CONGRUENCES AND IDEALS ON LEFT DIVISIBLE INVOLUTORY GROUPOIDS

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(Received May 10, 1995)

Abstract. In [1] ideals and congruences on semiloops were investigated. The aim of this paper is to generalize results obtained for semiloops to the case of left divisible involutory groupoids.

Keywords: ideal, congruence, left divisible involutory groupoid

MSC 1991: 08A30, 08B99

The concept of a semiloop was introduced in [1]: An algebra $\mathcal{A} = (A, \backslash, \cdot, e)$ of type $(2, 2, 0)$ is called a *semiloop* if it satisfies the identitites

$$\begin{aligned} x \backslash x &= e, \\ x \cdot (x \backslash y) &= y, \\ x \backslash (x \cdot y) &= y. \end{aligned}$$

In other words, semiloops are just right uniquely divisible groupoids with a right unit e. Of course, there exist semiloops which are not loops, see [1]. It was also shown that the variety $\mathcal{S}$ of all semiloops is congruence-permutable and congruence regular.

The concept of an ideal in an algebra with a nullary operation e was introduced in [4] and intensively studied by A. Ursini and H. P. Gumm in [3]. Recall that an $(n+m)$-ary term $p(x_1, \ldots, x_n, y_1, \ldots, y_m)$ of an algebra $\mathcal{A}$ with a nullary operation e is called an *ideal term in* $y_1, \ldots, y_m$ if $p(x_1, \ldots, x_n, e, \ldots, e) = e$ is an identity in $\mathcal{A}$. A subset I of A is called an *ideal of* $\mathcal{A}$ if for any ideal term $p(x_1, \ldots, x_n, y_1, \ldots, y_m)$ in $y_1, \ldots, y_m$ and for every $a_1, \ldots, a_n \in A, b_1, \ldots, b_m \in I$ we have $p(a_1, \ldots, a_n, b_1, \ldots, b_m) \in I$. An algebra $\mathcal{A}$ is said to be *ideal determined* if every ideal I of $\mathcal{A}$ is a kernel of a unique conguence $\Theta_I \in \operatorname{Con}\mathcal{A}$, i.e. $I = [e]_{\Theta_I}$. A variety $\mathcal{V}$ is called *ideal determined* if each $\mathcal{A} \in \mathcal{V}$ has this property.

Let us remark that for any $\Theta \in \operatorname{Con}\mathcal{A}$, $[e]_\Theta$ is an ideal of $\mathcal{A}$. Moreover, the set of all ideal terms of $\mathcal{A}$ forms a clone $\mathrm{IT}(\mathcal{A})$. We have either $\mathrm{IT}(\mathcal{A}) = \{e\}$, the trivial case, or $\mathrm{IT}(\mathcal{A})$ is infinite. If the clone $\mathrm{IT}(\mathcal{A})$ has a finite base, $\mathcal{A}$ is said to have a *good theory of ideals*, see [4].

It was also shown in [1] that every semiloop is ideal determined and has a good theory of ideals. Moreover, [1] contains the explicit description of the 4-element base of $\mathrm{IT}(\mathcal{A})$ for $\mathcal{A} \in \mathcal{S}$.

However, some considerations on groupoids have suggested that some properties of semiloops are superfluous for such a good congruence and ideal properties. So we try to weaken the properties of semiloops to obtain a simpler case of groupoids which still satisfy these conditions. Our work can be regarded as a way to obtain the essence of congruence and ideal properties.

Definition. Let $\mathcal{A} = (A, \backslash, \cdot, e)$ be an algebra of type $(2, 2, 0)$. $\mathcal{A}$ is called a *left divisible involutory groupoid* if the following identities hold in $\mathcal{A}$:

$$x \cdot (x \backslash y) = y,$$
$$x \backslash x = e.$$

Evidently, every loop or semiloop is a left divisible involutory groupoid but not vice versa. Denote by LDIG the variety of all left divisible involutory groupoids.

Now we can list some important congruence properties of varieties of LDIG. Recall that an algebra $\mathcal{A}$ with a nullary operation e is called *e-regular* if $\Theta = \Phi$ for each $\Theta, \Phi \in \operatorname{Con}\mathcal{A}$ whenever $[e]_\Theta = [e]_\Phi$. $\mathcal{A}$ is called *e-permutable* if $[e]_{\Theta\circ\Phi} = [e]_{\Phi\circ\Theta}$ for each $\Theta, \Phi \in \operatorname{Con}\mathcal{A}$.

In [3] it has been proved that a variety $\mathcal{V}$ with a nullary operation e is ideal determined if and only if it is e-regular and e-permutable.

Proposition 1. *The variety of all LDIG is e-permutable and e-regular, so it is ideal determined.*

P r o o f. It is a straightforward consequence of ideal determination of the variety of all semiloops. □

In what follows we will give an explicit construction of the congruence Θ_H corresponding to an ideal H and list the basis of the clone IT(LDIG). For $\mathcal{A} \in$ LDIG and $\emptyset \neq B$, $C \subseteq A$ denote

$$B \cdot C = \{b \cdot c;\ b \in B, c \in C\} \quad \text{and} \quad B \backslash C = \{b \backslash c;\ b \in B, c \in C\}.$$

If e.g. $B = \{b\}$ is a one-element set, then we will write briefly $b \cdot C$ instead of $\{b\} \cdot C$, etc.

Lemma 1. *Let Θ be na equivalence on an algebra $\mathcal{A} \in$ LDIG. Then Θ is a congruence iff $[a]_\Theta \cdot [b]_\Theta \subseteq [a \cdot b]_\Theta$ and $[a]_\Theta \setminus [b]_\Theta \subseteq [a \setminus b]_\Theta$ for every $a, b \in A$.*

Proof. The conditions are nothing else than the substitution property of the relation Θ. $\square$

Theorem 1. *Let $\emptyset \neq H \subseteq A$ for $\mathcal{A} \in$ LDIG. Then the following conditions are equivalent:*

(1) *the relation Θ_H defined by $\langle x, y\rangle \in \Theta_H$ iff $x \cdot H = y \cdot H$ is a congruence with the kernel H;*

(2) *H is a subalgebra of $\mathcal{A}$ satisfying the conditions*

$$(x \cdot y) \setminus [(x \cdot H) \cdot (y \cdot H)] \subseteq H \quad \text{and} \quad (x \setminus y) \setminus [(x \cdot H) \setminus (y \cdot H)] \subseteq H.$$

Proof. (1) $\Rightarrow$ (2): Evidently, Θ_H is a congruence with classes $x \cdot H$ for $x \in A$. Then by Lemma 1 we obtain

$$(x \cdot H) \cdot (y \cdot H) \subseteq (x \cdot y) \cdot H \quad \text{and} \quad (x \cdot H) \setminus (y \cdot H) \subseteq (x \setminus y) \cdot H.$$

The former inclusion gives

$$(x \cdot y) \setminus [(x \cdot H) \cdot (y \cdot H)] \subseteq (x \cdot y) \setminus [(x \cdot y) \cdot H].$$

If $z \in (x \cdot y) \cdot H$, then $(z, x \cdot y) \in \Theta_H$ and so

$$((x \cdot y) \setminus z, (x \cdot y) \setminus (x \cdot y)) = ((x \cdot y) \setminus z, e) \in \Theta_H,$$

i.e. $(x \cdot y) \setminus z \in [e]_{\Theta_H} = e \cdot H = H$ (since H is the kernel of Θ_H).

We have proved $(x \cdot y) \setminus [(x \cdot H) \cdot (y \cdot H)] \subseteq H$. Analogously, the inclusion $(x \cdot H) \setminus (y \cdot H) \subseteq (x \setminus y) \cdot H$ implies $(x \setminus y) \setminus [(x \cdot H) \setminus (y \cdot H)] \subseteq (x \setminus y) \setminus [(x \setminus y) \cdot H]$. If $z \in (x \setminus y) \cdot H$, then $(z, x \setminus y) \in \Theta_H$, so

$$((x \setminus y) \setminus z, (x \setminus y) \setminus (x \setminus y)) = ((x \setminus y) \setminus z, e) \in \Theta_H, \quad \text{i.e.} \quad (x \setminus y) \setminus z \in [e]_{\Theta_H} = H,$$

which proves $(x \setminus y) \setminus [(x \cdot H) \setminus (y \cdot H)] \subseteq H$.

(2) $\Rightarrow$ (1): Suppose $z \in (x \cdot H) \cdot (y \cdot H)$. Then by the conditions (2) we have $(x \cdot y) \setminus z = h \in H$. From this we conclude

$$(x \cdot y) \cdot ((x \cdot y) \setminus z) = z = (x \cdot y) \cdot h \in (x \cdot y) \cdot H,$$

i.e. $(x{\cdot}H){\cdot}(y{\cdot}H) \subseteq (x{\cdot}y){\cdot}H$. Analogously, if $z \in (x{\cdot}H)\backslash(y{\cdot}H)$, then $(x\backslash y)\backslash z = h \in H$, which yields

$$z = (x \setminus y) \cdot h \in (x \setminus y) \cdot H.$$

We have proved $(x \cdot H) \setminus (y \cdot H) \subseteq (x \setminus y) \cdot H$, so by Lemma 1 the relation Θ_H is a congruence.

It suffices to show $e \cdot H = [e]_{\Theta_H} = H$. Since H is a subalgebra of $\mathcal{A}$, $e \cdot H \subseteq H$. Conversely, let $h \in H$. Then there exists an element $a \in S$ with $e \cdot a = h$ (e.g. $a = e \setminus h$). But then $e, h \in H$ implies $a = e \setminus h \in H$, which proves $h \in e \cdot H$ and $e \cdot H \subseteq H$. □

Theorem 2. *Let $\mathcal{A} \in$LDIG and $\emptyset \neq I \subseteq A$. Then the following conditions are equivalent:*
(1) *I is an ideal of $\mathcal{A}$;*
(2) *I is closed under the following ideal terms:*

$$\begin{aligned} p_1(y_1, y_2) &= y_1 \cdot y_2, \\ p_2(y_1, y_2) &= y_1 \setminus y_2, \\ p_3(x_1, x_2, y_1, y_2) &= (x_1 \cdot x_2) \setminus [(x_1 \cdot y_1) \cdot (x_2 \cdot y_2)], \\ p_4(x_1, x_2, y_1, y_2) &= (x_1 \setminus x_2) \setminus [(x_1 \cdot y_1) \setminus (x_2 \cdot y_2)]. \end{aligned}$$

P r o o f. (1) ⇒ (2) is evident.

(2) ⇒ (1): By Theorem 1 it suffices to show that if I is closed under the terms p_1, p_2, p_3, p_4 then I is a subalgebra of $\mathcal{A}$ with $(x \cdot y) \setminus [(x \cdot H) \cdot (y \cdot H)] \subseteq H$ and $(x \setminus y) \setminus [(x \cdot H) \setminus (y \cdot H)] \subseteq H$. The terms p_1, p_2 ensure that I is a subalgebra of $\mathcal{A}$, the term p_3 gives $(x{\cdot}y)\backslash[(x{\cdot}I){\cdot}(y{\cdot}I)] \subseteq I$ and the term p_4 gives $(x\backslash y)\backslash[(x{\cdot}I)\backslash(y{\cdot}I)] \subseteq I$. □

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