsilon$. To show this, one can use, e.g., examples from [3], [8].

Remark 3. The theorems formulated above can be extended to the case where the solution of the given functional differential equation is an abstract function taking values in a Banach space ordered by a suitable closed cone.

## 4. Proofs

**4.1. An abstract theorem.**

We need the following statement on the unique solvability of an equation with the Lipschitz type non-linearity established in [6] (see also [7]). Let us consider the abstract operator equation

$$Fx = z, \tag{15}$$

where $F\colon E_1 \to E_2$ is a mapping, $\langle E_1, \|\cdot\|_{E_1}\rangle$ is a normed space, $\langle E_2, \|\cdot\|_{E_2}\rangle$ is a Banach space over the field $\mathbb{R}$, $K_i \subset E_i$, $i = 1, 2$, are closed cones, and $z$ is an arbitrary element from $E_2$.

The cones $K_i$, $i = 1, 2$, induce the natural partial orderings of the respective spaces. Thus, for each $i = 1, 2$, we write $x \leqslant_{K_i} y$ and $y \geqslant_{K_i} x$ if and only if $\{x, y\} \subset E_i$ and $y - x \in K_i$.

**Theorem 4** ([7, Theorem 49.4])**.** *Let the cone $K_2$ be normal and reproducing. Furthermore, let $B_k\colon E_1 \to E_2$, $k = 1, 2$, be additive and homogeneous operators such that $B_1^{-1}$ and $(B_1 + B_2)^{-1}$ exist and possess the properties*

$$B_1^{-1}(K_1) \subset K_2, \qquad (B_1 + B_2)^{-1}(K_1) \subset K_2, \tag{16}$$

*and, furthermore, let the order relation*

$$B_1(x - y) \leqslant_{K_2} Fx - Fy \leqslant_{K_2} B_2(x - y) \tag{17}$$

*be satisfied for any pair $(x, y) \in E_1^2$ such that $x \geqslant_{K_1} y$.*

*Then equation* (15) *has a unique solution $u \in E_1$ for an arbitrary element $z \in E_2$.*

Recall that the property of normality of the cone $K_2 \subset E_2$ is equivalent to the relation

$$\inf\{\gamma \in (0, +\infty);\ \|x\|_{E_2} \leqslant \gamma \|y\|_{E_2}\ \forall \{x, y\} \subset E_2\colon\ x \leqslant_{E_2} y\} < +\infty,$$

and the cone $K_1$ is reproducing in $E_1$ if and only if the equality

$$\{u - v;\ \{u, v\} \subset K_1\} = E_1$$

holds (see, e.g., [7], [5]).

**4.2. Lemmata.**

For the sake of convenience, we first establish several lemmata.

**Lemma 1.**

1. *If the semi-homogeneous problem* (1), (3) *is uniquely solvable for any* $\{q_k;\ k = 1, 2, \ldots, n\} \subset L_1([a,b], \mathbb{R})$, *then the same is true for problem* (1), (2) *for arbitrary* $\{q_k;\ k = 1, 2, \ldots, n\} \subset L_1([a,b], \mathbb{R})$ *and* $\{c_k;\ k = 1, 2, \ldots, n\} \subset \mathbb{R}$.
2. *If* $l$ *satisfies condition* (5), *then the semi-homogeneous problem* (1), (3) *has a unique solution* $u = (u_k)_{k=1}^n$ *for any* $\{q_k;\ k = 1, 2, \ldots, n\} \subset L_1([a,b], \mathbb{R})$. *Moreover, if all the* $q_k$, $k = 1, 2, \ldots, n$, *are non-negative almost everywhere on* $[a,b]$, *then the solution of* (1), (3) *has property* (4).

Proof. Consider the inhomogeneous problem (1), (2) and perform there the change of variable according to the formula

$$v := u - c. \tag{18}$$

If $u = (u_k)_{k=1}^n$ is a solution of (1), (2), then the function $v = (v_k)_{k=1}^n$ satisfies the relations

$$v_k'(t) = (l_k v)(t) + q_k(t) + (l_k c)(t), \qquad t \in [a,b],\ k = 1, 2, \ldots, n, \tag{19}$$

$$v_k(a) = 0, \qquad k = 1, 2, \ldots, n, \tag{20}$$

and vice versa. The assumption of the lemma guarantees the existence and uniqueness of a function $v = (v_k)_{k=1}^n \colon [a,b] \to \mathbb{R}^n$ satisfying the semi-homogeneous problem (19), (20) for all $\{q_k;\ k = 1, 2, \ldots, n\} \subset L_1([a,b], \mathbb{R})$ and $\{c_k;\ k = 1, 2, \ldots, n\} \subset \mathbb{R}$. The unique solution $u = (u_k)_{k=1}^n$ of problem (1), (2) is then determined from relation (18).

The second assertion of the lemma follows directly from the Definition 3 of the set $\mathcal{S}_a([a,b], \mathbb{R}^n)$ when one puts $c = 0$ in (19). □

The next lemma establishes the relation between the property described by Definition 3 and the positive invertibility of a certain operator.

**Lemma 2.** *If* $l = (l_k)_{k=1}^n \colon D([a,b], \mathbb{R}^n) \to L_1([a,b], \mathbb{R}^n)$ *is a positive linear operator such that inclusion* (5) *holds, then the operator* $V_l \colon D_0([a,b], \mathbb{R}^n) \to D_0([a,b], \mathbb{R}^n)$ *given by the formula*

$$D_0([a,b], \mathbb{R}^n) \ni u \longmapsto V_l u := u - \int_a^{\cdot} (lu)(t)\,\mathrm{d}t \tag{21}$$

*is continuously invertible and, moreover, its inverse* $V_l^{-1}$ *satisfies the inclusion*

$$V_l^{-1}(D_0^{++}([a,b], \mathbb{R}^n)) \subset D_0^{+}([a,b], \mathbb{R}^n). \tag{22}$$

P r o o f. Indeed, let the mapping $l$ satisfy condition (5). Given an arbitrary function $y = (y_k)_{k=1}^n$ from $D_0([a,b],\mathbb{R}^n)$, consider the equation

$$V_l u = y. \tag{23}$$

In view of notation 6, Sec. 2, we have

$$y_k(a) = 0, \qquad k = 1,2,\ldots,n. \tag{24}$$

By virtue of assumption (5), there exists a unique absolutely continuous $u = (u_k)_{k=1}^n$ such that

$$u_k'(t) = (l_k u)(t) + y_k'(t), \qquad t \in [a,b],\ k = 1,2,\ldots,n, \tag{25}$$

$$u_k(a) = 0, \qquad k = 1,2,\ldots,n. \tag{26}$$

Moreover, if the functions $y_k$, $k = 1,2,\ldots,n$, are non-negative and non-decreasing, then, by Lemma 1, the components of $u$ possess property (4). Integrating both parts of (25) and taking (24) and (26) into account, we find that $u = (u_k)_{k=1}^n$ is the unique solution of equation (23). □

Finally, we formulate our last simple

**Lemma 3.** *For arbitrary linear operators $p_i\colon D([a,b],\mathbb{R}^n) \to L_1([a,b],\mathbb{R}^n)$, $i = 1,2$, the identity*

$$V_{p_1} + V_{p_2} = 2V_{\frac{1}{2}(p_1+p_2)} \tag{27}$$

*is true.*

P r o o f. Equality (27) is obtained from formula (21) by direct computation. □

### 4.3. Proof of Theorem 1.

Consider the semi-homogeneous initial value problem (1), (3). Clearly, an absolutely continuous vector function $u = (u_k)_{k=1}^n\colon [a,b] \to \mathbb{R}^n$ is a solution of (1), (3) if and only if it satisfies the equation

$$u(t) = \int_a^t (lu)(s)\,\mathrm{d}s + \int_a^t q(s)\,\mathrm{d}s, \qquad t \in [a,b]. \tag{28}$$

Let us put $E_1 = E_2 = D_0([a,b],\mathbb{R}^n)$ and define the mapping $F\colon E_1 \to E_2$ by setting

$$(Fu)(t) := u(t) - \int_a^t (lu)(s)\,\mathrm{d}s, \qquad t \in [a,b], \tag{29}$$

for any $u$ from $D_0([a,b],\mathbb{R}^n)$. Then equation (28) takes on the form (15) with

$$z(t) := \int_a^t q(s)\,\mathrm{d}s, \qquad t \in [a,b]. \tag{30}$$

Recall that the sets $D_0([a,b],\mathbb{R}^n)$, $D_0^+([a,b],\mathbb{R}^n)$, and $D_0^{++}([a,b],\mathbb{R}^n)$ are defined by notation 6, Sec. 2. It is easy to see that $D_0([a,b],\mathbb{R}^n)$ is a closed subspace of $D([a,b],\mathbb{R}^n)$.

Assumption (9) means that the estimate

$$-(p_{0k}u)(t) + (p_{1k}u)(t) \leqslant (l_k u)(t) \leqslant (p_{0k}u)(t) + (p_{1k}u)(t), \quad t \in [a,b], \tag{31}$$

is true for any $u$ from $D_0^+([a,b],\mathbb{R}^n)$ and all $k = 1,2,\dots,n$. Therefore, for all such $u$, the relation

$$\begin{aligned} u_k'(t) - (p_{0k}u)(t) - (p_{1k}u)(t) &\leqslant u_k'(t) - (l_k u)(t) \\ &\leqslant u_k'(t) + (p_{0k}u)(t) - (p_{1k}u)(t), \quad k = 1,2,\dots,n, \end{aligned} \tag{32}$$

holds for almost every $t \in [a,b]$. Integrating (32) and taking property (3) of $u = (u_k)_{k=1}^n$ into account, we obtain that the inequality

$$\begin{aligned} u_k(t) - \int_a^t [(p_{0k}u)(\xi) + (p_{1k}u)(\xi)]\,\mathrm{d}\xi &\leqslant u_k(t) - \int_a^t (l_k u)(\xi)\,\mathrm{d}\xi \\ &\leqslant u_k(t) + \int_a^t [(p_{0k}u)(\xi) - (p_{1k}u)(\xi)]\,\mathrm{d}\xi, \quad t \in [a,b],\ k = 1,2,\dots,n, \end{aligned} \tag{33}$$

is true for all $u = (u_k)_{k=1}^n$ from the set $D_0^+([a,b],\mathbb{R}^n)$.

Let us define linear mappings $B_{ik}\colon E_1 \to E_2$, $i = 1,2$, $k = 1,2,\dots,n$, by putting

$$(B_{1k}u)(t) := u_k(t) - \int_a^t [(p_{0k}u)(\xi) + (p_{1k}u)(\xi)]\,\mathrm{d}\xi, \qquad t \in [a,b], \tag{34}$$

and

$$(B_{2k}u)(t) := u_k(t) + \int_a^t [(p_{0k}u)(\xi) - (p_{1k}u)(\xi)]\,\mathrm{d}\xi, \qquad t \in [a,b], \tag{35}$$

for an arbitrary $u$ from $D_0([a,b],\mathbb{R}^n)$ and let us construct the corresponding mappings $B_i\colon D_0([a,b],\mathbb{R}^n) \to D_0([a,b],\mathbb{R}^n)$, $i = 1,2$, according to the formula

$$D_0([a,b],\mathbb{R}^n) \ni u \longmapsto B_i u := \begin{pmatrix} B_{i1}u \\ B_{i2}u \\ \vdots \\ B_{in}u \end{pmatrix}, \qquad i = 1,2. \tag{36}$$

Then estimates (32) and (33), formula (21), and the definition of the sets $D_0^+([a,b], \mathbb{R}^n)$ and $D_0^{++}([a,b],\mathbb{R}^n)$ imply that

$$\{B_2 u - V_l u, V_l u - B_1 u\} \subset D_0^{++}([a,b],\mathbb{R}^n) \text{ for an arbitrary } u \text{ from } D_0^+([a,b],\mathbb{R}^n) \text{ possessing property (3).} \tag{37}$$

The last property means that mapping (29) satisfies condition (17) with

$$K_1 = D_0^+([a,b],\mathbb{R}^n),\ K_2 = D_0^{++}([a,b],\mathbb{R}^n). \tag{38}$$

It is not difficult to verify that sets (38) are cones in the Banach space $D_0([a,b], \mathbb{R}^n)$. Moreover, $D_0^+([a,b],\mathbb{R}^n)$ is reproducing and $D_0^{++}([a,b],\mathbb{R}^n)$ is normal.

It follows from Lemma 3 that the identity

$$V_{p_1-p_0} + V_{p_1+p_0} = 2V_{p_1}$$

is true. However, according to (34) and (35), we have $B_i = V_{p_1-(-1)^i p_0}$, $i=1,2$. Therefore, by virtue of assumption (8) and Lemma 2, we conclude that the inverse operators $B_1^{-1}$ and $(B_1+B_2)^{-1}$ exist and possess properties (16) with respect to cones (38). Applying Theorem 4, we establish the unique solvability of the semi-homogeneous problem (1), (3) for arbitrary $q_k$, $k=1,2,\dots,n$, from $L_1([a,b],\mathbb{R})$. Finally, to obtain the assertion required, it remains to refer to Lemma 1.

**4.4. Proof of Theorem 2.**

One can verify that under conditions (11) and (12), the operators $p_i\colon D([a,b], \mathbb{R}^n) \to L_1([a,b],\mathbb{R}^n)$, $i=0,1$, defined by the formulæ

$$p_0 := l_0 + \tfrac{1}{2} l_1,\ p_1 := -\tfrac{1}{2} l_1, \tag{39}$$

satisfy conditions (8) and (9) of Theorem 1. Indeed, it follows from assumption (12) and the positivity of the operator $l_1$ that for any $u$ from $D_0^+([a,b],\mathbb{R}^n)$, the relations

$$\begin{aligned}
|(l_k u)(t) + \tfrac{1}{2}(l_{1k}u)(t)| &= |(l_k u)(t) + (l_{1k}u)(t) - \tfrac{1}{2}(l_{1k}u)(t)| \\
&\leqslant (l_{0k}u)(t) + \tfrac{1}{2}|(l_{1k}u)(t)| \\
&= (l_{0k}u)(t) + \tfrac{1}{2}(l_{1k}u)(t), \qquad t\in[a,b],\ k=1,2,\dots,n,
\end{aligned}$$

are true. This means that $l$ admits estimate (9) with the operators $p_0$ and $p_1$ defined by formulae (39). Therefore, it remains only to note that assumption (11) ensures the validity of inclusion (8) for operators (39). Applying Theorem 1, we arrive at the required assertion.

**4.5. Proof of Theorem 3.** It is easy to see that under the conditions assumed, the operators $p_i\colon\ D([a,b],\mathbb{R}^n) \to L_1([a,b],\mathbb{R}^n)$, $i=0,1$, defined by the formulæ

$$p_0 := \tfrac{1}{2}(l_0 + l_1),\ p_1 := \tfrac{1}{2}(l_0 - l_1),$$

satisfy conditions (8) and (9) of Theorem 1.

Remark 4. Theorems 2 and 3 could also be proved directly by an argument similar to the proof of Theorem 1.

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