# A proof of uniqueness of the Gurarii space *

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**Abstract**

We present a short and elementary proof of isometric uniqueness of the Gurarii space.

## 1 Introduction

A *Gurarii space*, constructed by Gurarii [3] in 1965, is a separable Banach space $\mathbb{G}$ satisfying the following condition: given finite-dimensional Banach spaces $X \subseteq Y$, given $\varepsilon > 0$, and given an isometric linear embedding $f\colon X \to \mathbb{G}$ there exists an injective linear operator $g\colon Y \to \mathbb{G}$ extending $f$ and satisfying $\|g\| \cdot \|g^{-1}\| < 1+\varepsilon$. It is not hard to prove straight from this definition that such a space is unique up to isomorphism of norm arbitrarily close to one. The question whether the Gurarii space is unique up to isometry remained open for some time. It was answered affirmatively by Lusky [6] in 1976 using deep techniques developed by Lazar and Lindenstrauss [5]. Subsequently, another proof of uniqueness was given by Henson using model theoretic methods of continuous logic. (This proof remains unpublished.) The natural question whether there is an elementary proof of uniqueness occurred to several mathematicians. This question was made current by recent increased interest in universal, homogeneous structures and their automorphism groups; see, for example, [4] and [7]. The aim of this note is to provide just such a simple and elementary proof of isometric uniqueness of the Gurarii space. This proof is given in Section 2. In Section 3, we give an elementary argument showing isometric universality of the Gurarii space among separable Banach spaces.

In order to state the theorem precisely, we introduce some notions. Let $X$, $Y$ be Banach spaces, $\varepsilon > 0$. A linear operator $f\colon X \to Y$ is an *$\varepsilon$-isometry* if

$$(1+\varepsilon)^{-1} \cdot \|x\| < \|f(x)\| < (1+\varepsilon) \cdot \|x\|.$$

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holds for every $x \in X \setminus \{0\}$. We use strict inequalities for the sake of convenience. In particular, in the case of finite dimensional spaces, every $\varepsilon$-isometry is an $\varepsilon'$-isometry for some $0 < \varepsilon' < \varepsilon$. Note that the inverse of a bijective $\varepsilon$-isometry is again an $\varepsilon$-isometry. By an *isometry* we mean a linear operator $f\colon X \to Y$ that is an $\varepsilon$-isometry for every $\varepsilon > 0$, that is, $\|f(x)\| = \|x\|$ holds for every $x \in X$. (A word of caution about our terminology may be in place: in the literature, such functions are often called *isometric embeddings*, with the word "isometry" reserved for a *bijective* isometric embedding.)

We will give a proof of the following theorem.

**Theorem 1.1.** *Let $E$, $F$ be separable Gurarii spaces, $0 < \varepsilon < 1$. Assume $X \subseteq E$ is a finite dimensional space and $f\colon X \to F$ is an $\varepsilon$-isometry. Then there exists a bijective isometry $h\colon E \to F$ such that $\|h \restriction X - f\| < 2\varepsilon$.*

By taking $X$ to be the trivial space, we obtain the following corollary.

**Corollary 1.2** (Lusky [6])**.** *The Gurarii space is unique up to a bijective isometry.*

## 2 Proof of uniqueness of the Gurarii space

**Lemma 2.1.** *Let $X$, $Y$ be finite dimensional Banach spaces and let $f\colon X \to Y$ be an $\varepsilon$-isometry, where $0 < \varepsilon < 1$. Consider the algebraic sum $X \oplus Y$ and the canonical embeddings $i\colon X \to X \oplus Y$ and $j\colon Y \to X \oplus Y$. Then there exists a norm $\|\cdot\|$ on $X \oplus Y$ such that*

$$\|j \circ f - i\| < 2\varepsilon$$

*and both $i$ and $j$ are isometries.*

*Proof.* We will denote by $\|\cdot\|_X$, $\|\cdot\|_Y$ the norms of $X$ and $Y$, respectively. Given $x^* \in S_X^*$, denote by $\overline{x}^*$ a fixed functional on $Y$ satisfying $\overline{x}^* \circ f = x^*$ and

$$\|\overline{x}^*\|_Y^* = \|x^* f^{-1}\|_{f[X]}^*.$$

The existence of $\overline{x}^*$ is a direct consequence of Hahn-Banach's Theorem.

Now define

$$\varphi_X(x, y) = \sup_{x^* \in S_X^*} \left| x^*(x) + \frac{1}{\|\overline{x}^*\|_Y^*} \overline{x}^*(y) \right|.$$

It is clear that $\varphi_X$ is a seminorm on $X \oplus Y$. Observe that $\varphi_X(x, 0) = \|x\|_X$ and $\varphi_X(0, y) \leqslant \|y\|_Y$. Next, define

$$\varphi_Y(x, y) = \sup_{y^* \in S_Y^*} \left| \frac{1}{\|y^* f\|_X^*} y^* f(x) + y^*(y) \right|.$$

Again, $\varphi_Y$ is a seminorm on $X \oplus Y$ such that $\varphi_Y(x,0) \leqslant \|x\|_X$ and $\varphi_Y(0,y) = \|y\|_Y$. Finally, define

$$\|(x,y)\| = \max\Big\{\varphi_X(x,y), \varphi_Y(x,y), \varepsilon\|x\|_X, \varepsilon\|y\|_Y\Big\}.$$

Now $\|\cdot\|$ is a norm on $X \oplus Y$ and, since $\varepsilon < 1$, we have that $\|(x,0)\| = \|x\|_X$ and $\|(0,y)\| = \|y\|_Y$. Hence, $i$ and $j$ are isometries with respect to $\|\cdot\|$. It remains to check that $\|jf(x) - i(x)\| < 2\varepsilon\|x\|_X$.

Fix $x \in S_X$ and let $u = jf(x) - i(x) = (-x, f(x)) \in X \oplus Y$. Note that, by compactness, in the definitions of $\varphi_X$, $\varphi_Y$ the supremum can be replaced by the maximum. So fix $x^* \in S_X^*$ and $y^* \in S_Y^*$ such that

$$\varphi_X(u) = \left| x^*(-x) + \frac{1}{\|\overline{x}^*\|_Y^*}\overline{x}^* f(x)\right|$$

and

$$\varphi_Y(u) = \left|\frac{1}{\|y^* f\|_X^*} y^* f(-x) + y^* f(x)\right|.$$

Since $\overline{x}^* f(x) = x^*(x)$, we have

$$\varphi_X(u) = \left|\frac{1}{\|x^* f^{-1}\|_{f[X]}^*} - 1\right| \cdot |x^*(x)| = \left|\frac{1}{\|x^* f^{-1}\|_{f[X]}^*} - 1\right|.$$

Similarly,

$$\varphi_Y(u) = \left|1 - \frac{1}{\|y^* f\|_X^*}\right| \cdot |y^* f(x)| < (1+\varepsilon) \cdot \left|1 - \frac{1}{\|y^* f\|_X^*}\right|.$$

Now recall that both $f$ and $f^{-1}$ are $\varepsilon$-isometries and $\|x^*\|_X^* = 1 = \|y^*\|_Y^*$, therefore $(1+\varepsilon)^{-1} < \|x^* f^{-1}\|_{f[X]}^* < 1 + \varepsilon$ and $(1+\varepsilon)^{-1} < \|y^* f\|_X^* < 1 + \varepsilon$. It follows that $\varphi_X(u) < \varepsilon$ and $\varphi_Y(u) < \varepsilon(1+\varepsilon) < 2\varepsilon$. Finally, since $\varepsilon\|x\|_X < \varepsilon$, we have that $\|u\| = \max\{\varphi_X(u), \varphi_Y(u)\} < 2\varepsilon$. This completes the proof. □

**Lemma 2.2.** *Let $E$ be a Gurarii space and let $f\colon X \to Y$ be an $\varepsilon$-isometry, where $X$ is a finite dimensional subspace of $E$ and $0 < \varepsilon < 1$. Then for every $\delta > 0$ there exists a $\delta$-isometry $g\colon Y \to E$ such that $\|gf(x) - x\| < 2\varepsilon\|x\|$ for every $x \in X$.*

*Proof.* Use Lemma 2.1 together with the definition of a Gurarii space. □

*Proof of Theorem 1.1.* Let $\{X_n\}_{n\in\mathbb{N}}$ be an increasing sequence of finite dimensional subspaces of $E$ such that $X_0 = X$ and $\bigcup_{n\in\mathbb{N}} X_n$ is dense in $E$. Similarly, let $\{Y_n\}_{n\in\mathbb{N}}$ be a chain of finite dimensional subspaces of $F$ such that $Y_0 = f[X]$ and $\bigcup_{n\in\mathbb{N}} Y_n$ is dense in $F$. Fix a strictly decreasing sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ of positive real numbers. The precise conditions on $\{\varepsilon_n\}_{n\in\mathbb{N}}$ will be specified later. We define inductively two sequences of linear operators $\{f_n\}_{n\in\mathbb{N}}$, $\{g_n\}_{n\in\mathbb{N}}$ so that the following conditions are satisfied.

(0) $X_0 = X$, $Y_0 = f[X]$, and $f_0 = f$;

(1) $f_n \colon X_{k_n} \to Y_{\ell_n}$ is an $\varepsilon_{2n}$-isometry and $k_n < \ell_n$;

(2) $g_n \colon Y_{\ell_n} \to X_{k_{n+1}}$ is an $\varepsilon_{2n+1}$-isometry and $\ell_n < k_{n+1}$;

(3) $\|g_n f_n(x) - x\| < 2\varepsilon_{2n}\|x\|$ for $x \in X_{k_n}$;

(4) $\|f_{n+1} g_n(y) - y\| < 2\varepsilon_{2n+1}\|y\|$ for $y \in Y_{\ell_n}$.

Condition (0) tells us how to start the inductive construction. Here we pick $\varepsilon_0 > 0$ so that (1) holds for $n = 0$ and $\varepsilon_0 < \varepsilon$. Suppose $f_i$, $g_i$ have been constructed for $i < n$. We easily find $f_n$ and $g_n$ using Lemma 2.2. Thus, the construction can be carried out.
Fix $n \in \mathbb{N}$ and $x \in X_{k_n}$ with $\|x\| = 1$. Using (4), we get

$$\|f_{n+1} g_n f_n(x) - f_n(x)\| < 2\varepsilon_{2n+1}\|f_n(x)\| \leqslant 2\varepsilon_{2n+1}(1 + \varepsilon_{2n}) < 4\varepsilon_{2n+1}.$$

Using (3), we get

$$\|f_{n+1} g_n f_n(x) - f_{n+1}(x)\| \leqslant \|f_{n+1}\| \cdot \|g_n f_n(x) - x\| < (1 + \varepsilon_{2n+2}) \cdot 2\varepsilon_{2n} < 2(\varepsilon_{2n} + \varepsilon_{2n+2}).$$

These inequalities give

$$(\dagger) \qquad \|f_n(x) - f_{n+1}(x)\| < 2(\varepsilon_{2n} + 2\varepsilon_{2n+1} + \varepsilon_{2n+2}).$$

Now it is clear that if the series $\sum_{n\in\mathbb{N}} \varepsilon_n$ converges, then the sequence $\{f_n(x)\}_{n\in\mathbb{N}}$ is Cauchy. Let us make a stronger assumption, namely that

$$(\ddagger) \qquad 2(2\varepsilon_1 + \varepsilon_2) + \sum_{n=1}^{\infty} 2(\varepsilon_{2n} + 2\varepsilon_{2n+1} + \varepsilon_{2n+2}) < 2\varepsilon - 2\varepsilon_0.$$

Given $x \in \bigcup_{n\in\mathbb{N}} X_n$, define $h(x) = \lim_{n\geqslant m} f_n(x)$, where $m$ is such that $x \in X_{k_m}$. Then $h$ is an $\varepsilon_n$-isometry for every $n \in \mathbb{N}$, hence it is an isometry. Consequently, it uniquely extends to an isometry on $E$, which we denote also by $h$. Furthermore, (†) and (‡) give

$$\|f(x) - h(x)\| \leqslant \sum_{n=0}^{\infty} 2(\varepsilon_{2n} + 2\varepsilon_{2n+1} + \varepsilon_{2n+2}) < 2\varepsilon.$$

It remains to see that $h$ is a bijection. To this end, we check as before that $\{g_n(y)\}_{n\geqslant m}$ is a Cauchy sequence for every $y \in Y_{\ell_m}$. Once this is done, we obtain an isometry $g_\infty$ defined on $F$. Conditions (3) and (4) tell us that $g_\infty \circ h = \mathrm{id}_E$ and $h \circ g_\infty = \mathrm{id}_F$. This completes the proof. □

## 3 On universality of the Gurarii space

It is known that the Gurarii space is isometrically universal among separable Banach spaces. Indeed, as pointed out by Gevorkjan [2], universality follows from the results of Lazar and Lindenstrauss [5] and Michael and Pełczyński [8]: the dual of the Gurarii space is a non-separable $L_1$ space, therefore the Gurarii space contains an isometric copy of $C([0,1])$. The reader may also consult the recent paper [1] for another approach.

We conclude with applying our method to proving universality directly, without referring to the structure of the dual or to universality of other Banach spaces.

**Lemma 3.1.** *Let $X_0, X_1, Y_0$ be finite-dimensional Banach spaces such that $X_0 \subseteq X_1$ and let $f\colon X_0 \to Y_0$ be an $\varepsilon$-isometry, where $\varepsilon > 0$. Then there exist a finite-dimensional Banach space $Y_1$ containing $Y_0$ and an isometry $g\colon X_1 \to Y_1$ such that*

$$\|g \restriction X_0 - f\| < 2\varepsilon.$$

*Proof.* A standard and well known amalgamation property for Banach spaces says that there exist $W \supseteq Y_0$ and an $\varepsilon$-isometry $f'\colon X_1 \to W$ such that $f' \restriction X_0 = f$. More precisely, $W = (X_1 \oplus Y_0)/\Delta$, where $X_1 \oplus Y_0$ is endowed with the $\ell_1$-norm and

$$\Delta = \{(z, -f(z))\colon z \in X_0\}.$$

The space $Y_0$ is naturally identified with the subspace of $W$ and $f'(x)$ is the equivalence class of $(x, 0)$ (where $x \in X_1$).

Finally, the desired isometry $g$ is provided by Lemma 2.1. □

**Theorem 3.2.** *Every separable Banach space can be isometrically embedded into the Gurarii space.*

*Proof.* Let $\mathbb{G}$ denote the Gurarii space. Fix a separable Banach space $X$ and let $\{X_n\}_{n\in\mathbb{N}}$ be a chain of finite-dimensional spaces such that $X_0 = \{0\}$ and $\bigcup_{n\in\mathbb{N}} X_n$ is dense in $X$. In case $X$ is finite-dimensional, we set $X_n = X$ for $n > 0$. We inductively define $f_n\colon X_n \to \mathbb{G}$ so that

(i) $f_n$ is a $2^{-n}$-isometry,

(ii) $\|f_{n+1} \restriction X_n - f_n\| < 3 \cdot 2^{-n}$,

for every $n \in \mathbb{N}$. We set $f_0 = 0$. Suppose $f_n$ has already been defined. Let $Y = f_n[X_n]$. Using Lemma 3.1, we find a finite-dimensional space $W \supseteq Y$ and an isometry $g\colon X_{n+1} \to W$ such that $\|g \restriction X_n - f_n\| < 2 \cdot 2^{-n}$. Using the property of the Gurarii space, we find a $2^{-(n+1)}$-isometry $h\colon W \to \mathbb{G}$ such that $h \restriction Y$ is the inclusion $Y \subseteq \mathbb{G}$. Now set $f_{n+1} := h \circ g$. Given $x \in X_n$ with $\|x\| = 1$, we have that $\|g(x) - f_n(x)\| < 2 \cdot 2^{-n}$ and hence

$$\|f_{n+1}(x) - f_n(x)\| = \|h(g(x)) - h(f_n(x))\| < (1 + 2^{-(n+1)}) \cdot 2 \cdot 2^{-n} \leqslant 3 \cdot 2^{-n}.$$

This shows (ii). Finally, we obtain a sequence $\{f_n\}_{n\in\mathbb{N}}$ that is pointwise Cauchy on each $X_n$. By (i) and (ii), $f_\infty(x) := \lim_{n\to\infty} f_n(x)$ is a well-defined linear isometry on $\bigcup_{n\in\mathbb{N}} X_n$. This isometry extends uniquely to an isometry $f\colon X \to \mathbb{G}$. $\square$

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