NEW CHARACTERIZATION OF MORREY SPACE

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Abstract. In this paper we prove that

$$\|f\|_{\mathcal{M}_{p,\lambda}} \approx \|f\|_{\widetilde{\mathcal{M}}_{p,\lambda}},$$

where

$$\|f\|_{\widetilde{\mathcal{M}}_{p,\lambda}} := \sup \left\{ \int_{\mathbb{R}^n} |fg| : \inf_{x\in\mathbb{R}^n} \int_0^\infty r^{\frac{n-\lambda}{p}-1} \|g\|_{L_{p'}(\complement B(x,r))} dr \le 1 \right\}.$$

1. Introduction

The well-known Morrey spaces $\mathcal{M}_{p,\lambda}$ introduced by C. Morrey in 1938 [8] in relation to the study of partial differential equations, were widely investigated during last decades, including the study of classical operators of harmonic analysis - maximal, singular and potential operators - in generalizations of these spaces ("so-called" Morrey-type spaces). These spaces appeared to be quite useful in the study of the local behavior of the solutions to partial differential equations, apriori estimates and other topics in the theory of PDE.

In [2] local Morrey-type spaces $LM_{p\theta,\omega}$ and global Morrey-type spaces $GM_{p\theta,\omega}$ were defined and some properties of these spaces were studied. Authors investigated the boundedness of the Hardy-Littlewood maximal operator in these spaces. After this paper was intensive study of boundedness of other classical operators such as fractional maximal operator, Riesz potential and Calderón-Zygmund singular integral operator (see, for instance [4], for references).

Later in [3] "so-called" complementary local Morrey-type spaces ${}^{\complement}LM_{p\theta,\omega}$ were introduced and the boundedness of fractional maximal operator from complementary local Morrey-type space ${}^{\complement}LM_{p\theta,\omega}$ into local Morrey-type space $LM_{p\theta,\omega}$ was investigated. As in the definition of the space ${}^{\complement}LM_{p\theta,\omega}$ was used complement of ball instead of ball, it was named complementary local Morrey-type space and no relation between $LM_{p\theta,\omega}$ and ${}^{\complement}LM_{p\theta,\omega}$ was studied.

In [6] associated spaces and dual spaces of local Morrey-type spaces and complementary local Morrey-type spaces were characterized. More precisely, it was shown that associated spaces of local Morrey-type spaces are complementary local Morrey-type spaces. Moreover, it was proved that for some values of parameters these associated spaces are dual of local Morrey-type spaces. Namely, it was proved that the space ${}^{\complement}LM_{p'\theta',\widetilde{\omega}}$ is dual space of the

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space $LM_{p\theta,\omega}$, where $1 \le p,\ \theta < \infty$, p' and θ' are conjugate exponents of p and θ, respectively, and $\widetilde{\omega}(t) = \omega^{\theta-1}(t) \left(\int_t^\infty \omega^\theta(s)ds\right)^{-1}$.

In the present paper we show that similar results holds true for $LM_{p\theta,\omega}^{\{x\}}$ and ${}^{\complement}LM_{p\theta,w}^{\{x\}}$ spaces for any $x \in \mathbb{R}^n$ (see, Section 2, for definitions). By means of these results we prove Hölder's inequality for the classical Morrey spaces.

The paper is organized as follows. We start with notations and give some preliminaries in Section 2. In Section 3 we recall some results on associate spaces of local Morrey-type spaces and complementary local Morrey-type spaces. New characterization of the Morrey space was given in Section 4.

2. Notations and Preliminaries

Now we make some conventions. Throughout the paper, we always denote by c and C a positive constant which is independent of main parameters, but it may vary from line to line. By $A \lesssim B$ we mean that $A \le cB$ with some positive constant c independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent. Constant, with subscript such as c_1, does not change in different occurrences. For a measurable set E, χ_E denotes the characteristic function of E.

Unless a special remarks is made, the differential element dx is omitted when the integrals under consideration are the Lebesgue integrals.

For a fixed p with $p \in [1,\infty)$, p' denotes the conjugate exponent of p, namely,

$$p' := \begin{cases} \frac{p}{1-p} & \text{if} \quad 0 < p < 1, \\ +\infty & \text{if} \quad p = 1, \\ \frac{p}{p-1} & \text{if} \quad 1 < p < +\infty, \\ 1 & \text{if} \quad p = +\infty, \end{cases}$$

and $1/(+\infty) = 0$, $0/0 = 0$, $0 \cdot (\pm\infty) = 0$.

If E is a nonempty measurable subset on $\mathbb{R}^n$ and f is a measurable function on E, then we put

$$\|g\|_{L_p(E)} : = \left(\int_E |f(y)|^p dy\right)^{\frac{1}{p}},\ 0 < p < +\infty,$$
$$\|f\|_{L_\infty(E)} : = \sup\{\alpha : |\{y \in E : |f(y)| \ge \alpha\}| > 0\}.$$

If I is a nonempty measurable subset on $(0,+\infty)$ and g is a measurable function on I, then we define $\|g\|_{L_p(I)}$ and $\|g\|_{L_\infty(I)}$, correspondingly.

For $x \in \mathbb{R}^n$ and $r > 0$, let $B(x,r)$ be the open ball centered at x of radius r and ${}^{\complement}B(x,r) := \mathbb{R}^n \backslash B(x,r)$.

Morrey spaces $\mathcal{M}_{p,\lambda}$ were introduced by C. Morrey in 1938 [8] and defined as follows: *for* $0 \le \lambda \le n$, $1 \le p \le \infty$, $f \in \mathcal{M}_{p,\lambda}$ *if* $f \in L_p^{\text{loc}}(\mathbb{R}^n)$ *and*

$$\|f\|_{\mathcal{M}_{p,\lambda}} \equiv \|f\|_{\mathcal{M}_{p,\lambda}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n,\ r>0} r^{\frac{\lambda-n}{p}} \|f\|_{L_p(B(x,r))} < \infty,$$

where $B(x,r)$ is the open ball centered at x of radius r.

Note that $\mathcal{M}_{p,0} = L_\infty(\mathbb{R}^n)$ and $\mathcal{M}_{p,n} = L_p(\mathbb{R}^n)$. If $\lambda < 0$ or $\lambda > n$, then $\mathcal{M}_{p,\lambda} = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

In [1] D.R.Adams introduced a variant of Morrey-type spaces as follows: For $0 \le \lambda \le n$, $1 \le p, \theta \le \infty$, $f \in \mathcal{M}_{p\theta,\lambda}$ if $f \in L_p^{\rm loc}(\mathbb{R}^n)$ and

$$\|f\|_{\mathcal{M}_{p\theta,\lambda}} \equiv \|f\|_{\mathcal{M}_{p\theta,\lambda}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n} \|r^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,r))}\|_{L_\theta(0,\infty)} < \infty.$$

(If $\theta = \infty$, then $\mathcal{M}_{p\theta,\lambda} = \mathcal{M}_{p,\lambda}$.)

Let us recall definitions of local Morrey-type space and complementary local Morrey-type space.

Definition 2.1. ([2]) Let $0 < p, \theta \le \infty$ and let w be a non-negative measurable function on $(0,\infty)$. We denote by $LM_{p\theta,\omega}$ the local Morrey-type space, the space of all functions $f \in L_p^{\rm loc}(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{LM_{p\theta,\omega}} \equiv \|f\|_{LM_{p\theta,\omega}(\mathbb{R}^n)} = \left\|w(r)\|f\|_{L_p(B(0,r))}\right\|_{L_\theta(0,\infty)}.$$

Definition 2.2. ([3]) Let $0 < p, \theta \le \infty$ and let w be a non-negative measurable function on $(0,\infty)$. We denote by ${}^{\complement}LM_{p\theta,\omega}$ the complementary local Morrey-type space, the space of all functions $f \in L_p({}^{\complement}B(0,t))$ for all $t > 0$ with finite quasinorm

$$\|f\|_{{}^{\complement}LM_{p\theta,\omega}} \equiv \|f\|_{{}^{\complement}LM_{p\theta,\omega}(\mathbb{R}^n)} = \left\|w(r)\|f\|_{L_p({}^{\complement}B(0,r))}\right\|_{L_\theta(0,\infty)}.$$

Definition 2.3. Let $0 < p, \theta \le \infty$. We denote by Ω_θ the set all non-negative measurable functions ω on $(0,\infty)$ such that

$$0 < \|\omega\|_{L_\theta(t,\infty)} < \infty, \; t > 0,$$

and by ${}^{\complement}\Omega_\theta$ the set all non-negative measurable functions ω on $(0,\infty)$ such that

$$0 < \|\omega\|_{L_\theta(0,t)} < \infty, \; t > 0.$$

It is convenient to define local Morrey-type spaces and complementary local Morrey-type spaces at any fixed point $x \in \mathbb{R}^n$.

Definition 2.4. Let $0 < p, \theta \le \infty$ and let w be a non-negative measurable function on $(0,\infty)$. For any fixed $x \in \mathbb{R}^n$ we denote by $LM_{p\theta,\omega}^{\{x\}}$, the local Morrey-type space: the space of all functions $f \in L_p^{\rm loc}(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{LM_{p\theta,\omega}^{\{x\}}} \equiv \|f\|_{LM_{p\theta,\omega}^{\{x\}}(\mathbb{R}^n)} := \left\|w(r)\|f\|_{L_p(B(x,r))}\right\|_{L_\theta(0,\infty)} = \|f(x+\cdot)\|_{LM_{p\theta,\omega}}.$$

Definition 2.5. Let $0 < p, \theta \le \infty$ and let w be a non-negative measurable function on $(0,\infty)$. For any fixed $x \in \mathbb{R}^n$ we denote by ${}^{\complement}LM_{p\theta,w}^{\{x\}}$ the complementary local Morrey-type space, the space of all functions $f \in L_p({}^{\complement}B(x,t))$ for all $t > 0$ with finite quasinorm

$$\|f\|_{{}^{\complement}LM_{p\theta,w}^{\{x\}}} \equiv \|f\|_{{}^{\complement}LM_{p\theta,w}^{\{x\}}(\mathbb{R}^n)} := \left\|w(r)\|f\|_{L_p({}^{\complement}B(x,r))}\right\|_{L_\theta(0,\infty)} = \|f(x+\cdot)\|_{{}^{\complement}LM_{p\theta,\omega}}.$$

Note by $LM_{p\theta,\omega} = LM_{p\theta,\omega}^{\{0\}}$ and ${}^{\complement}LM_{p\theta,\omega} = {}^{\complement}LM_{p\theta,w}^{\{0\}}$.

3. Associate and dual spaces of local Morrey-type and complementary local Morrey-type spaces

Let $(\mathcal{R},\mu)$ be a totally σ-finite non-atomic measure space. Let $\mathfrak{M}(\mathcal{R},\mu)$ be the set of all μ-measurable a.e. finite real functions on $\mathcal{R}$.

Definition 3.1. Let X be a set of functions from $\mathfrak{M}(\mathcal{R},\mu)$, endowed with a positively homogeneous functional $\|\cdot\|_X$, defined for every $f\in\mathfrak{M}(\mathcal{R},\mu)$ and such that $f\in X$ is and only if $\|f\|_X<\infty$. We define the associate space X' of X as the set of all functions $f\in\mathfrak{M}(\mathcal{R},\mu)$ such that $\|f\|_{X'}<\infty$, where

$$\|f\|_{X'}=\sup\left\{\int_{\mathcal{R}}|fg|d\mu:\|g\|_X\le 1\right\}.$$

In what follows we assume $\mathcal{R}=\mathbb{R}^n$ and $d\mu=dx$.

In [6] the associate spaces of local Morrey-type and complementary local Morrey-type spaces were calculated. Our method of characterization of the Morrey space mainly based on these results. For the sake of completeness we recall some statements from [6].

Theorem 3.2. ([6], Theorem 4.5) *Assume* $1\le p<\infty$, $0<\theta\le\infty$. *Let* $\omega\in{}^{\complement}\Omega_\theta$. *Set* $X={}^{\complement}LM_{p\theta,\omega}$.

(i) Let $0<\theta\le 1$. Then

$$\|f\|_{X'}\approx\sup_{t\in(0,\infty)}\|f\|_{L_{p'}(B(0,t))}\|\omega\|^{-1}_{L_\theta(0,t)},$$

with the positive constant in equivalency independent of f.

(ii) Let $1<\theta\le\infty$. Then

$$\|f\|_{X'}\approx\left(\int_{(0,\infty)}\|f\|^{\theta'}_{L_{p'}(B(0,t))}d\left(-\|\omega\|^{-\theta'}_{L_\theta(0,t+)}\right)\right)^{\frac{1}{\theta'}}+\frac{\|f\|_{L_{p'}(\mathbb{R}^n)}}{\|\omega\|_{L_\theta(0,\infty)}},$$

with the positive constant in equivalency independent of f.

Theorem 3.3. ([6], Theorem 4.6) *Assume* $1\le p<\infty$, $0<\theta\le\infty$. *Let* $\omega\in\Omega_\theta$. *Set* $X=LM_{p\theta,\omega}$.

(i) Let $0<\theta\le 1$. Then

$$\|f\|_{X'}\approx\sup_{t\in(0,\infty)}\|f\|_{L_{p'}({}^{\complement}B(0,t))}\|\omega\|^{-1}_{L_\theta(t,\infty)},$$

with the positive constant in equivalency independent of f.

(ii) Let $1<\theta\le\infty$. Then

$$\|f\|_{X'}\approx\left(\int_{(0,\infty)}\|f\|^{\theta'}_{L_{p'}({}^{\complement}B(0,t))}d\|\omega\|^{-\theta'}_{L_\theta(t-,\infty)}\right)^{\frac{1}{\theta'}}+\frac{\|f\|_{L_{p'}(\mathbb{R}^n)}}{\|\omega\|_{L_\theta(0,\infty)}},$$

with the positive constant in equivalency independent of f.

In fact more general results, which are important for our applications, are true.

Theorem 3.4. *Assume* $1\le p<\infty$, $0<\theta\le\infty$. *Let* $\omega\in{}^{\complement}\Omega_\theta$. *For any fixed* $x\in\mathbb{R}^n$ *set* $X={}^{\complement}LM^{\{x\}}_{p\theta,\omega}$.

(i) Let $0 < \theta \le 1$. Then

$$\|f\|_{X'} \approx \sup_{t\in(0,\infty)} \|f\|_{L_{p'}(B(x,t))} \|\omega\|^{-1}_{L_\theta(0,t)},$$

with the positive constant in equivalency independent of f and x.

(ii) Let $1 < \theta \le \infty$. Then

$$\|f\|_{X'} \approx \left(\int_{(0,\infty)} \|f\|^{\theta'}_{L_{p'}(B(x,t))} d\left(-\|\omega\|^{-\theta'}_{L_\theta(0,t+)}\right)\right)^{\frac{1}{\theta'}} + \frac{\|f\|_{L_{p'}(\mathbb{R}^n)}}{\|\omega\|_{L_\theta(0,\infty)}},$$

with the positive constant in equivalency independent of f and x.

Proof. Let x be any fixed point in $\mathbb{R}^n$. Then

$$\begin{aligned}\|f\|_{X'} = \|f\|_{\left({}^{\complement}LM^{\{x\}}_{p\theta,\omega}\right)'} &= \sup\left\{\int_{\mathbb{R}^n} |f(y)g(y)|dy : \|g\|_{{}^{\complement}LM^{\{x\}}_{p\theta,\omega}} \le 1\right\}\\ &= \sup\left\{\int_{\mathbb{R}^n} |f(x+y)g(x+y)|dy : \|g(x+\cdot)\|_{{}^{\complement}LM_{p\theta,\omega}} \le 1\right\}\\ &= \sup\left\{\int_{\mathbb{R}^n} |f(x+y)g(y)|dy : \|g\|_{{}^{\complement}LM_{p\theta,\omega}} \le 1\right\}\\ &= \|f(x+\cdot)\|_{\left({}^{\complement}LM_{p\theta,\omega}\right)'}.\end{aligned}$$

It remains to apply Theorem 3.2. □

Theorem 3.5. *Assume* $1 \le p < \infty$, $0 < \theta \le \infty$. *Let* $\omega \in \Omega_\theta$. *For any fixed* $x \in \mathbb{R}^n$ *set* $X = LM^{\{x\}}_{p\theta,\omega}$.

(i) Let $0 < \theta \le 1$. Then

$$\|f\|_{X'} \approx \sup_{t\in(0,\infty)} \|f\|_{L_{p'}({}^{\complement}B(x,t))} \|\omega\|^{-1}_{L_\theta(t,\infty)},$$

with the positive constant in equivalency independent of f and x.

(ii) Let $1 < \theta \le \infty$. Then

$$\|f\|_{X'} \approx \left(\int_{(0,\infty)} \|f\|^{\theta'}_{L_{p'}({}^{\complement}B(x,t))} d\|\omega\|^{-\theta'}_{L_\theta(t-,\infty)}\right)^{\frac{1}{\theta'}} + \frac{\|f\|_{L_{p'}(\mathbb{R}^n)}}{\|\omega\|_{L_\theta(0,\infty)}},$$

with the positive constant in equivalency independent of f and x.

The proof of Theorem 3.5 is similar to that of Theorem 3.4 (we only need to apply Theorem 3.3 instead of Theorem 3.2) and we omit it.

It was shown in [6] that for some values of parameters the dual spaces coincide with the asssociated spaces. Namely, the following theorems were proved.

Theorem 3.6. ([6], Theorem 5.1) *Assume* $1 \le p < \infty$, $1 \le \theta < \infty$. *Let* $\omega \in \Omega_\theta$ *and* $\|\omega\|_{L_\theta(0,\infty)} = \infty$. *Then*

$$(LM_{p\theta,\omega})^* = {}^{\complement}LM_{p'\theta',\widetilde{\omega}}\,, \tag{3.1}$$

where $\widetilde{\omega}(t) = \omega^{\theta-1}(t)\left(\int_t^\infty \omega^\theta(s)ds\right)^{-1}$, *under the following pairing:*

$$< f, g >= \int_{\mathbb{R}^n} fg.$$

Moreover $\|f\|_{{}^{\complement}LM_{p'\theta',\widetilde{\omega}}} = \sup_g \left|\int_{\mathbb{R}^n} fg\right|$, *where the supremum is taken over all functions* $g \in LM_{p\theta,\omega}$ *with* $\|g\|_{LM_{p\theta,\omega}} \le 1$.

Theorem 3.7. ([6], Theorem 5.2) *Assume* $1 \le p < \infty$, $1 \le \theta < \infty$. *Let* $\omega \in {}^{\complement}\Omega_\theta$ *and* $\|\omega\|_{L_\theta(0,\infty)} = \infty$. *Then*

$$\left({}^{\complement}LM_{p\theta,\omega}\right)^* = LM_{p'\theta',\overline{\omega}}, \tag{3.2}$$

where $\overline{\omega}(t) = \omega^{\theta-1}(t)\left(\int_0^t \omega^\theta(s)ds\right)^{-1}$, *under the following pairing:*

$$< f, g >= \int_{\mathbb{R}^n} fg.$$

Moreover $\|f\|_{LM_{p'\theta',\overline{\omega}}} = \sup_g \left|\int_{\mathbb{R}^n} fg\right|$, *where the supremum is taken over all functions* $g \in {}^{\complement}LM_{p\theta,\omega} : \|g\|_{{}^{\complement}LM_{p\theta,\omega}} \le 1$.

In fact more general results hold true.

Theorem 3.8. *Assume* $1 \le p < \infty$, $1 \le \theta < \infty$. *Let* $\omega \in \Omega_\theta$ *and* $\|\omega\|_{L_\theta(0,\infty)} = \infty$. *Then for any* $x \in \mathbb{R}^n$

$$\left(LM_{p\theta,\omega}^{\{x\}}\right)^* = {}^{\complement}LM_{p'\theta',\widetilde{\omega}}^{\{x\}}, \tag{3.3}$$

where $\widetilde{\omega}(t) = \omega^{\theta-1}(t)\left(\int_t^\infty \omega^\theta(s)ds\right)^{-1}$, *under the following pairing:*

$$< f, g >= \int_{\mathbb{R}^n} fg.$$

Moreover $\|f\|_{{}^{\complement}LM_{p'\theta',\widetilde{\omega}}^{\{x\}}} = \sup_g \left|\int_{\mathbb{R}^n} fg\right|$, *where the supremum is taken over all functions* $g \in LM_{p\theta,\omega}^{\{x\}}$ *with* $\|g\|_{LM_{p\theta,\omega}^{\{x\}}} \le 1$.

Theorem 3.9. *Assume* $1 \le p < \infty$, $1 \le \theta < \infty$. *Let* $\omega \in {}^{\complement}\Omega_\theta$ *and* $\|\omega\|_{L_\theta(0,\infty)} = \infty$. *Then for any* $x \in \mathbb{R}^n$

$$\left({}^{\complement}LM_{p\theta,\omega}^{\{x\}}\right)^* = LM_{p'\theta',\overline{\omega}}^{\{x\}}, \tag{3.4}$$

where $\overline{\omega}(t) = \omega^{\theta-1}(t)\left(\int_0^t \omega^\theta(s)ds\right)^{-1}$, *under the following pairing:*

$$< f, g >= \int_{\mathbb{R}^n} fg.$$

Moreover $\|f\|_{LM_{p'\theta',\overline{\omega}}^{\{x\}}} = \sup_g \left|\int_{\mathbb{R}^n} fg\right|$, *where the supremum is taken over all functions* $g \in {}^{\complement}LM_{p\theta,\omega}^{\{x\}} : \|g\|_{{}^{\complement}LM_{p\theta,\omega}^{\{x\}}} \le 1$.

Proofs of Theorem 3.8 and Theorem 3.9 are analogous to proofs of Theorem 3.6 and Theorem 3.7, respectively and we omit them.

4. New characterization of Morrey space

In this section we give new characterization of the classical Morrey space.

Note that $g \mapsto \inf\limits_{x\in\mathbb{R}^n} \int_0^\infty r^{\frac{n-\lambda}{p}-1}\|g\|_{L_{p'}(\complement_{B(x,r)})}dr$ is positively homogeneous functional on $\bigcup_{x\in\mathbb{R}^n} {}^{\complement}LM^{\{x\}}_{p'1,\frac{n-\lambda}{p}-1}$.

Denote by

$$\widetilde{\mathcal{M}}_{p,\lambda} := \left\{ f \in \mathfrak{M}(\mathbb{R}^n, dx) : \|f\|_{\widetilde{\mathcal{M}}_{p,\lambda}} < \infty \right\} \tag{4.1}$$

the associate space of the set of functions $\bigcup_{x\in\mathbb{R}^n} {}^{\complement}LM^{\{x\}}_{p'1,\frac{n-\lambda}{p}-1}$, where

$$\|f\|_{\widetilde{\mathcal{M}}_{p,\lambda}} := \sup\left\{ \int_{\mathbb{R}^n} |fg| : \inf_{x\in\mathbb{R}^n} \int_0^\infty r^{\frac{n-\lambda}{p}-1}\|g\|_{L_{p'}(\complement_{B(x,r)})}dr \le 1 \right\}. \tag{4.2}$$

To study properties of the space $\widetilde{\mathcal{M}}_{p,\lambda}$ the following Hölder's inequality for the classical Morrey spaces are useful.

Lemma 4.1. *Let $1 \le p < \infty$ and $0 < \lambda < n$. Then the inequality*

$$\int_{\mathbb{R}^n} |fg| \le C\|f\|_{\mathcal{M}_{p,\lambda}} \inf_{x\in\mathbb{R}^n} \int_0^\infty r^{\frac{n-\lambda}{p}-1}\|g\|_{L_{p'}(\complement_{B(x,r)})}dr, \tag{4.3}$$

holds with positive constant C independent of functions f and g.

Proof. For $\theta = \infty$ and $w(t) = t^{\frac{\lambda-n}{p}}$ Corollary 3.5 (part (ii)) implies the following inequality

$$\int_{\mathbb{R}^n} |fg| \le C \sup_{t>0} t^{\frac{\lambda-n}{p}} \|f\|_{L_p(B(x,t))} \int_0^\infty r^{\frac{n-\lambda}{p}-1}\|g\|_{L_{p'}(\complement_{B(x,r)})}dr, \tag{4.4}$$

with constant C independent of f, g and $x \in \mathbb{R}^n$. Therefore

$$\begin{aligned} \int_{\mathbb{R}^n} |fg| &\le C \sup_{x\in\mathbb{R}^n,\, t>0} t^{\frac{\lambda-n}{p}} \|f\|_{L_p(B(x,t))} \int_0^\infty r^{\frac{n-\lambda}{p}-1}\|g\|_{L_{p'}(\complement_{B(x,r)})}dr \\ &= C\|f\|_{\mathcal{M}_{p,\lambda}} \int_0^\infty r^{\frac{n-\lambda}{p}-1}\|g\|_{L_{p'}(\complement_{B(x,r)})}dr. \end{aligned} \tag{4.5}$$

In view of arbitrariness of x we arrive at (4.3). □

Lemma 4.2. *Let $1 \le p < \infty$ and $0 < \lambda < n$. Then*

$$\inf_{x\in\mathbb{R}^n} \int_0^\infty r^{\frac{n-\lambda}{p}-1}\|g\|_{L_{p'}(\complement_{B(x,r)})}dr = 0 \tag{4.6}$$

if and only if $g = 0$ on $\mathbb{R}^n$.

Proof. Obviously, $\inf\limits_{x\in\mathbb{R}^n} \int_0^\infty r^{\frac{n-\lambda}{p}-1}\|g\|_{L_{p'}(\complement_{B(x,r)})}dr = 0$, when $g = 0$ a.e. on $\mathbb{R}^n$.

Now assume that $\inf\limits_{x\in\mathbb{R}^n} \int_0^\infty r^{\frac{n-\lambda}{p}-1}\|g\|_{L_{p'}(\complement_{B(x,r)})}dr = 0$. For any fixed $R > 0$ consider the function $f = \chi_{B(0,R)}$. Obviously, $f \in \mathcal{M}_{p,\lambda}$, since $\|\chi_{B(0,R)}\|_{\mathcal{M}_{p,\lambda}} \approx R^{\lambda/p}$. Then by the inequality (4.3), we have $\int_{B(0,R)} |f| = 0$, therefore, $f = 0$ a.e. on $B(0,R)$. From arbitrariness of R, we get that $f = 0$ a.e. on $\mathbb{R}^n$.

□

Lemma 4.3. *Let* $1 \le p < \infty$ *and* $0 < \lambda < n$. *Then*

$$\bigcup_{x \in \mathbb{R}^n} {}^{\complement}LM^{\{x\}}_{p'1,\frac{n-\lambda}{p}-1} \subset L_1^{\mathrm{loc}}(\mathbb{R}^n).$$

Proof. Let g be any function from $\bigcup_{x \in \mathbb{R}^n} {}^{\complement}LM^{\{x\}}_{p'1,\frac{n-\lambda}{p}-1}$. Then there exists $x \in \mathbb{R}^n$ such that $g \in {}^{\complement}LM^{\{x\}}_{p'1,\frac{n-\lambda}{p}-1}$. Let R be any fixed positive number. Since the function $f = \chi_{B(x,R)} \in \mathcal{M}_{p,\lambda}$ and $\|f\|_{\mathcal{M}_{p,\lambda}} \approx R^{\lambda/p}$, by the inequality (4.5) we get

$$\int_{B(x,R)} |g(y)|dy \le CR^{\frac{\lambda}{p}} \|g\|_{{}^{\complement}LM^{\{x\}}_{p'1,\frac{n-\lambda}{p}-1}} < \infty.$$

In view of arbitrariness of R we get that $g \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$. □

Lemma 4.4. *Assume* $1 \le p < \infty$ *and* $0 < \lambda < n$. *Moreover, let* $f \in L_{p'}^{\mathrm{loc}}(\mathbb{R}^n)$. *Then for any fixed* $x \in \mathbb{R}^n$ *and* $R > 0$

$$f\chi_{B(x,R)} \in {}^{\complement}LM^{\{x\}}_{p'1,\frac{n-\lambda}{p}-1}.$$

Proof. Indeed, for any fixed $x \in \mathbb{R}^n$ and $R:\, 0 < R < \infty$, we get

$$\begin{aligned}
\|f\chi_{B(x,R)}\|_{{}^{\complement}LM^{\{x\}}_{p'1,\frac{n-\lambda}{p}-1}} &= \int_0^\infty r^{\frac{n-\lambda}{p}-1} \|f\chi_{B(x,R)}\|_{L_{p'}({}^{\complement}B(x,r))} dr \\
&= \int_0^\infty r^{\frac{n-\lambda}{p}-1} \left(\int_{{}^{\complement}B(x,r) \cap B(x,R)} |f|^{p'} \right)^{\frac{1}{p'}} dr \\
&= \int_0^R r^{\frac{n-\lambda}{p}-1} \left(\int_{{}^{\complement}B(x,r) \cap B(x,R)} |f|^{p'} \right)^{\frac{1}{p'}} dr \\
&\le \left(\int_{B(x,R)} |f|^{p'} \right)^{\frac{1}{p'}} \int_0^R r^{\frac{n-\lambda}{p}-1} dr \\
&= c_1 R^{\frac{n-\lambda}{p}} \left(\int_{B(x,R)} |f|^{p'} \right)^{\frac{1}{p'}} < \infty.
\end{aligned}$$

□

Our main result in this section reads as follows.

Theorem 4.5. *Assume* $1 \le p < \infty$ *and* $0 < \lambda < n$. *Then*

$$\|f\|_{\mathcal{M}_{p,\lambda}} \approx \|f\|_{\widetilde{\mathcal{M}}_{p,\lambda}}. \tag{4.7}$$

Proof. By Lemma 4.1, it is easy to see that

$$\|f\|_{\widetilde{\mathcal{M}}_{p,\lambda}} \lesssim \|f\|_{\mathcal{M}_{p,\lambda}}.$$

Let us to prove opposite estimate $\|f\|_{\mathcal{M}_{p,\lambda}} \lesssim \|f\|_{\widetilde{\mathcal{M}}_{p,\lambda}}$. If $\|f\|_{\widetilde{\mathcal{M}}_{p,\lambda}} = \infty$, then there is nothing to prove. Assume that $\|f\|_{\widetilde{\mathcal{M}}_{p,\lambda}} < \infty$.

Observe that for $g \in L_{p'}^{\mathrm{loc}}(\mathbb{R}^n)$ the inequality

$$\int_{B(x,R)} |fg| \le CR^{\frac{n-\lambda}{p}} \left(\int_{B(x,R)} |g|^{p'} \right)^{\frac{1}{p'}} \|f\|_{\widetilde{\mathcal{M}}_{p,\lambda}} \tag{4.8}$$

holds with constant $C > 0$ independent of f, g, x and R. Indeed, let x be any fixed point in $\mathbb{R}^n$ and $R > 0$. When $\int_{B(x,R)} |g|^{p'} = 0$ there is nothing to prove, since in this case $g = 0$ a.e. on $B(x,R)$. Assume that $\int_{B(x,R)} |g|^{p'} > 0$. Denote by

$$h(y) = \frac{g(y)\chi_{B(x,R)}(y)}{c_1 R^{\frac{n-\lambda}{p}} \left(\int_{B(x,R)} |g|^{p'} \right)^{\frac{1}{p'}}}. \tag{4.9}$$

By Lemma 4.4

$$h \in {}^{\complement}LM^{\{x\}}_{p'1,\frac{n-\lambda}{p}-1},$$

and moreover, $\|h\|_{{}^{\complement}LM^{\{x\}}_{p'1,\frac{n-\lambda}{p}-1}} \le 1$. Consequently,

$$\inf_{x\in\mathbb{R}^n} \int_0^\infty r^{\frac{n-\lambda}{p}-1} \|h\|_{L_{p'}({}^{\complement}B(x,r))} dr \le 1.$$

Therefore

$$\int_{\mathbb{R}^n} |hf| \le \|f\|_{\widetilde{\mathcal{M}}_{p,\lambda}}, \tag{4.10}$$

and from (4.9), we get (4.8).

The inequality (4.8) implies that $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$. By Theorem of Resonance (see [9, Lemma 27, p.283]) we get that $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$. The function $g := |f|^{p-1}\chi_{B(x,R)} \in L_{p'}^{\mathrm{loc}}(\mathbb{R}^n)$, and if we put the function g in the inequality (4.8), we obtain

$$\int_{B(x,R)} |f|^p \le cR^{\frac{n-\lambda}{p}} \left(\int_{B(x,R)} |f|^p \right)^{\frac{1}{p'}} \|f\|_{\widetilde{\mathcal{M}}_{p,\lambda}}.$$

Therefore,

$$R^{\frac{\lambda-n}{p}} \left(\int_{B(x,R)} |f|^p \right)^{\frac{1}{p}} \le c\|f\|_{\widetilde{\mathcal{M}}_{p,\lambda}}.$$

Since a constant c is independent of x and R, we get

$$\|f\|_{\mathcal{M}_{p,\lambda}} \le c\|f\|_{\widetilde{\mathcal{M}}_{p,\lambda}}.$$

□

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