NOTE ON FUNCTIONS SATISFYING THE INTEGRAL HÖLDER CONDITION

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(Received March 21, 1995)

Abstract. Given a modulus of continuity ω and $q \in [1, \infty[$ then H_q^ω denotes the space of all functions f with the period 1 on $\mathbb{R}$ that are locally integrable in power q and whose integral modulus of continuity of power q (see(1)) is majorized by a multiple of ω. The moduli of continuity ω are characterized for which H_q^ω contains "many" functions with infinite "essential" variation on an interval of length 1.

Keywords: integral modulus of continuity, variation of a function

MSC 1991: 26A15, 26A45, 26A16

By a modulus of continuity we understand a continuous nondecreasing function $\omega\colon [0,\infty[\to [0,\infty[$ which is subadditive, i.e.

$$\omega(t_1 + t_2) \leqslant \omega(t_1) + \omega(t_2), \quad t_1, t_2 \geqslant 0$$

and satisfies the requirements

$$\omega(0) = 0, \qquad \omega(t) > 0 \qquad \text{for } t > 0.$$

In what follows ω will always stand for a fixed modulus of continuity. If $f\colon \mathbb{R} \to \mathbb{R}^- \equiv \mathbb{R} \cup \{-\infty, \infty\}$ is a Lebesgue measurable function with period 1 and $q \geqslant 1$ we denote

$$\| f \|_q = \left[\int_0^1 |f(x)|^q \,\mathrm{d}x \right]^{1/q}$$

and in case $\| f \|_q < \infty$ we define its modulus of continuity of power q by

$$\omega(f,t)_q := \sup_{|h| \leqslant t} \left[\int_0^1 |f(x+h) - f(x)|^q \,\mathrm{d}x \right]^{1/q}. \tag{1}$$

Any two such functions are considered equivalent if their difference is equal to a constant function almost everywhere on $\mathbb{R}$. H_q^ω denotes the set of all such classes of mutually equivalent functions f for which there exists a $c \in [0, \infty[$ such that

$$\omega(f,t)_q \leqslant c\omega(t), \quad t > 0;$$

the least c with this property will be denoted by

$$\| f \|_q^\omega := \sup_{t>0} \omega(f,t)_q / \omega(t).$$

As usual, the elements of H_q^ω will be identified with functions (representing the whole class of mutually equivalent functions). Then H_q^ω is a linear space over $\mathbb{R}$ and $\| \, \|_q^\omega$ is the norm in this factor space. The space H_q^ω normed by $\| \, \|_q^\omega$ is a Banach space.

Let us denote by $C_0^{(1)}$ the set of all continuously differentiable functions on $\mathbb{R}$ vanishing outside the interval $[0,1]$ and let us define for any $f \in H_q^\omega$ its essential variation on $]0,1[$ by

$$\operatorname{var}(f) = \sup \Big\{ \int_0^1 f(x)\varphi'(x) \, \mathrm{d}x ; \, \varphi \in C_0^{(1)}, |\varphi| \leqslant 1 \Big\}.$$

It is easy to see that $\operatorname{var}(f)$ does not actually depend on the choice of the representing function in the class of functions equivalent to f. It is possible to prove that $\operatorname{var}(f) < \infty$ iff there exists a g equivalent to f with a finite total variation on $[0,1]$ defined in the usual way as the least upper bound of all sums of the form

$$\sum_{i=1}^{n} |g(t_i) - g(t_{i-1})|,$$

where $0 = t_0 < t_1 < \ldots < t_n = 1$ ranges over all subdivisions of the interval $[0,1]$.

Conditions on the modulus of continuity ω sufficient for the existence of an $f \in H_q^\omega$ with $\operatorname{var}(f) = \infty$ have been investigated by O. Kováčik. He showed in [1] by a direct construction that

$$\sum_{n=1}^{\infty} n^{-\alpha} \omega(\tfrac{1}{n}) = \infty \tag{2}$$

with an $\alpha \in]0,1]$ represents such a sufficient condition. We shall show in this note using method of the Baire category (see [2], [3]) that this result can be sharpened.

Denoting $\omega'_+(0) := \liminf_{t\to 0} \omega(t)/t$, we shall prove that H_q^ω contains an f with $\operatorname{var}(f) = \infty$ iff

$$\omega'_+(0) = \infty. \tag{3}$$

More precisely, we have the following results.

Theorem 1. *If* (3) *holds then the set*

$$\{f \in H_q^\omega;\ \operatorname{var}(f) < \infty\} \tag{4}$$

is of the first category in H_q^ω *(and, consequently, its complement in* H_q^ω *is non-void); in the opposite case* $\omega'_+(0) < \infty$ *the set* (4) *coincides with the whole space* H_q^ω.

Before going into the proof of this theorem we shall establish several simple auxiliary results.

Lemma 1. *If* $\mathbf{1}$ *stands for the constant function equal to 1 on* $\mathbb{R}$ *and, for* $f \in H_q^\omega$,

$$m(f) = \int_0^1 f(x)\,\mathrm{d}x, \tag{5}$$

then

$$\|f - m(f)\mathbf{1}\|_q \leqslant \|f\|_q^\omega \left[\int_0^1 \omega(h)^q\,\mathrm{d}h\right]^{1/q}. \tag{6}$$

P r o o f 1. Let f be a function with period 1 which is locally integrable in power q; using the notation (5) we have

$$\begin{aligned}
\|f - m(f)\mathbf{1}\|_q &= \left[\int_0^1 \left|f(x) - \int_0^1 f(t)\,\mathrm{d}t\right|^q \mathrm{d}x\right]^{1/q}\\
&= \left[\int_0^1 \left|\int_0^1 [f(x) - f(t)]\,\mathrm{d}t\right|^q \mathrm{d}x\right]^{1/q}\\
&\leqslant \left[\int_0^1 \int_0^1 |f(x) - f(t)|^q\,\mathrm{d}t\,\mathrm{d}x\right]^{1/q}\\
&= \left[\int_0^1 \int_0^1 |f(t+h) - f(t)|^q\,\mathrm{d}h\,\mathrm{d}t\right]^{1/q}\\
&\leqslant \left[\int_0^1 [\omega(f,h)_q]^q\,\mathrm{d}h\right]^{1/q}.
\end{aligned}$$

If $f \in H_q^\omega$ then the inequality $\omega(f,h)_q \leqslant \|f\|_q^\omega \omega(h)$ implies (6). □

Lemma 2. *The function* $\operatorname{var}: f \to \operatorname{var}(f)$ *is lower semicontinuous on the space* H_q^ω.

Proof 2. Let $\{f_n\}_{n=1}^\infty$ be an arbitrary sequence of functions in H_q^ω converging to $f_0 \in H_q^\omega$ with respect to the norm $\|\ldots\|_q^\omega$.

We wish to verify that

$$\operatorname{var}(f_0) \leqslant \liminf_{n\to\infty} \operatorname{var}(f_n). \tag{7}$$

To this purpose we choose an arbitrary $c < \operatorname{var}(f_0)$. Then there exists a $\varphi \in C_0^{(1)}$ such that $|\varphi| \leqslant 1$ and

$$\int_0^1 f_0(x)\varphi'(x)\,\mathrm{d}x > c.$$

Let $c_n = m(f_n) - m(f_0)$. According to Lemma 1 the functions $f_n - c_n\mathbf{1}$ converge to f_0 with respect to the norm $\|\ldots\|_q$ and, consequently, also with respect to $\|\ldots\|_1$.

Hence

$$\int_0^1 f_n(x)\varphi'(x)\,\mathrm{d}x = \int_0^1 [f_n(x) - c_n]\varphi'(x)\,\mathrm{d}x \to \int_0^1 f_0(x)\varphi'(x)\,\mathrm{d}x$$

as $n \to \infty$, so that

$$\operatorname{var}(f_n) \geqslant \int_0^1 f_n(x)\varphi'(x)\,\mathrm{d}x > c$$

for all sufficiently large n. Thus (7) is verified. □

Lemma 3. *For each* $n \in N$ *let us define the function* ω_n *on* $\mathbb{R}$ *so that* ω_n *has period* $\frac{1}{n}$ *and*

$$\omega_n(t) = \begin{cases} \omega(t) & \text{for } 0 \leqslant t \leqslant \frac{1}{2n} \\ \omega(\frac{1}{n} - t) & \text{for } \frac{1}{2n} \leqslant t \leqslant \frac{1}{n}. \end{cases}$$

Then $\omega_n \in H_q^\omega$, $\operatorname{var}(\omega_n) = 2n\omega(\frac{1}{2n})$ *and* $\|\omega_n\|_q^\omega \leqslant 1$.

Proof 3. Since ω_n is continuous and monotonous on each of the intervals $[\frac{k}{2n}, \frac{(k+1)}{2n}]$, $0 \leqslant k < 2n$, which are mapped onto an interval of length $\omega(\frac{1}{2n})$, we have $\operatorname{var}(\omega_n) = 2n\omega(\frac{1}{2n})$. We can see from the definition of ω_n that

$$|\omega_n(x+h) - \omega_n(x)| \leqslant \omega(|h|)$$

for x, $h \in \mathbb{R}$, whence

$$\left[\int_0^1 |\omega_n(x+h) - \omega_n(x)|^q\,\mathrm{d}x\right]^{1/q} \leqslant \omega(|h|),$$

so that $\|\omega_n\|_q^\omega \leqslant 1$.

Now we are in a position to present the following. □

Proof 4 of the Theorem 1. Assume (3) and put for $k \in N$

$$B_k = \{f \in H_q^\omega;\ \mathrm{var}(f) \leqslant k\}.$$

It follows from Lemma 2 that B_k is closed in H_q^ω. In order to show that B_k is nowhere dense we shall verify that for each $f_0 \in H_q^\omega$ and any $\varepsilon > 0$ there is an $f \in H_q^\omega \setminus B_k$ such that $\|f - f_0\|_q^\omega \leqslant \varepsilon$. If $f_0 \in H_q^\omega \setminus B_k$ we may, of course, choose $f = f_0$; so let $\mathrm{var}(f_0) \leqslant k$. Choose $n \in N$ so large that

$$2n\omega(\tfrac{1}{2n}) > 2\tfrac{k}{\varepsilon}$$

and put

$$f = f_0 + \varepsilon\omega_n.$$

According to Lemma 3 we have $\|f - f_0\|_q^\omega \leqslant \varepsilon$ and $\mathrm{var}(f) \geqslant \varepsilon\,\mathrm{var}(\omega_n) - \mathrm{var}(f_0) > \varepsilon 2n\omega(\frac{1}{2n}) - k > k$, so that $f \in H_q^\omega \setminus B_k$ as required. Hence $\bigcup\limits_{k\in N} B_k$ coinciding with (4) is of the first category in H_q^ω.

Conversely, let now

$$\omega'_+(0) < \infty. \tag{8}$$

Since ω is a modulus of continuity we have then

$$\sup_{t>0} \omega(t)/t \leqslant 2\omega'_+(0),$$

which follows e.g. from the inequality (6) in Section 3.2.4. in [4]. If $\varphi \in C_0^{(1)}$ then

$$[\varphi(x+h) - \varphi(x)]/h \to \varphi'(x)$$

uniformly with respect to $x \in \mathbb{R}$ as $h \to 0$. Choosing an arbitrary $f \in H_q^\omega$ we have then

$$\begin{aligned}\int_0^1 f(x)\varphi'(x)\,\mathrm{d}x &= \lim_{h\downarrow 0}\int_0^1 f(x)[\varphi(x+h) - \varphi(x)]/h\,\mathrm{d}x\\ &= \lim_{h\downarrow 0} h^{-1}\int_0^1 [f(x-h) - f(x)]\varphi(x)\,\mathrm{d}x\\ &\leqslant \sup_{h\neq 0}|h|^{-1}\omega(f,|h|)_q\left[\int_0^1 |\varphi(x)|^p\,\mathrm{d}x\right]^{1/p},\end{aligned}$$

where p is the Hölder conjugate exponent of q $(1/p + 1/q = 1)$. Hence we obtain

$$\mathrm{var}(f) \leqslant 2\omega'_+(0)\|f\|_q^\omega < \infty.$$

We observe that in this case (4) coincides with the whole space H_q^ω. □

R e m a r k 1. It follows from the above proof that, under the condition (8),the natural embedding of H^{ω}_q into the factor space (modulo constant functions) of periodic functions with $\operatorname{var}(f) < \infty$ (normed by $\operatorname{var}(\ldots)$) is continuous; in the case $\omega(t) = t$ and $q = 1$ these spaces can be identified (cf. [5]). In case $q > 1$ and $\omega'_+(0) < \infty$ the reasoning from the end of the previous proof implies continuity of the natural embedding of H^{ω}_q into the factor space (modulo constant functions) formed by periodic functions that are absolutely continuous and satisfy

$$\infty > \left[\int_0^1 |f'(x)|^q \,\mathrm{d}x\right]^{1/q} \equiv \sup\left\{\int_0^1 f(x)\varphi'(x)\,\mathrm{d}x;\ \varphi \in C_0^{(1)},\ \int_0^1 |\varphi(x)|^p \,\mathrm{d}x \leqslant 1\right\};$$

the norm in this space is given by $\|f'\|_q$ (cf. [4], 3.12.13).

R e m a r k 2. The theorem established above holds also for $q = \infty$ provided the expressions of the form $\left[\int_0^1 |f(x)|^q \,\mathrm{d}x\right]^{1/q}$ occurring in the construction of H^{ω}_q are replaced by the essential norm formed by $\inf\{\alpha \geqslant 0;\ \operatorname{meas}(\{x;\ |f(x)| \geqslant \alpha\}) = 0\}$, where meas(...) is the Lebesgue measure on the real line.

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