SOME FULL CHARACTERIZATIONS OF THE STRONG McSHANE INTEGRAL

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(Received November 17, 2003)

Abstract. Some full characterizations of the strong McShane integral are obtained.

Keywords: strong McShane integral, strong absolute continuity, McShane variational measure

MSC 2000: 26B30, 26A36, 26A39

1. Introduction

It is now a classical result that if a function F is absolutely continuous on a compact interval $[a,b] \subset \mathbb{R}$, then F is differentiable almost everywhere on $[a,b]$ and F is the indefinite Lebesgue integral of F'. Since the McShane integral represents a Riemann-type definition of the Lebesgue integral [5], [12], it is not surprising that a similar result holds also for the McShane integral. Although the McShane integral is conceptually simpler than the Lebesgue integral, it is rather surprising to note that the McShane integrability of the derivative F' of the above function F is obtained via convergence theorems for the Lebesgue integral. See, for example, [10, Theorem 5.3.15]. In view of this observation, it seems that none of the previous authors has used the full strength of Riemann sums to obtain the McShane integrability of F'. In this paper, we will use Riemann sums to obtain some complete characterizations of the strong McShane integral, which is equivalent to the Bochner integral [11]. In particular, we extend the following results: [3, Proposition 1], [3, Corollary 1], [10, Theorem 5.3.15], [10, Theorem 9.3.7] and [9, Proposition 4, Remark 6] valid for the McShane integral to the strong McShane integral. See Theorem 3.5 for more details. Moreover, we deduce a Radon-Nikodým Theorem for the McShane variational measure (Theorem 3.9). Our extensions are possible because our techniques, unlike those employed by previous authors, use neither convergence theorems nor the extended real-valuedness property of integrable functions.

2. Preliminaries

Unless stated otherwise, the following conventions and notation will be used. The set of all real numbers is denoted by $\mathbb{R}$, and the ambient space of this paper is $\mathbb{R}^m$, where m is a fixed positive integer. The norm in $\mathbb{R}^m$ is the maximum norm $\|\cdot\|$, where $\|(x_1, x_2, \ldots, x_m)\| = \max\limits_{i=1,\ldots,m} |x_i|$. For $x \in \mathbb{R}^m$ and $r > 0$, set $B(x, r) := \{y \in \mathbb{R}^m : \|y - x\| < r\}$. Let $E := \prod\limits_{i=1}^{m} [a_i, b_i]$ be a fixed interval in $\mathbb{R}^m$. For a set $A \subseteq E$, we denote by χ_A, $\operatorname{diam}(A)$ and $\mu_m^*(A)$ the characteristic function, diameter and m-dimensional Lebesgue outer measure of A, respectively. Moreover, we denote the interior and closure of $A \subseteq E$ with respect to the subspace topology of E by $\operatorname{int}(A)$ and $\overline{A}$, respectively. The distance between $Y \subseteq E$ and $Z \subseteq E$ will be denoted by $\operatorname{dist}(Y, Z)$. Given two subsets X, Y of E, the symmetric difference of Y and Z is denoted by $Y \Delta Z$. A set $A \subset E$ is called *negligible* whenever $\mu_m^*(A) = 0$. We say that two sets are nonoverlapping if their intersection is negligible. Let X be a Banach space equipped with a norm $\|\cdot\|$. A function is always X-valued. When no confusion is possible, we do not distinguish between a function defined on a set Z and its restriction to a set $W \subset Z$.

An *interval* in $\mathbb{R}^m$ is the cartesian product of m one dimensional compact intervals. $\mathcal{I}$ denote the family of all nondegenerate subintervals of E. If $I \in \mathcal{I}$, we write $\mu_m^*(I)$ as $|I|$, the volume of I. For each $J \in \mathcal{I}$, the *regularity* of an m-dimensional interval $J \subseteq E$, denoted by $\operatorname{reg}(J)$, is the ratio of its shortest and longest sides. A function F defined on $\mathcal{I}$ is said to be additive if $F(I \cup J) = F(I) + F(J)$ for each pair nonoverlapping intervals $I, J \in \mathcal{I}$ with $I \cup J \in \mathcal{I}$. In particular, if we follow the proof of [7, Corollary 6.2.4], then we can verify that if F is an additive interval function on $\mathcal{I}$ with $J \in \mathcal{I}$ and $\{K_1, K_2, \ldots, K_r\}$ is a finite collection of non-overlapping nondegenerate subintervals of J with $\bigcup\limits_{i=1}^{r} K_i = J$, then $F(J) = \sum\limits_{i=1}^{r} F(K_i)$.

A *partition* P is a finite collection $\{(I_i, \xi_i)\}_{i=1}^{p}$, where $I_1, I_2, \ldots, I_p$ are pairwise nonoverlapping nondegenerate subintervals of E. Given $Z \subseteq E$, a positive function δ on Z is called a *gauge* on Z. We say that a partition $\{(I_i, \xi_i)\}_{i=1}^{p}$ is

(i) a partition *in* Z if $\bigcup\limits_{i=1}^{p} I_i \subseteq Z$,

(ii) a partition *of* Z if $\bigcup\limits_{i=1}^{p} I_i = Z$,

(iii) *anchored* in Z if $\{\xi_1, \xi_2, \ldots, \xi_p\} \subset Z$,

(iv) *δ-fine* if $I_i \subset B(\xi_i, \delta(\xi_i))$ for each $i = 1, 2, \ldots, p$,

(v) *Perron* if $\xi_i \in I_i$ for each $i = 1, 2, \ldots, p$,

(vi) *McShane* if ξ_i need not belong to I_i for all $i = 1, 2, \ldots, p$,

(vii) α-regular for some $0 < \alpha \leqslant 1$ if $\operatorname{reg}(I_i) \geqslant \alpha$ for each $i = 1, 2, \ldots, p$.

According to Cousin's Lemma [7, Lemma 6.2.6], for any given gauge δ on E, δ-fine Perron partitions of E exist. Hence the following definition is meaningful.

Definition 2.1. A function $f\colon E \longrightarrow X$ is said to be strongly McShane integrable on E if there exists an additive interval function $F\colon \mathcal{I} \longrightarrow X$ with the following property: for each $\varepsilon > 0$ there exists a gauge δ on E such that

$$\sum_{i=1}^{p} \|f(\xi_i)\,|I_i| - F(I_i)\| < \varepsilon$$

for each δ-fine McShane partition $\{(I_i, \xi_i)\}_{i=1}^{p}$ in E. The function F is called the indefinite strong McShane integral of f on E.

Remark 2.2. When $X = \mathbb{R}$, the reader can verify that Definition 2.1 is equivalent to the classical definition of the McShane integral. See, for example, [5], [7], [8] [10], [11], [12].

3. Main results

Let F be an interval function on $\mathcal{I}$. If δ is a gauge defined on $W \subseteq E$, we set

$$V_L(F, W, \delta) := \sup \sum \|F(I_i)\|,$$

where the supremum is taken over all δ-fine McShane partitions anchored in W, and

$$V_{\mathcal{L}}F(W) := \inf_{\delta} V_L(F, W, \delta),$$

where the infimum is taken over all gauges on W. It is not difficult to see that $V_{\mathcal{L}}F(\cdot)$ is a metric outer measure. See [3] or [13].

Theorem 3.1. *If F is the indefinite strong McShane integral of a strongly McShane integrable function f on E, then $V_{\mathcal{L}}F$ is absolutely continuous with respect to the Lebesgue measure μ_m.*

Proof. It is similar to that given in [3, Proposition 2]. $\square$

In order to proceed further, we need the following definitions.

Definition 3.2. Given an additive interval function F on $\mathcal{I}$ and $0 < \alpha < 1$, we say that F is derivable at $x \in E$ if there exists an element of X, denoted by $F'(x)$, independent of α such that for $\varepsilon > 0$ there exists $\eta = \eta(x, \alpha) > 0$ such that

$$\left\| F'(x) - \frac{F(I)}{|I|} \right\| < \varepsilon$$

whenever $I \in \mathcal{I}$ with $\operatorname{reg}(I) \geqslant \alpha$ and $x \in I \subset B(x, \eta)$.

Definition 3.3 [8]**.** An additive interval function F on $\mathcal{I}$ is said to be strongly absolutely continuous if given $\varepsilon > 0$, there exists $\eta > 0$ such that

$$\sum_{i=1}^{q} \|F(I_i)\| < \varepsilon$$

whenever $\{I_1, I_2, \ldots, I_q\}$ is a collection of nonoverlapping subintervals of E with $\sum\limits_{i=1}^{q} |I_i| < \eta$.

Definition 3.4 [9]**.** A gauge δ on E is said to be nearly upper semicontinuous if there exists a negligible set $Z \subset E$ such that $\delta\,|_{E \setminus Z}$ is upper semicontinuous on E.

In view of Remark 2.2, the next theorem generalizes the results [3, Proposition 1], [3, Corollary 1], [10, Theorem 5.3.15], [10, Theorem 9.3.7] and [9, Proposition 4, Remark 6] valid for the McShane integral to the strong McShane integral.

Theorem 3.5 (Main Result)**.** *Suppose that an additive interval function F is derivable at almost all $x \in E$. Then the following conditions are equivalent:*

(i) *$V_{\mathcal{L}}F$ is absolutely continuous with respect to the Lebesgue measure μ_m.*

(ii) *F is strongly absolutely continuous on $\mathcal{I}$.*

(iii) *There exists a function $f\colon E \longrightarrow X$ satisfying the following condition: given $\varepsilon > 0$, there exists a nearly upper semicontinuous gauge δ on E such that*

$$\sum_{i=1}^{p} \|f(\xi_i)\,|I_i| - F(I_i)\| < \varepsilon$$

for each δ-fine McShane partition $\{(I_i, \xi_i)\}_{i=1}^{p}$ in E.

Proof. (i) $\Longleftrightarrow$ (ii). The proof is similar to that of [3, Proposition 1].

(ii) $\Longrightarrow$ (iii). In this case, there exists a negligible relative G_δ-set $Z \subset E$ such that F is derivable at each $x \in E \setminus Z$. Set $f(x) = F'(x)$ if $x \in E \setminus Z$, and $f(x) = 0$ if $x \in Z$.

Let $\varepsilon > 0$ and $\varepsilon_0 := \frac{\varepsilon}{3+5|E|}$. Then there exists a gauge δ_0 on $E \setminus Z$ such that the inequality

$$\|f(x)\,|I| - F(I)\| < \varepsilon_0\,|I|$$

holds whenever $x \in I \cap (E \setminus Z)$, $I \in \mathcal{I}$ and $I \subset B(x, \delta_0(x))$.

Since Z is a relative G_δ subset of E, we may fix an increasing sequence $\{Y_k\}_{k=1}^{\infty}$ of closed sets such that $\bigcup\limits_{k=1}^{\infty} Y_k = E \setminus Z$. For each positive integer k we set $W_k := \{x \in Y_k \colon \|f(x)\| \leqslant k\varepsilon_0$ and $\delta_0(x) \geqslant \frac{1}{k}\}$ and $X_k = \overline{W}_k$. As $\{X_k\}$ is an increasing sequence of closed sets whose union is $E \setminus Z$, we may assume that each X_k is nonempty.

By our choice of $\delta_0(\cdot)$ we have $\|f(x) - f(y)\| < 2\varepsilon_0$ whenever $x, y \in E \setminus Z$ with $\|x - y\| < \min\{\delta_0(x), \delta_0(y)\}$. Since Y_k is closed with $X_k \subseteq Y_k$, we see that $\|f\|\chi_{X_k}$ is bounded. Hence we may choose an open set G_k containing X_k such that $\mu_m(G_k \setminus X_k) < \min\{\frac{\varepsilon_0}{(2^k)\|f\chi_{X_k}\|_\infty}, \eta_k\}$, where $\eta_k > 0$ corresponds to $\frac{\varepsilon_0}{2^k}$ in the definition of the strong absolute continuity of F.

Since Z is negligible, we select an open set U containing Z such that $\mu_m(U) < \eta_1$. Put $X_0 := \emptyset$ and define a gauge δ on E as follows:

$$\delta(x) = \begin{cases} \min\{\frac{1}{k}, \operatorname{dist}(x, X_{k-1} \cup (E \setminus G_k))\} \text{ if } x \in X_k \setminus X_{k-1} \text{ for some } k \in \{1, 2, \ldots\}, \\ \operatorname{dist}(x, E \setminus U) \text{ if } x \in Z. \end{cases}$$

Then δ is nearly upper semicontinuous on E. We shall prove that F is the indefinite McShane integral of f using this gauge δ. Select any δ-fine McShane partition $P = \{(I_i, x_i)\}_{i=1}^{p}$ anchored in E. Since $F(\cdot)$ and $\mu_m(\cdot)$ are both additive interval functions, an application of [10, Proposition 7.3.3] shows that P may be assumed to be $\frac{1}{2}$-regular. Then we have

$$\begin{aligned} (1) \qquad \sum_{i=1}^{p} \|f(x_i)\,|I_i| - F(I_i)\| &\leqslant \sum \{\|f(x_i)\,|I_i| - F(I_i)\| \colon x_i \in Z\} \\ &+ \sum_{k=1}^{\infty} \sum \{\|f(x_i)\,|I_i| - F(I_i)\| \colon x_i \in X_k \text{ and } I \cap X_k = \emptyset\} \\ &+ \sum_{k=1}^{\infty} \sum \{\|f(x_i)\,|I_i| - F(I_i)\| \colon x_i \in X_k \text{ and } I \cap X_k \neq \emptyset\}. \end{aligned}$$

By our choice of U, we see that the first term is less than ε_0. Likewise, the second term is less than $\sum\limits_{k=1}^{\infty} \{(\sup\limits_{x \in X_k} \|f(x)\|)\mu_m(G_k \setminus X_k) + \frac{\varepsilon_0}{2^k}\}$. It remains to prove that the last term is less than $5\varepsilon_0\,|E|$.

Let $\{(I, x)\}$ be a $\frac{1}{k}$-fine $\frac{1}{2}$-regular partition anchored in $\{x\} \subseteq X_k$ such that $I \cap X_k \neq \emptyset$, from which we may select and fix $y \in I \cap X_k$. As both x, y belong to

X_k, there exist two sequences $\{x_n\}, \{y_n\}$ in W_k such that $\|x_n - x\| + \|y_n - y\| \to 0$ as $n \to \infty$. For each positive integer n, we let I_n be the smallest interval containing y_n and I. Then $\operatorname{diam}(I_n) \to \operatorname{diam}(I)$ and $\operatorname{reg}(I_n) \to \operatorname{reg}(I)$ as $n \to \infty$. Thus

$$\left\|f(x) - \frac{F(I)}{|I|}\right\| \leqslant 2\varepsilon_0 + \limsup_{n\to\infty} \|f(x_n) - f(y_n)\| + \limsup_{n\to\infty} \left\|f(y_n) - \frac{F(I_n)}{|I_n|}\right\| \leqslant 5\varepsilon_0,$$

showing that the last term in (1) is less than $5\varepsilon_0\,|E|$. In view of the above inequalities, the sum $\sum\limits_{i=1}^{p} \|f(x_i)\,|I_i| - F(I_i)\|$ is less than $\varepsilon_0 + 2\sum\limits_{k=1}^{\infty} \frac{\varepsilon_0}{2^k} + 5\varepsilon_0\,|E| = \varepsilon$, proving that F is the indefinite strong McShane integral of f over E.

(iii) $\Longrightarrow$ (i). This follows from Theorem 3.1. The proof is complete. $\square$

R e m a r k 3.6. Since the strong McShane integral is equivalent to the Bochner integral [11], it is not difficult to see that Theorem 3.5 extends [2, Theorem 6].

Lemma 3.7. *Let F be an additive interval function on $\mathcal{I}$. If $V_{\mathcal{L}}F$ is absolutely continuous with respect to the Lebesgue measure μ_m, then it is finite.*

P r o o f. By modifying the proof of [3, Proposition 1] we see that $V_{\mathcal{L}}F(E)$ is finite. Since $V_{\mathcal{L}}F(\cdot)$ is an outer measure, we see that $V_{\mathcal{L}}F(S)$ is finite for each subset S of E. The proof is complete. $\square$

Lemma 3.8. *If f is strongly McShane integrable on E and F is the indefinite strong McShane integral of f over E, then $\|f\|$ is McShane integrable on each $I \in \mathcal{I}$ with*

$$\int_E \|f\|\chi_I = V_{\mathcal{L}}F(I).$$

P r o o f. Given $\varepsilon > 0$, choose a gauge δ_1 on E that corresponds to $\frac{\varepsilon}{3}$ in the definition of the strong McShane integral of f on E.

In view of Lemma 3.7 there exists a gauge δ_2 on I such that

$$\left|\sum_{i=1}^{q} \|F(J_i')\| - V_{\mathcal{L}}F(I)\right| < \frac{\varepsilon}{3}$$

for some δ_2-fine McShane partition $\{(J_i', x_i)\}_{i=1}^{q}$ of I.

By following the proofs of [11, Lemma 8] and [5, Theorem 10.8], there exists a gauge δ_3 on I such that

$$\left|\sum_{i=1}^{s} \|f(t_i)\|\,|K_i| - \sum_{i=1}^{r} \|f(y_i)\|\,|J_i|\right| < \frac{\varepsilon}{3}$$

whenever $\{(K_i, t_i)\}_{i=1}^s$ and $\{(J_i, y_i)\}_{i=1}^r$ are δ_3-fine McShane partitions of I. Clearly, we have

$$\left| \sum_{i=1}^{p} \|f(\xi_i)\| \, |I_i| - V_{\mathcal{L}}(I) \right| < \varepsilon$$

for each $\min\{\delta_1, \delta_2, \delta_3\}$-fine McShane partition $\{(I_i, \xi_i)\}_{i=1}^p$ of I. This proves the lemma. $\square$

Using the above lemmas and Theorem 3.5, we can now state and prove the Radon-Nikodým Theorem for the McShane variational measure.

Theorem 3.9. *Suppose that F is an additive interval function on $\mathcal{I}$ such that F is derivable at almost all $x \in E$. If $V_{\mathcal{L}}F$ is absolutely continuous with respect to the Lebesgue measure μ_m with*

$$f(x) = \begin{cases} F'(x) \text{ if } F'(x) \text{ exists,} \\ 0 \text{ otherwise,} \end{cases}$$

then $\|f\|$ is McShane integrable on E, and the equality

$$\int_E \|f\| \chi_Y = V_{\mathcal{L}}F(Y)$$

holds for each measurable set $Y \subset E$.

Proof. By virtue of Lemma 3.8, the proof is similar to that of [4, Theorem 2E]. $\square$

We conclude this paper by giving another application of Theorem 3.5 to obtain two results for the strong McShane integral, extending the corresponding results [9, Proposition 4, Remark 6], [9, Corollary 5, Remark 6] for the McShane integral. Our proofs, unlike the ones employed in [9], do not depend on the fact that the McShane integrable function is real-valued.

Theorem 3.10. *If f is strongly McShane integrable on E, then the gauge function in the definition of the strong McShane integral of f can be chosen to be nearly upper semicontinuous on E.*

Proof. This follows from Theorems 3.1 and 3.5. $\square$

References

[1] *A. M. Bruckner, J. B. Bruckner, B. S. Thomson*: Real Analysis. Prentice-Hall, 1997.
[2] *J. A. Clarkson*: Uniformly convex spaces. Trans. Amer. Math. Soc. *40* (1936), 396–414.
[3] *L. Di Piazza*: Variational measures in the theory of the integration in $\mathbb{R}^m$. Czechoslovak Math. J. *51* (2001), 95–110.
[4] *D. H. Fremlin, J. Mendoza*: On the integration of vector-valued functions. Illinois J. Math. *38* (1994), 127–141.
[5] *R. A. Gordon*: The Integrals of Lebesgue, Denjoy, Perron, and Henstock. Amer. Math. Soc., Providence, 1994.
[6] *J. Jarník, J. Kurzweil*: Perron-type integration on n-dimensional intervals and its properties. Czechoslovak Math. J. *45* (1995), 79–106.
[7] *Lee Peng Yee, R. Výborný*: The Integral, An Easy Approach after Kurzweil and Henstock. Australian Mathematical Society Lecture Ser. 14, Cambridge University Press, 2000.
[8] *Lee Tuo-Yeong*: Every absolutely Henstock-Kurzweil integrable function is McShane integrable: an alternative proof. To appear in Rocky Mountain J. Math.
[9] *W. F. Pfeffer*: A note on the generalized Riemann integral. Proc. Amer. Math. Soc. *103* (1988), 1161–1166.
[10] *W. F. Pfeffer*: The Riemann Approach to Integration. Cambridge Univ. Press, Cambridge, 1993.
[11] *Š. Schwabik, Ye Guoju*: On the strong McShane integral of functions with values in a Banach space. Czechoslovak Math. J. *51* (2001), 819–828.
[12] *C. Swartz*: Introduction to the Gauge Integrals. World Scientific, 2001.
[13] *B. S. Thomson*: Derivates of Interval Functions. Mem. Amer. Math. Soc. 452, 1991.

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