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ON THE MINIMUM OF THE WORK OF INTERACTION FORCES BETWEEN A PSEUDOPLATE AND A RIGID OBSTACLE

Igor Bock, Ján Lovíšek, Bratislava

Dedicated to Prof. J. Nečas on the occasion of his 70th birthday

Abstract. An optimization problem for the unilateral contact between a pseudoplate and a rigid obstacle is considered. The variable thickness of the pseudoplate plays the role of a control variable. The cost functional is a regular functional only in the smooth case. The existence of an optimal thickness is verified. The penalized optimal control problem is considered in the general case.

Keywords: elliptic variational inequality, pseudoplate, thickness, optimal control, penalization

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1. Introduction

We will deal with an optimization problem for the unilateral contact between an elastic pseudoplate and a rigid obstacle. The pseudoplate is a plate acting only upon shear stresses and a perpendicular load. The state problems are second order elliptic variational inequalities. The variable thickness of the plate appearing also on the right-hand side will play the role of control variables. The inner obstacle and the variable thickness imply that the convex set of admissible states depends on the control parameters. The cost functional represents the resultant of transverse contact forces between the pseudoplate and the obstacle. This resultant is a regular function only in the case of a sufficiently smooth obstacle. It is a Radon measure in the general case. Hence the existence result has only a generalized form. We

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shall investigate both the regular and the singular goal function. The penalized optimal control problem appears in the singular case. We shall verify the existence of an optimal thickness and the convergence of a sequence of penalized optimal controls and their corresponding states. We have considered the optimal design problems for the elastic and the viscoelastic plate with a variable thickness and a inner obstacle in [1], [2] with a control parameter dependent convex sets of admissible states and regular goal functions. Problems of these types for Reissner-Mindlin plate models have been solved in [5]. The G-convergence approach for nondifferentiable optimization problems was used by A. Myslinski and J. Sokolowski in the paper [9]. Contact problems of two elastic or elasto-plastic plates with variable thicknesses were investigated in the book of A. M. Khludnev and J. Sokolowski [6].

2. The elastic pseudoplate

We consider an isotropic plate occupying the domain

$$Q = \{(x,z) \in \mathbb{R}^3;\ x = (x_1, x_2) \in \Omega;\ -e(x) < z < e(x)\},$$

where Ω is a bounded simply connected domain in $\mathbb{R}^2$ with a Lipschitz boundary $\partial\Omega$. It is subjected to a perpendicular distributed load $f\colon\ \Omega \to \mathbb{R}$ and to its own weight. Considering shear stresses only we arrive at the stress-strain relations

$$\sigma_{i3} = \frac{kEe}{2(1+\mu)}\varepsilon_{i3}, \quad i = 1,2 \tag{1}$$

with a Young modulus $E > 0$, Poisson ratio $\mu \in (0, \frac{1}{2})$ and a shear correction coefficient $k > 0$.

We do not consider the angles of rotation of cross sections $x_i = \text{const.}$, $i = 1,2$. The strain-displacement relations have then the form

$$\varepsilon_{i3} = \frac{1}{2}\frac{\partial w}{\partial x_i}, \quad i = 1,2 \tag{2}$$

with a function $w\colon\ \Omega \to \mathbb{R}$ characterizing the deflection of the middle surface. The plate is clamped on its boundary.

The bending energy of the plate has the form

$$P_b = \frac{1}{2}\int_\Omega \sigma_{i3}\varepsilon_{i3}\,\mathrm{d}x = \frac{1}{2}\int_\Omega Ge(x)|\nabla w|^2\,\mathrm{d}x, \quad G = \frac{kE}{2(1+\mu)}. \tag{3}$$

Assuming the unilateral inner obstacle $\Phi\colon\ \Omega \to \mathbb{R}$ we obtain that the deflection $w\colon\ \Omega \to \mathbb{R}$ of the plate minimizes the functional of the total potential energy over the convex set

$$K(e) = \{v \in V\colon\ v \geqslant \Phi(x) + e(x) \text{ for a.e. } x \in \Omega\}, \tag{4}$$

where

$$V \equiv H_0^1(\Omega) = \{v \in H^1(\Omega)\colon\ v = 0 \text{ on } \partial\Omega \text{ in the sense of traces}\}$$

is a Hilbert space with the inner product and the norm

$$((u,v)) = \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x, \quad \|v\| = ((v,v))^{\frac12},\ u,v \in V.$$

Let us denote by $U = C(\overline{\Omega})$ the Banach space of all continuous functions $e\colon\ \overline{\Omega} \to \mathbb{R}$ with the norm

$$\|e\|_U = \max_{x\in\overline{\Omega}} |e(x)|,\ e \in U.$$

Further we introduce the set of admissible thickness-functions

$$\begin{aligned} U_{\mathrm{ad}} = \Big\{&e \in C^{0,1}(\overline{\Omega})\colon\ 0 < e_{\min} \leqslant e(x) \leqslant e_{\max} \text{ on } \Omega, \\ &\Big|\frac{\partial e}{\partial x_i}\Big| \leqslant C_i,\ i=1,2;\ \text{a.e. on } \Omega,\ \int_\Omega e(x)\,\mathrm{d}x = C_3,\ e(s) = \xi(s) \text{ on } \partial\Omega\Big\}, \end{aligned} \tag{5}$$

where C_1, C_2, C_3 are given constants, ξ is a given continuous function and $C^{0,1}(\overline{\Omega})$ is the set of all Lipschitz-continuous functions on $\overline{\Omega}$. Due to the Ascoli-Arzela theorem is the set U_{ad} compact in the Banach space U. We suppose that the obstacle function fulfils the conditions

$$\Phi \in H^1(\Omega) \cap C(\overline{\Omega}), \quad \Phi(s) + \xi(s) \leqslant 0 \quad \text{for all } s \in \partial\Omega. \tag{6}$$

The convex set $K(e)$ of admissible states is then closed and nonempty for every $e \in U_{\mathrm{ad}}$, because we have

$$u(e) \in K(e), \quad u(e) = \max\{0, \Phi + e\}.$$

Assuming the own weight of the plate with the density $\varrho > 0$ and the perpendicular load $f \in L_2(\Omega)$ acting at the upper plane we obtain the deflection

$$w \equiv w(e) \in K(e)$$

as a solution of the elliptic variational inequality

$$\begin{aligned}\int_\Omega & Ge(x)\nabla w(x)\cdot\nabla(v-w)(x)\,\mathrm{d}x \\ &\geqslant \int_\Omega (f(x)-2\varrho e(x))(v-w)(x)\,\mathrm{d}x \quad \text{for all } v\in K(e)\end{aligned} \tag{7}$$

or in the operator form

$$\langle A(e)w(e), v-w(e)\rangle \geqslant \langle L(e), v-w(e)\rangle \quad \text{for all } v\in K(e) \tag{8}$$

with the operators $A(e)\colon V\to V^*$ and $L(e)\in V^*$ (V^* is the dual of V) defined by

$$\langle A(e)u, v\rangle = \int_\Omega Ge(x)\nabla u(x)\cdot\nabla v(x)\,\mathrm{d}x,\quad u,v\in V, \tag{9}$$

$$\langle L(e), v\rangle = \int_\Omega (f(x)-2\varrho e(x))v(x)\,\mathrm{d}x,\quad v\in V. \tag{10}$$

The following theorem about the existence, uniquenes and the continuous dependence $e\to w(e)$ holds:

Theorem 1. *For every $e\in U_{\mathrm{ad}}$ there exists a unique solution $w(e)\in K(e)$ of the variational inequality* (8). *A function $w(\cdot)\colon U_{\mathrm{ad}}\to V$, ($U_{\mathrm{ad}}\subset U$) is continuous in the weak topology of V i.e.*

$$e_n\to e_0 \text{ in } U \Longrightarrow w(e_n)\rightharpoonup w(e_0) \text{ in } V. \tag{11}$$

Proof. The existence and uniqueness of $w(e)$ is a classical result from the theory of elliptic variational inequalities ([7]). The continuity result was verified in Lemma 2.3 of the paper [1]. □

The regularity of a solution will play an important role in the optimal control problem formulated in the next section. The regularity depends in the same way as in the linear case on the smoothness of the boundary, coefficients and the right-hand side, but the essential difference is that the obstacle constraint does not allow to surpass the regularity $w\in W^{2,p}(\Omega)$ for a solution $w\equiv w(e)$.

Let us recall some connections between continuous functionals and Borel measures. We denote by $C_c(\Omega)$ the space of all continuous functions with a compact support in Ω. A sequence $\varphi_n\in C_c(\Omega)$ converges to $\varphi\in C_c(\Omega)$, if the supports of the functions φ_n belong to a compact subset of Ω and φ_n converges to φ uniformly on Ω.

By the representation theorem due to Riesz and Schwartz [11] every continuous linear functional T over $C_c(\Omega)$ can be represented by an integral

$$\langle T, \varphi\rangle = \int_\Omega \varphi \, \mathrm{d}\mu, \quad \forall \varphi \in C_c(\Omega), \tag{12}$$

where μ belongs to the set $M(\Omega)$ of regular Borel measures defined on Ω. Any signed measure can be in a unique way expressed as the difference of two positive disjoint measures $\mu = \mu^+ - \mu^-$. A linear continuous functional T on $C_c(\Omega)$ is said to be positive, if $T(\varphi) \geqslant 0$ for all $\varphi \in C_c(\Omega)$, $\varphi(x) \geqslant 0$. A linear continuous functional on $C_c(\Omega)$ is positive if and only if it is represented by a nonnegative Borel measure μ. Positive functionals on $C_c(\Omega)$ possess an important property verified in [6]:

Theorem 2. *Let a functional T be linear and positive on $C_c(\Omega)$. Then T is continuous and can be represented in the form* (12) *with a nonnegative measure μ.*

Let us assume that a solution of the variational inequality (7) possesses the regularity $w \in H^2(\Omega)$. Let $\mathcal{D}(\Omega)$ be the set of all infinitely times differentiable functions $\varphi\colon \Omega \to \mathbb{R}$ with compact supports in Ω. Setting $v = w + \varphi$, $\varphi \in \mathcal{D}(\Omega)$, $\varphi \geqslant 0$ it can be verified applying the Green theorem that the variational inequality (7) is equivalent to the complementary problem

$$\begin{aligned} &w(x) - \Phi(x) - e(x) \geqslant 0, \ -\mathrm{div}(Ge(x)\nabla w(x)) - f(x) + \varrho e(x) \geqslant 0 \text{ and} \\ &(w(x) - \Phi(x) - e(x))(-\mathrm{div}(Ge(x)\nabla w(x)) - f(x) + \varrho e(x)) = 0 \text{ a.e. in } \Omega. \end{aligned} \tag{13}$$

In the generalized case (8) the difference $A(e)w(e) - L(e)$ can be expressed as a positive Borel measure as included in the next regularity theorem due to Rodriguez [10].

Theorem 3. *Let the operator $A(e)\colon V \to V^*$ and the functional $L(e) \in V^*$ be defined by* (9), (10) *and let*

$$e \in U_{\mathrm{ad}}, \ \Phi \in H^1(\Omega) \cap C(\overline{\Omega}), \ \Phi(s) + e(s) \leqslant 0 \text{ on } \partial\Omega. \tag{14}$$

(i) *If*

$$A(e)(\Phi + e) = \nu \in M(\Omega), \tag{15}$$

then $A(e)w(e) - L(e) = \mu(e) \in M(\Omega) \cap V^*$ and

$$0 \leqslant \mu(e) \leqslant (\nu - L(e))^+ \text{ in } M(\Omega). \tag{16}$$

Simultaneously $w(e) \in C(\overline{\Omega}) \cap V$ *and the nonegative measure* $A(e)w(e) - L(e)$ *is such that*

$$\operatorname{supp}(A(e)w(e) - L(e)) \subset I \equiv \{x \in \Omega\colon\ w(e)(x) = \Phi(x) + e(x)\}, \tag{17}$$

$$A(e)w(e) = L(e) \text{ in } \Omega \setminus I = \{x \in \Omega\colon\ w(e)(x) > \Phi(x) + e(x)\}. \tag{18}$$

(ii) *If* $(A(e)(\Phi + e) - f + \varrho e)^+ \in L^p(\Omega)$, $1 \leqslant p \leqslant \infty$ *then*

$$A(e)w(e) \in L^p(\Omega) \tag{19}$$

and

$$0 \leqslant A(e)w(e) - f + \varrho e \leqslant (A(e)(\Phi + e) - f + \varrho e)^+ \text{ a.e. in } \Omega. \tag{20}$$

Moreover, $w(e) \in W^{2,p}(\Omega) \cap V$, $1 < p < \infty$ *if* $\partial\Omega \in C^{1,1}$.

3. Optimal design problems

Let us impose the regularity assumption $\Phi \in H^2(\Omega)$ on the function characterizing the obstacle. We introduce the following goal function characterizing the work of interaction forces:

$$\begin{aligned} j(e) &= \int_\Omega [A(e)w(e) - L(e)](x)\,\mathrm{d}x \\ &= \int_\Omega [-\operatorname{div}(Ge\nabla w(e)) - f + \varrho e](x)\,\mathrm{d}x,\ \ e \in \widetilde{U}_{\mathrm{ad}}, \end{aligned} \tag{21}$$

where

$$\widetilde{U}_{\mathrm{ad}} = \{e \in U_{\mathrm{ad}} \cap H^2(\Omega)\colon\ \|e\|_2 \leqslant C_4\}. \tag{22}$$

The integrals in (21) are well defined due to the assertion $A(e)w(e) \in L^2(\Omega)$ from Theorem 3 (ii). We formulate
Optimal Design Problem P:

$$e_0 = \arg\min_{e \in \widetilde{U}_{\mathrm{ad}}} j(e). \tag{23}$$

Theorem 4. *There exists a solution* e_0 *of the Optimal Design Problem P.*

P r o o f. We have $j(e) \geqslant 0$ for all $e \in \widetilde{U}_{\mathrm{ad}}$. Let $\{e_n\} \subset \widetilde{U}_{\mathrm{ad}}$ be the minimizing sequence:

$$\lim_{n\to\infty} j(e_n) = \inf_{e\in\widetilde{U}_{\mathrm{ad}}} j(e). \tag{24}$$

There exists a subsequence of $\{e_n\}$ (again denoted by $\{e_n\}$) such that

$$e_n \rightharpoonup e_0 \text{ (weakly) in } H^2(\Omega). \tag{25}$$

We have simultaneously

$$\frac{\partial e_n}{\partial x_i} \rightharpoonup^* \frac{\partial e_0}{\partial x_i} \text{ (weakly-star) in } L^\infty(\Omega), \tag{26}$$

$$e_n \to e_0 \text{ (uniformly) in } U = C(\overline{\Omega}) \tag{27}$$

and hence $e_0 \in \widetilde{U}_{\mathrm{ad}}$.

Theorem 1 then implies the relation

$$w(e_n) \rightharpoonup w(e_0) \text{ (weakly) in } V. \tag{28}$$

Simultaneously we obtain the convergence of the corresponding sequence of operators

$$A(e_n) \to A(e_0) \text{ in } \mathcal{L}(V, V^*), \tag{29}$$

where $\mathcal{L}(V, V^*)$ is the Banach space of all linear continuous operators operating from V into V^*. Let us denote

$$F(e) \equiv A(e)w(e) - L(e) = -\operatorname{div}(Ge\nabla w(e)) - f + \varrho e \in L_2(\Omega), \quad e \in \widetilde{U}_{\mathrm{ad}}.$$

We have the relation

$$\int_\Omega F(e)v \,\mathrm{d}x = \int_\Omega [Ge(x)\nabla w(e)(x) \cdot \nabla v(x) - (f - \varrho e)(x)v(x)] \,\mathrm{d}x, \quad v \in V. \tag{30}$$

The convergences (28), (29) imply

$$\lim_{n\to\infty} \int_\Omega F(e_n)v \,\mathrm{d}x = \int_\Omega F(e)v \,\mathrm{d}x \text{ for all } v \in V. \tag{31}$$

The set $\widetilde{U}_{\mathrm{ad}}$ is bounded in $H^2(\Omega)$. The upper estimate (20) then implies the boundedness of the set $\{F(e)\}$ in $L_2(\Omega)$:

$$\Big[\int_\Omega F(e)^2 \,\mathrm{d}x\Big]^{1/2} = \|F(e)\|_0 \leqslant C \quad \text{for all } e \in \widetilde{U}_{\mathrm{ad}}. \tag{32}$$

The set V is dense in $L_2(\Omega)$, the sequence $\{F(e_n)\}$ is bounded in $L_2(\Omega)$ and we obtain from (31), (32) that the sequence $\{F(e_n)\}$ is weakly convergent in $L_2(\Omega)$. We can then set $v \equiv 1$ in (31) and the relations

$$\lim_{n\to\infty} j(e_n) = \lim_{n\to\infty} \int_\Omega F(e_n)\,\mathrm{d}x = \int_\Omega F(e_0)\,\mathrm{d}x = j(e_0) \tag{33}$$

follow. The convergence (24) implies

$$j(e_0) = \min_{e\in\widetilde{U}_{\mathrm{ad}}} j(e) \tag{34}$$

and hence e_0 is a solution of the Optimal Design Problem P, which concludes the proof. □

We shall continue with the case of the admissible set of controls U_{ad} defined in (5) and the obstacle function Φ fulfilling only (6).

The regularity $A(e)w(e) \in L_2(\Omega)$ does not hold in general. We consider instead of the cost functional $j(e)$ from (21) the functional corresponding to the penalized problem

$$A(e)w_\varepsilon(e) + \tfrac{1}{\varepsilon}\beta(e, w_\varepsilon(e)) = L(e),\ \varepsilon > 0; \tag{35}$$

where

$$\beta(e,v)(x) = -(v - \Phi - e)_-(x),\ v \in V. \tag{36}$$

The operator $A(e) + \frac{1}{\varepsilon}\beta(e,.)\colon V \to V^*$ is bounded, hemicontinuous, monotone and coercive and due to Theorem 2.2.1 from [8] for every $e \in U_{\mathrm{ad}}$ there exists a solution $w_\varepsilon(e) \in V$ of the penalized equation (35). We can then formulate
Optimal Design Problem P_ε:

$$e_\varepsilon = \arg\min_{e\in U_{\mathrm{ad}}} j_\varepsilon(e), \tag{37}$$

where

$$\begin{aligned} j_\varepsilon(e) &= \int_\Omega [A(e)w_\varepsilon(e) - L(e)](x)\,\mathrm{d}x \\ &= \int_\Omega [-\mathrm{div}(Ge\nabla w_\varepsilon(e)) - f + \varrho e](x)\,\mathrm{d}x,\ e \in U_{\mathrm{ad}}. \end{aligned} \tag{38}$$

The integrals are well defined, because $w_\varepsilon \in V$ fulfils the equation (35) with $\frac{1}{\varepsilon}\beta(e, w_\varepsilon(e)) - L(e) \in L_2(\Omega)$. Using the standard compactness and monotonicity methods we can derive for $\varepsilon \to 0+$ the strong convergence in V of the sequence $\{w_{\varepsilon_n}(e)\}$ of solutions of the penalized problem (35) with $\varepsilon \equiv \varepsilon_n$ to a solution $w(e) \in K(e)$ of the original state variational inequality (8). We shall use a similar approach in order to analyze the convergence of sequences $\{e_{\varepsilon_n}\}$ of penalized optimal controls and the corresponding state solutions $\{w_{\varepsilon_n}(e_{\varepsilon_n})\}$.

Theorem 5. *For every $\varepsilon > 0$ there exists a solution e_ε of the Optimal Design Problem P_ε.*

If $\{\varepsilon_n\}$ is a sequence fulfilling $\varepsilon_n > 0$, $\varepsilon_n \to 0$, then there exists its subsequence (again denoted by ε_n) such that

$$e_{\varepsilon_n} \to e_* \in U_{\text{ad}} \quad \textit{in } U = C(\overline{\Omega}), \tag{39}$$

$$w_{\varepsilon_n}(e_{\varepsilon_n}) \to w(e_*) \textit{ in } V, \tag{40}$$

where $w_{\varepsilon_n}(e_{\varepsilon_n}) \in V$ fulfils the penalized equation

$$A(e_{\varepsilon_n})w_{\varepsilon_n}(e_{\varepsilon_n}) + \tfrac{1}{\varepsilon_n}\beta(e_{\varepsilon_n}, w_{\varepsilon_n}(e_{\varepsilon_n})) = L(e_{\varepsilon_n}) \tag{41}$$

and $w(e_*) \in K(e_*)$ solves the variational inequality

$$\langle A(e_*)w(e_*), v - w(e_*)\rangle \geqslant \langle L(e_*), v - w(e_*)\rangle \textit{ for all } v \in K(e_*). \tag{42}$$

Proof. Let $\{e_n\} \subset U_{\text{ad}}$ be a minimizing sequence for the functional j_ε. The set U_{ad} is compact in the Banach space $U = C(\overline{\Omega})$ and hence it contains a subsequence (again denoted by $\{e_n\}$) fulfilling

$$e_n \to e_\varepsilon \in U_{\text{ad}} \text{ in } U = C(\overline{\Omega}). \tag{43}$$

Let $w_\varepsilon(e_n)$ be the corresponding sequence of solutions to the penalized problem

$$A(e_n)w_\varepsilon(e_n) + \tfrac{1}{\varepsilon}\beta(e_n, w_\varepsilon(e_n)) = L(e_n). \tag{44}$$

Applying the inequality

$$(\beta(e_n, w_\varepsilon(e_n)), w_\varepsilon(e_n) - \Phi - e_n)_0 = \int_\Omega \beta(e_n, w_\varepsilon(e_n))(w_\varepsilon(e_n) - \Phi - e_n)\,\mathrm{d}x \geqslant 0$$

and the uniform coercivity of the operators $\{A(e_n)\}$ we obtain the boundedness of the sequence $\{w_\varepsilon(e_n)\}$ in the space V. Then there exists its subsequence (again denoted by $\{w_\varepsilon(e_n)\}$) fulfilling

$$w_\varepsilon(e_n) \rightharpoonup w_\varepsilon \text{ weakly in } V \tag{45}$$

and

$$w_\varepsilon(e_n) \to w_\varepsilon \ \text{ strongly in } L_2(\Omega). \tag{46}$$

Expressing the penalty functional in the form

$$\beta(e,w) = -(w-\Phi-e)_- = \tfrac{1}{2}(w-\Phi-e-|w-\Phi-e|)$$

we conclude that it is continuous as a function $\beta(\cdot,\cdot)\colon\, U \times L_2(\Omega) \to L_2(\Omega)$. The equality

$$A(e_\varepsilon)w_\varepsilon + \tfrac{1}{\varepsilon}\beta(e_\varepsilon, w_\varepsilon) = L(e_\varepsilon) \tag{47}$$

then follows from

$$A(e_n)w_\varepsilon(e_n) \rightharpoonup A(e_\varepsilon)w_\varepsilon(e_\varepsilon) \text{ in } V^*$$

and hence $w_\varepsilon \equiv w_\varepsilon(e_\varepsilon)$.

Simultaneously we have $A(e_\varepsilon)w_\varepsilon(e_\varepsilon) \in L_2(\Omega)$ and

$$A(e_n)w_\varepsilon(e_n) \rightharpoonup A(e_\varepsilon)w_\varepsilon(e_\varepsilon) \text{ in } L_2(\Omega). \tag{48}$$

The last convergence is a consequence of the weak convergence of the same sequence in the dual space V^*, its boundedness in $L_2(\Omega)$ and the density of the space V in $L_2(\Omega)$. Then

$$\lim_{n\to\infty} j_\varepsilon(e_n) = j_\varepsilon(e_\varepsilon)$$

and $e_\varepsilon \in U_{\mathrm{ad}}$ is a solution of the Optimal Design Problem P_ε.

Let $\varepsilon_n > 0,\ n = 1,2,\ldots$ fulfil $\varepsilon_n \to 0$. The sequence $\{e_{\varepsilon_n}\}$ of penalized optimal thicknesses is bounded in U_{ad} and contains a subsequence (again denoted by $\{e_{\varepsilon_n}\}$) fulfilling the convergence (39). Simultaneously we have

$$(e_{\varepsilon_n} - e_*) \rightharpoonup 0 \text{ in } V. \tag{49}$$

We have the identity

$$\begin{aligned}\langle A(e_{\varepsilon_n})&w_{\varepsilon_n}(e_{\varepsilon_n}), w_{\varepsilon_n}(e_{\varepsilon_n}) - (\Phi + e_{\varepsilon_n})_+\rangle\\ &- \tfrac{1}{\varepsilon_n}((w_{\varepsilon_n}(e_{\varepsilon_n}) - \Phi - e_{\varepsilon_n})_-, w_{\varepsilon_n}(e_{\varepsilon_n}) - (\Phi + e_{\varepsilon_n})_+)_0\\ &= \langle L(e_{\varepsilon_n}, w_{\varepsilon_n}(e_{\varepsilon_n}) - (\Phi + e_{\varepsilon_n})_+\rangle.\end{aligned}$$

The penalizing member of the previous identity is nonnegative and the sequence $\{w_{\varepsilon_n}(e_{\varepsilon_n})\}$ of solutions to the penalized problem (41) is bounded in V. Then there exists its subsequence (again denoted by $\{w_{\varepsilon_n}(e_{\varepsilon_n})\}$) and an element $u \in V$ such that

$$w_{\varepsilon_n}(e_{\varepsilon_n}) \rightharpoonup u \text{ (weakly) in } V, \tag{50}$$

$$w_{\varepsilon_n}(e_{\varepsilon_n}) \to u \text{ (strongly) in } L_2(\Omega). \tag{51}$$

The boundedness of the sequence $\{w_{\varepsilon_n}(e_{\varepsilon_n})\}$ in V implies the estimate

$$\|\beta(e_{\varepsilon_n}, w_{\varepsilon_n}(e_{\varepsilon_n})\|_* \leqslant C\varepsilon_n,\ C > 0,\ n = 1, 2, \ldots$$

and combining it with (39), (51) we obtain

$$\beta(e_*, u) = (u - \Phi - e_*)_- = 0, \quad u \in K(e_*). \tag{52}$$

Let $w(e_*) \in K(e_*)$ be a solution of the variational inequality (42). Then we have the inequality

$$\langle A(e_*)w(e_*), u - w(e_*)\rangle \geqslant \langle L(e_*), u - w(e_*)\rangle. \tag{53}$$

We have simultaneously the identity

$$\begin{aligned}
&\langle A(e_{\varepsilon_n})w_{\varepsilon_n}(e_{\varepsilon_n}), w_{\varepsilon_n}(e_{\varepsilon_n}) - w(e_*) + e_{\varepsilon_n} - e_*\rangle \\
&\qquad - \tfrac{1}{\varepsilon_n}((w_{\varepsilon_n}(e_{\varepsilon_n}) - \Phi - e_{\varepsilon_n})_-, w_{\varepsilon_n}(e_{\varepsilon_n}) - \Phi - e_{\varepsilon_n})_0 \\
&\qquad + \tfrac{1}{\varepsilon_n}((w_{\varepsilon_n}(e_{\varepsilon_n}) - \Phi - e_{\varepsilon_n})_-, w(e_*) - \Phi - e_*)_0 \\
&\quad = \langle L(e_{\varepsilon_n}, w_{\varepsilon_n}(e_{\varepsilon_n}) - w(e_*) + e_{\varepsilon_n} - e_*\rangle.
\end{aligned} \tag{54}$$

After adding (53) and (54) we obtain the inequality

$$\begin{aligned}
&\langle A(e_{\varepsilon_n})[w_{\varepsilon_n}(e_{\varepsilon_n}) - w(e_*)], w_{\varepsilon_n}(e_{\varepsilon_n}) - w(e_*)\rangle + \langle A(e_*)w(e_*), w_{\varepsilon_n}(e_{\varepsilon_n}) - u\rangle \\
&\quad + \langle [A(e_{\varepsilon_n}) - A(e_*)]w(e_*), w_{\varepsilon_n}(e_{\varepsilon_n}) - w(e_*)\rangle + \langle A(e_{\varepsilon_n})w(e_*), e_{\varepsilon_n} - e_*\rangle \\
&\quad + \langle A(e_{\varepsilon_n})[w_{\varepsilon_n}(e_{\varepsilon_n}) - w(e_*)], e_{\varepsilon_n} - e_*\rangle \\
&\leqslant \langle L(e_{\varepsilon_n}) - L(e_*), w_{\varepsilon_n}(e_{\varepsilon_n}) - w(e_*)\rangle \\
&\quad + \langle L(e_*), w_{\varepsilon_n}(e_{\varepsilon_n}) - u\rangle + \langle L(e_{\varepsilon_n}), e_{\varepsilon_n} - e_*\rangle.
\end{aligned}$$

The members with fractions $\frac{1}{\varepsilon_n}$ in (54) are nonnegative and hence we can omit them. The uniform coerciveness of $\{A(e)\}$, the convergences (39), (49), (50) and the continuity properties of the operators $A(\cdot)\colon U \to \mathcal{L}(V, V^*)$, $L(\cdot)\colon U \to V^*$ then imply $u = w(e_*)$ and the convergence (40), which concludes the proof. $\square$

R e m a r k 1. The limit thickness-function e_* from the previous theorem can be regarded as the generalized optimal control for the Optimal Control Problem P. Using the Radom measure expression from Theorem 3 we can express it as a solution of

O p t i m a l D e s i g n P r o b l e m P_μ:

$$e_* = \arg \min_{e \in U_{\mathrm{ad}}} \int_\Omega \mathrm{d}\mu(e). \tag{55}$$

R e m a r k 2. The finite element approximations and their convergence can be stated and verified in a similar way as in the paper [4] or in the monograph [3]. Bicubic polynomials have to be used to approximate both the variable thicknesses and the deflections in the case of Optimal Design Problem P and a rectangular division of the region Ω. The thicknesses in the penalized Problem P_ε can be approximated by bilinear polynomials.

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Authors' addresses: *Igor Bock*, Department of Mathematics, Faculty of Electr. Engineering and Inform. Tech., Slovak University of Technology, Ilkovičova 3, 812 19 Bratislava, Slovakia, e-mail: `bock@kmat.elf.stuba.sk`; *Ján Lovíšek*, Department of Mechanics, Faculty of Civil Engineering, Slovak University of Technology, Radlinského 11, 813 68 Bratislava, Slovakia, e-mail: `lovisek@svf.stuba.sk`.