STRATIDISTANCE IN STRATIFIED GRAPHS

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Abstract. A graph G is a stratified graph if its vertex set is partitioned into classes (each of which is a stratum or a color class). A stratified graph with k strata is k-stratified. If G is a connected k-stratified graph with strata S_i ($1 \leqslant i \leqslant k$) where the vertices of S_i are colored X_i ($1 \leqslant i \leqslant k$), then the X_i-proximity $\varrho_{X_i}(v)$ of a vertex v of G is the distance between v and a vertex of S_i closest to v. The strati-eccentricity $\mathrm{se}(v)$ of v is $\max\{\varrho_{X_i}(v) \mid 1 \leqslant i \leqslant k\}$. The minimum strati-eccentricity over all vertices of G is the stratiradius $\mathrm{sr}(G)$ of G; while the maximum strati-eccentricity is its stratidiameter $\mathrm{sd}(G)$. For positive integers a, b, k with $a \leqslant b$, the problem of determining whether there exists a k-stratified graph G with $\mathrm{sr}(G) = a$ and $\mathrm{sd}(G) = b$ is investigated.

A vertex v in a connected stratified graph G is called a straticentral vertex if $\mathrm{se}(v) = \mathrm{sr}(G)$. The subgraph of G induced by the straticentral vertices of G is called the straticenter of G. It is shown that every ℓ-stratified graph is the straticenter of some k-stratified graph. Next a stratiperipheral vertex v of a connected stratified graph G has $\mathrm{se}(v) = \mathrm{sd}(G)$ and the subgraph of G induced by the stratiperipheral vertices of G is called the stratiperiphery of G. Almost every stratified graph is the stratiperiphery of some k-stratified graph. Also, it is shown that for a k_1-stratified graph H_1, a k_2-stratified graph H_2, and an integer $n \geqslant 2$, there exists a k-stratified graph G such that H_1 is the straticenter of G, H_2 is the stratiperiphery of G, and $d(H_1, H_2) = n$.

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1. Introduction

Dividing the vertex set of a graph into classes according to some prescribed rule is a fundamental process in graph theory. The vertices of a graph can be divided into cut-vertices and non-cut-vertices. Equivalently, the vertices of a tree are divided into its leaves and non-leaves. The vertex set of a graph is partitioned according to the degrees of its vertices. When studying distance, the vertices of a connected graph are partitioned according to their eccentricities. Also, in a connected rooted graph, the vertices are partitioned according to their distance from the root. Perhaps the best known example of this process, however, is graph coloring, where the vertex set of a graph is partitioned into classes each of which is independent in the graph.

In VLSI design, the design of computer chips often yields a division of the nodes into several layers each of which must induce a planar subgraph. So here too the vertex set of a graph is divided into classes. Motivated by these observations, Rashidi [3] defined a graph G to be a stratified graph if its vertex set is partitioned into classes.

Formally, then, a graph G is a *stratified graph* if its vertex set $V(G)$ is partitioned into classes, called *strata*. Each class then is a *stratum*. If there are k strata, then G is called a *k-stratified graph*. A 1-stratified graph is then simply a graph, as is an n-stratified graph of order n. Normally, we denote the strata of a k-stratified graph by $S_1, S_2, \ldots, S_k$. The strata are also referred to as *color classes*, where the vertices of S_i are colored X_i $(1 \leqslant i \leqslant k)$. When specific colors are employed, we use red (R) for X_1, blue (B) for X_2, and yellow (Y) for X_3. So the vertices of S_1 are colored red.

In [3] Rashidi studied a number of problems involving stratified graphs; while distance in stratified graphs was investigated in Chartrand, Holley, Rashidi, and Sherwani [2] and Chartrand, Eroh, Rashidi, Schultz, and Sherwani [1].

In the present paper we are interested in problems concerning distance in stratified graphs, which also have sociological applications. To illustrate such an application, suppose that a councilperson in a large city is looking for a location for his or her office. This conscientious public servant wishes to serve and be available to all of the various ethnic groups within the city. Typically, ethnic groups live in clusters of neighborhoods in various parts of the city. Each ethnic group feels that their concerns are of sufficient importance that the councilperson's office should be located in close proximity to some neighborhood in which the ethnic group lives. If we consider the street intersections of the city as vertices, street segments as edges, and a vertex colored according to the ethnic group most notably represented by the particular neighborhood involved, then we are led to a new application of stratified graphs.

Let G be a connected k-stratified graphs with strata $S_1, S_2, \ldots, S_k$ the colors of whose vertices are denoted by $X_1, X_2, \ldots, X_k$, respectively. For a vertex v of G, the *X_i-proximity* $\varrho_{X_i}(v)$ is the distance between v and a vertex of S_i closest to v. Clearly, $\varrho_{X_i}(v) = 0$ if and only if v is colored X_i. The *proximity vector* $\varrho(v)$ of v is the k-vector $(\varrho_{X_1}(v), \varrho_{X_2}(v), \ldots, \varrho_{X_k}(v))$, which, then, has exactly one coordinate equal to 0. The *strati-eccentricity* or, more simply, the *s-eccentricity* $\mathrm{se}(v)$ of v is defined by

$$\mathrm{se}(v) = \max\{\varrho_{X_i}(v) \mid 1 \leqslant i \leqslant k\}.$$

The minimum s-eccentricity among all vertices of G is called the *stratiradius* or *s-radius* $\mathrm{sr}(G)$ of G; while the maximum s-eccentricity is the *stratidiameter* or *s-diameter* $\mathrm{sd}(G)$. Clearly, $\mathrm{sr}(G) \leqslant \mathrm{sd}(G)$ for every connected stratified graph G. The vertices of the 3-stratified graph G of Figure 1 are labeled with their s-eccentricities. Consequently for this graph G, $\mathrm{sr}(G) = 2$ and $\mathrm{sd}(G) = 5$.

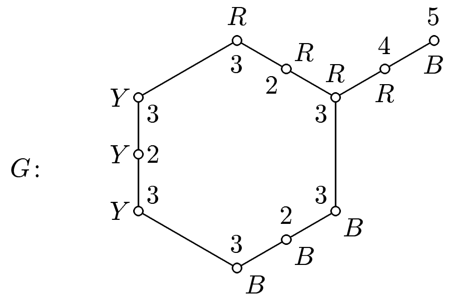

Figure 1. The s-eccentricities of the vertices of a 3-stratified graph

In this section, we consider the question: For which positive integers a and b with $a \leqslant b$, does there exist a k-stratified graph G $(k \geqslant 2)$ with $\mathrm{sr}(G) = a$ and $\mathrm{sd}(G) = b$? We begin with $a = 1$. Before continuing, the following notation will be useful. For positive integers s and t, the *comet* $C_{s,t}$ denotes the tree obtained by identifying the center of the star $K_{1,s}$ with an end-vertex of the path P_t of length $t-1$. So, $C_{s,1} = K_{1,s}$ and $C_{1,n-1} = P_n$.

Proposition 1. *For positive integers k and b with $k \geqslant 2$, there exists a k-stratified graph G with $\mathrm{sr}(G) = 1$ and $\mathrm{sd}(G) = b$.*

Proof. If $b = 1$, then a k-coloring of a complete graph of order k produces a k-stratified graph G with $\mathrm{sr}(G) = \mathrm{sd}(G) = 1$. For $b \geqslant 2$, color the vertices of $C_{k-1,b}$ as follows: color the b vertices on the tail P_b, including the center of the star, of

$C_{k-1,b}$ with the color X_1 and assign the remaining $k-1$ colors $X_2, X_3, \ldots, X_k$ to the end-vertices of the star. This produces a k-stratified graph G with $\operatorname{sr}(G) = 1$ and $\operatorname{sd}(G) = b$. □

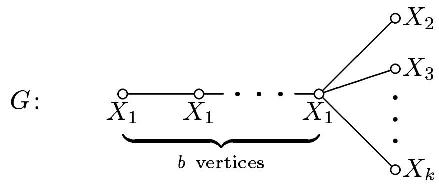

Figure 2. A k-stratified graph G with $\operatorname{sr}(G) = 1$ and $\operatorname{sd}(G) = b$

Next we characterize the s-radius and s-diameter of a 2-stratified graph.

Theorem 2. *For positive integers a and b with $b \geqslant a \geqslant 1$, there exists a connected 2-stratified graph G with $\operatorname{sr}(G) = a$ and $\operatorname{sd}(G) = b$ if and only if $a = 1$ and b is any positive integer.*

Proof. Let G be a connected 2-stratified graph whose strata are colored red and blue. Necessarily, since G is connected, a red vertex must be adjacent to a blue vertex and thus strad $G = 1$.

By proposition 1, for every positive integer b, there exists a 2-stratified graph G such that $\operatorname{sr}(G) = 1$ and $\operatorname{sd}(G) = b$. □

Hence in what follows, we restrict our attention to k-stratified graphs for $k \geqslant 3$. We begin with 3-stratified graphs and show that a 3-stratified graph G with $\operatorname{sr}(G) = a$ and $\operatorname{sd}(G) = b$ must satisfy $b \geqslant 2a-2$. Furthermore, we show that for every pair a, b of positive integers with $b \geqslant 2a-2$, there exists a 3-stratified graph G with $\operatorname{sr}(G) = a$ and $\operatorname{sd}(G) = b$. First, we show that for all positive integers $k \geqslant 3$ and $a \geqslant 2$, there exists a k-stratified graph with s-radius a and s-diameter $2a-2$.

Proposition 3. *For positive integers a and k with $k \geqslant 3$ and $a \geqslant 2$, there exists a k-stratified graph G with $\operatorname{sr}(G) = a$ and $\operatorname{sd}(G) = 2a-2$.*

Proof. Let $X_1, X_2, \ldots, X_k$ denote k distinct colors. We construct a k-stratified graph G as follows. Take a u-v path P on $4a-4$ vertices. Color the first $2a-2$ vertices on the path, including u, with X_1 and color the remaining $2a-2$ vertices, including v, with X_2. Join u and v with $k-2$ edges and then subdivide each of these edges $2a-2$ times. Let $Q_3, Q_4, \ldots, Q_k$ denote the resulting $k-2$ u-v paths of length $2a-1$. For $i = 3, 4, \ldots, k$, color the $2a-2$ internal vertices of the path Q_i with X_i. Let G denote the resulting k-stratified graph. Note that $\operatorname{se}(u) = \operatorname{se}(v) = 2a-2$ and the vertex x colored X_1 at distance $a-1$ from u has $\mathrm{se}(x)=a$ while every other vertex of G has s-eccentricity at least a and at most $2a-2$. Thus $\mathrm{sr}(G)=a$ and $\mathrm{sd}(G)=2a-2$. □

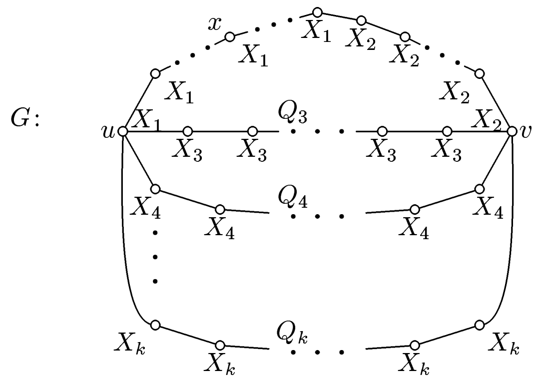

Figure 3. A k-stratified graph G with $\mathrm{sr}(G)=a$ and $\mathrm{sd}(G)=2a-2$

Next we show that for positive integers a and b with $b \geqslant 2a-1$, there exists a k-stratified ($k \geqslant 3$) graph G with $\mathrm{sr}(G)=a$ and $\mathrm{sd}(G)=b$.

Proposition 4. *For positive integers a,b,k with $k \geqslant 3$ and $b \geqslant 2a-1 \geqslant 3$, there exists a k-stratified graph G with $\mathrm{sr}(G)=a$ and $\mathrm{sd}(G)=b$.*

Proof. Let $X_1, X_2, \ldots, X_k$ denote k distinct colors. Consider the comet $C_{k-2,b}$. Color the first $b-2a+2$ vertices on the tail P_b of the comet with X_1, and color the remaining $2a-2$ vertices on the tail, including the center of the star, with X_2. Then assign the $k-2$ colors $X_3, \ldots, X_k$ to the $k-2$ end-vertices of the star (see Figure 4) to produce a k-stratified graph G with $\mathrm{sr}(G)=a$ and $\mathrm{sd}(G)=b$. □

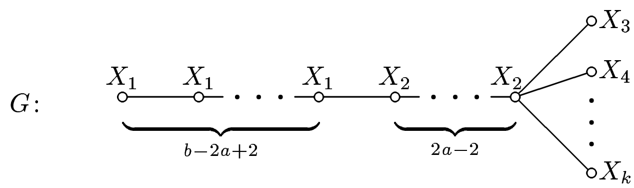

Figure 4. A k-stratified graph G with $\mathrm{sr}(G)=a$ and $\mathrm{sd}(G)=b$ where $b \geqslant 2a-1$

An immediate corollary of Propositions 3 and 4 now follows.

Corollary 5. *For all integers a, b, and k with $k \geqslant 3$ and $b \geqslant 2a-2 \geqslant 2$, there exists a k-stratified graph G with $\mathrm{sr}(G)=a$ and $\mathrm{sd}(G)=b$.*

Thus for all integers $k \geqslant 3$ and $b \geqslant 2$, there exists a k-stratified G with $\mathrm{sr}(G) = 2$ and $\mathrm{sd}(G) = b$. Recall that in Proposition 1, it was shown that for every positive integer b, there exists a k-stratified graph G with $\mathrm{sr}(G) = 1$ and $\mathrm{sd}(G) = b$. Hence we restrict our attention to k-stratified graphs G with $\mathrm{sr}(G) = a$ where $a \geqslant 3$.

Theorem 6. *For integers a and b with $3 \leqslant a \leqslant b$, there exists a 3-stratified graph G with $\mathrm{sr}(G) = a$ and $\mathrm{sd}(G) = b$ if and only if $b \geqslant 2a - 2$.*

Proof. Let G be a 3-stratified graph with $\mathrm{sr}(G) = a$ and $\mathrm{sd}(G) = b$. Let R (red), B (blue), and Y (yellow) denote the three color classes of G. Let v be a vertex of G with $\mathrm{se}(v) = a$. Without loss of generality, we may assume that v is colored red and that $\mathrm{se}(v) = \varrho_B(v)$. Then there exists a vertex u colored blue with $d(v, u) = a$. Let P denote a shortest v-u path (necessarily of length a) and let u' be the vertex adjacent to u on P. Then u is the only vertex of P colored blue and furthermore, no interval vertex of P is colored yellow, for otherwise such a vertex would have s-eccentricity less than a. Hence every vertex of P different from u is colored red.

Consider the vertex u. Since $\varrho_R(u) = 1$, if follows that $\mathrm{se}(u) = \varrho_Y(u) = \ell$, where ℓ is a positive integer such that $a \leqslant \ell \leqslant b$. Let w be a vertex colored yellow with $d(u, w) = \ell$ and let Q be a shortest u-w path. Then, as before, w is the only vertex of Q colored yellow and every internal vertex of Q is colored red or blue. Let x be the vertex of Q at distance $a - 2$ from u. Since $\ell \geqslant a \geqslant 3$, it follows that x is an internal vertex of Q. If x is colored blue, then $\varrho_R(x) \leqslant d(x, u') \leqslant a - 1$ and hence $a \leqslant \varrho_Y(x) \leqslant d(x, w) = \ell - a + 2$. Thus $b \geqslant \ell \geqslant 2a - 2$. On the other hand, if x is colored red, then $\varrho_B(x) \leqslant d(x, u) = a - 2$ and hence $a \leqslant \varrho_Y(x) \leqslant d(x, w) = \ell - a + 2$. Again, $b \geqslant 2a - 2$. The sufficiency follows from Corollary 5 with $k = 3$. □

An immediate corollary of Proposition 1, Corollary 5, and Theorem 6 now follows.

Corollary 7. *There exists a 3-stratified graph G with $\mathrm{sr}(G) = \mathrm{sd}(G) = a$ if and only if $a = 1$ or $a = 2$.*

We now wish to determine a precise bound on the s-diameter of a k-stratified graph in terms of the s-radius. Based on the examples we have seen, we have the following conjecture.

Conjecture. For a k-stratified graph G ($k \geqslant 3$) with s-radius a and s-diameter b,

$$b \geqslant \frac{k-1}{k-2}(a-1).$$

Note that the conjecture is true when $k = 3$, as shown in Theorem 6. Furthermore, we have a class of examples that give rise to this bound. Suppose first that k is even, say $k = 2r$ for some positive integer r, and let t be a positive integer. Let C be a cycle of length kt; say C: $v_{1,1}, v_{1,2}, \ldots, v_{1,t}, v_{2,1}, v_{2,2}, \ldots, v_{2,t}, \ldots, v_{k,1}, v_{k,2}, \ldots, v_{k,t}, v_{1,1}$. Furthermore, for $i = 1, 2, \ldots, k$, let $v_{i,1}, v_{i,2}, \ldots, v_{i,t}$ be colored X_i. We now determine the s-radius a and s-diameter b of C. Observe that we need only determine the s-eccentricities of the vertices of one color class. First, we determine $\mathrm{se}(v_{1,1})$. Now, the color class furthest from $v_{1,1}$ is X_{r+1} and, in fact, the distance from $v_{1,1}$ to a vertex colored X_{r+1} is $d(v_{1,1}, v_{r+1,t}) = t(r-1)+1$. Hence, $\mathrm{se}(v_{1,1}) = t(r-1)+1$. Then $\mathrm{se}(v_{1,2}) = t(r-1)+2$, $\mathrm{se}(v_{1,3}) = t(r-1)+3, \ldots, \mathrm{se}(v_{1,\lceil t/2\rceil}) = t(r-1)+\lceil t/2\rceil$, while $\mathrm{se}(v_{1,t}) = d(v_{1,t}, v_{r+1,1}) = t(r-1)+1$, $\mathrm{se}(v_{1,t-1}) = t(r-1)+2, \ldots, \mathrm{se}(v_{1,\lceil t/2\rceil}) = t(r-1)+\lceil t/2\rceil$. Thus $a = t(r-1)+1$ and $b = t(r-1)+\lceil t/2\rceil$. So $b = (a-1)+\lceil t/2\rceil$ and solving for t in terms of a and r yields $t = (a-1)/(r-1)$ and thus $b \geqslant (a-1)+(a-1)/2(r-1)$. Since $k = 2r$, we have that $b \geqslant (a-1)+(a-1)/(k-2)$ or

$$b \geqslant \frac{k-1}{k-2}(a-1).$$

Finally, suppose that k is odd, say $k = 2r+1$ for some positive integer r and let t be a positive integer. As before, let C be a cycle of length kt; say C: $v_{1,1}, v_{1,2}, \ldots, v_{1,t}, v_{2,1}, v_{2,2}, \ldots, v_{2,t}, \ldots, v_{k,1}, v_{k,2}, \ldots, v_{k,t}, v_{1,1}$, where for $i = 1, 2, \ldots, k$, the vertices $v_{i,1}, v_{i,2}, \ldots, v_{i,t}$ are colored X_i. We determine the s-radius a and s-diameter b of C and to do this, we need only compute the s-eccentricities of the vertices of one color class. Consider the color class X_1. The color class furthest from $v_{1,1}$ is X_{r+1}, while the color class furthest from $v_{1,t}$ is X_{r+2}. Now the distance from $v_{1,1}$ to a vertex colored X_{r+1} is tr as is the distance from $v_{1,t}$ to a vertex colored X_{r+2}. Hence $\mathrm{se}(v_{1,1}) = \mathrm{se}(v_{1,t}) = tr$. Now $v_{1,2}$ is closer to a vertex colored X_{r+1} so, in fact, $\mathrm{se}(v_{1,2}) = tr-1$. Similarly, $\mathrm{se}(v_{1,t-1}) = tr-1$. Next, $\mathrm{se}(v_{1,3}) = \mathrm{se}(v_{1,t-2}) = tr-2, \ldots, \mathrm{se}(v_{1,\lceil t/2\rceil}) = tr - \lfloor (t-1)/2\rfloor$. Thus $a = tr - \lfloor (t-1)/2\rfloor$ and $b = tr$. So, $b = a + \lfloor (t-1)/2\rfloor = a + \lceil (t-2)/2\rceil$ or $b = a + \lceil t/2\rceil - 1$. Solving for t in terms of b and r, we obtain $t = b/r$, or since $k = 2r+1$, we have that $t = 2b/(k-1)$ so $b = a + \lceil b/(k-1)\rceil - 1$ or $b \geqslant (a-1)+b/(k-1)$. Hence $b - b/(k-1) \geqslant (a-1)$ or $b \geqslant [(k-1)/(k-2)](a-1)$.

Observe that if t is even, then in each case (k even or k odd) we have equality, i.e., $b = [(k-1)/(k-2)](a-1)$. Therefore, we have a class of k-stratified graphs with s-radius a and s-diameter $[(k-1)/(k-2)](a-1)$. Next we have the following lemma.

Lemma 8. *For a 4-stratified graph G with s-radius a, every internal vertex on a path of length a from a vertex v colored X_i to a vertex u colored X_j where $\mathrm{se}(v) = \varrho_{X_j}(v)$ is colored X_i.*

Proof. Let P denote a shortest v-u path (of length a). Since $\varrho_{X_j}(v) = a$, it follows that u is the only vertex of P colored X_j. Let X_k and X_ℓ be the colors of G different from X_i and X_j. Suppose that some vertex of P is colored X_k. Let w be the vertex at distance 2 from v on P. Then w is at distance at most $a-2$ from vertices of color X_i, X_j and X_k. Thus, $\operatorname{se}(w) = \varrho_{X_\ell}(w)$. Now let x be the vertex adjacent to w on a shortest path (of length a) from w to a vertex of color X_ℓ. Then x has s-eccentricity less than a, which is impossible. Hence no vertex of P is colored X_k. Similarly, no vertex of P is colored X_ℓ. □

Theorem 9. *If G is a 4-stratified graph with $\operatorname{sr}(G) = \operatorname{sd}(G) = a$, then $a \leqslant 4$.*

Proof. Suppose that $a \geqslant 5$. Let X_1, X_2, X_3, X_4 denote the four colors used to color the vertices of G. Let v be a vertex of G colored X_1. Without loss of generality, we may assume that $\operatorname{se}(v) = \varrho_{X_2}(v)$. Then there exists a vertex u colored X_2 with $d(v,u) = a$. By Lemma 1, every internal vertex of a shortest v-u path is colored X_1. In particular, $\varrho_{X_1}(u) = 1$. Without loss of generality, we may assume that $\operatorname{se}(u) = \varrho_{X_3}(u)$. Then there exists a vertex w colored X_3 with $d(u,w) = a$. Let P be a shortest u-w path (of length a). By Lemma 8, every internal vertex of P is colored X_2. Let x be the vertex at distance 2 from u on P. Since $a \geqslant 5$, x is at distance at most $a-2$ from vertices of color X_1, X_2 and X_3. Thus, $\operatorname{se}(x) = \varrho_{X_4}(x) = a$. Now let y be the vertex adjacent to x on shortest path (of length a) from x to a vertex of color X_4. Then x has s-eccentricity less than a, which is impossible. Hence $a \leqslant 4$. □

3. Straticenters and stratiperiphery in stratified graphs

A vertex v in a connected k-stratified graph G is called a *straticentral* or *s-central vertex* of G if $\operatorname{se}(v) = \operatorname{sr}(G)$. The subgraph of G induced by the s-central vertices of G is called the *straticenter* or *s-center* $\operatorname{SC}(G)$ of G. First, we note that the s-center of every connected 2-stratified graph is 2-stratified; while the s-center of every connected k-stratified graph is ℓ-stratified for some ℓ with $1 \leqslant \ell \leqslant k$.

Theorem 10. *Let ℓ and k be integers with $1 \leqslant \ell \leqslant k$ and $k \geqslant 3$. For every ℓ-stratified graph H, there exists a k-stratified graph G such that $SC(G) = H$.*

Proof. Let H be an ℓ-stratified graph and let k be an integer satisfying $k \geqslant \ell$ and $k \geqslant 3$. Let the strata of H be colored $X_1, X_2, \ldots, X_\ell$. For each vertex v of H and for each $i = 1, 2, \ldots, \ell$ join a new vertex $w_{v,i}$ colored X_i to v if $\varrho_{X_i}(v) > 1$. Let F denote the resulting ℓ-stratified graph. We will obtain a k-stratified graph G from

F. Suppose first that $k = \ell$. We proceed depending on whether H is connected. If H is connected, then let $G = F$ and observe that every vertex of H has s-eccentricity 1 while every other vertex $w_{v,i}$, for some $v \in V(H)$ and $1 \leqslant i \leqslant \ell$, has $\mathrm{se}(w_{v,i}) \geqslant 2$. Thus $\mathrm{SC}(G) = H$. If H is disconnected, then necessarily F is disconnected and we obtain G from F by adding a new vertex z colored X_1 and joining z to a vertex colored X_1 in each component of F. As before, every vertex of H has s-eccentricity 1 while every other vertex of G has s-eccentricity at least 2. Hence $\mathrm{SC}(G) = H$.

Finally, suppose that $k > \ell$. Let $X_{\ell+1}, X_{\ell+2}, \ldots, X_k$ denote $k - \ell$ new colors. For each vertex v of H and for each $j = \ell+1, \ell+2, \ldots, k$, join a new vertex $z_{v,j}$ colored X_j to v in F. Next let the vertices colored X_k induce a path, necessarily of length $n-1$, where n denotes the order of H. The resulting k-stratified graph is G. Every vertex of H has s-eccentricity 1, while every other vertex of G has s-eccentricity at least 2. Therefore, $\mathrm{SC}(G) = H$. □

A vertex v is a *stratiperipheral* or *s-peripheral vertex* of a connected stratified graph G if $\mathrm{se}(v) = \mathrm{sd}(G)$. The subgraph induced by the s-peripheral vertices of G is called the *stratiperiphery* $\mathrm{SP}(G)$ of G. We now show that every ℓ-stratified graph is the s-periphery of some k-stratified graph if $k > \ell$ and not every ℓ-stratified graph is the s-periphery of some ℓ-stratified graph.

Theorem 11. *Let H be an ℓ-stratified graph. Then for every positive integer k with $k > \ell$, there exists a k-stratified graph G such that $SP(G) = H$. Furthermore, there exists an ℓ-stratified graph G with $SP(G) = H$ if and only if no vertex in any component of H has s-eccentricity 1 or every vertex of H has s-eccentricity 1.*

Proof. Let H be an ℓ-stratified graph of order n, say $V(H) = \{v_1, v_2, \ldots, v_n\}$, and let k be a positive integer such that $k > \ell$. Furthermore, let $X_1, X_2, \ldots, X_\ell$ denote the color classes of H and let $X_{\ell+1}, X_{\ell+2}, \ldots, X_k$ denote $k - \ell$ new colors. For $i = 1, 2, \ldots, n$, let $G_i = K_k$ where G_i is a k-stratified graph with exactly one vertex belonging to each color class. For $i = 1, 2, \ldots, n$ join v_i to that vertex of G_i belonging to the same color class as v_i. Next let the vertices colored X_k induce a path, necessarily of length $n-1$, and denote the resulting k-stratified graph by G. Then, in G, every vertex of H has s-eccentricity 2 while every other vertex of G has s-eccentricity 1. Thus $\mathrm{SP}(G) = H$.

Next, suppose that no vertex in any component of H has s-eccentricity 1. As before, for $i = 1, 2, \ldots, n$, let $G_i = K_\ell$ where G_i is an ℓ-stratified graph with exactly one vertex belonging to each color class and join v_i to that vertex of G_i belonging to the same color class as v_i. Finally, let the n vertices of $G_1 \cup G_2 \cup \ldots \cup G_n$ colored X_ℓ induce a path. Let G denote the resulting ℓ-stratified graph. Then in G, every vertex of H has s-eccentricity 2 while every other vertex of G has s-eccentricity 1.

Hence $\mathrm{SP}(G) = H$. On the other hand, if every vertex of H has s-eccentricity 1, then $\mathrm{SP}(H) = H$ and H is the s-periphery of itself.

For the converse, assume that H is an ℓ-stratified graph for which some but not all vertices have s-eccentricity 1 or some vertex in a component of H has s-eccentricity 1, and suppose, to the contrary, that H is the s-periphery of some ℓ-stratified graph G. Since some vertex of H has s-eccentricity 1 or some vertex in a component of H has s-eccentricity 1, it follows that $\mathrm{sd}(G) = 1$. Thus every vertex of G has s-eccentricity 1, and hence $\mathrm{SP}(G) = G$ or $G = H$. Thus every vertex of H has s-eccentricity 1 and H is connected, producing a contradiction. □

We now show that any two stratified graphs can be s-center and s-periphery of some k-stratified graph.

Theorem 12. *Let H_1 be a k_1-stratified graph and let H_2 be a k_2-stratified graph. Then there exists a k-stratified graph G for some positive integer k such that $SC(G) = H_1$ and $SP(G) = H_2$.*

Proof. Let $k = \max\{k_1, k_2\}+1$ and let $X_1, X_2, \ldots, X_k$ denote k distinct colors. Furthermore, assume that the vertices of H_1 are colored $X_1, X_2, \ldots, X_{k_1}$ and that the vertices of H_2 are colored $X_1, X_2, \ldots, X_{k_2}$. To construct the graph G, we begin by making the s-eccentricity 1 of each vertex of H_1. So for $i = 1, 2, \ldots, k$ and for each vertex v of H_1, we join a new vertex $w_{v,i}$ colored X_i to v if $\varrho_{X_i}(v) > 1$. Next fix a vertex x of H_1 colored X_j $(1 \leqslant j \leqslant k_1)$ and join a new vertex y colored X_j to x. For each vertex w of H_2, join w to y. If H_1 is connected, then let G denote the resulting k-stratified graph. If H_1 is disconnected, then add edges among the vertices $w_{v,k}$ where $v \in V(H_1)$ so that the subgraph induced by $\{w_{v,k} \mid v \in V(H_1)\}$ is a path, and let G denote the resulting k-stratified graph. Since each vertex of H_1 is adjacent to a vertex of each strata in G, it follows that $\mathrm{se}(v) = 1$ for every vertex v of H_1. Also for each vertex $w_{v,i}$ where $v \in V(H_1)$ and $1 \leqslant i \leqslant k$, the strata furthest away from $w_{v,i}$ is colored X_k and thus $\mathrm{se}(w_{v,i}) = 2$. Since y is at distance at most 2 from any strata and $\varrho_{X_k}(y) = 2$, it follows that $\mathrm{se}(y) = 2$. Finally, for each vertex w of H_2, a strata furthest away from w is colored X_k and, in fact, $\varrho_{X_k}(w) = 3$ and hence $\mathrm{se}(w) = 3$. Thus $\mathrm{SC}(G) = H_1$ while $\mathrm{SP}(G) = H_2$. □

The *distance between two subgraphs* G_1 and G_2 of a graph is defined by $d(G_1, G_2) = \min\{d(v_1, v_2) \mid v_1 \in V(G_1), v_2 \in V(G_2)\}$. It turns out that not only can we specify the s-center and s-periphery, but we can also make these two stratified graphs arbitrarily far apart, as the next theorem shows.

Theorem 13. *Let H_1 be a k_1-stratified graph, let H_2 be a k_2-stratified graph, and let $n \geqslant 2$ be an integer. Then there exists a k-stratified graph G for some positive integer k such that $SC(G) = H_1$, $SP(G) = H_2$, and $d(H_1, H_2) = n$.*

P r o o f. Let $k = \max\{k_1, k_2\} + 1$ and let $X_1, X_2, \ldots, X_k$ denote distinct colors. As in the proof of Theorem 12, assume that the vertices of H_1 are colored $X_1, X_2, \ldots, X_{k_1}$ and that the vertices of H_2 are colored $X_1, X_2, \ldots, X_{k_2}$. Now for $i = 1, 2, \ldots, k$ and for each vertex v of H_1, we join a new vertex $w_{v,i}$ colored X_i to v if $\varrho_{X_i}(v) > 1$. Fix a vertex x of H_1 colored X_j $(1 \leqslant j \leqslant k_1)$. Let $y_1, y_2, \ldots, y_{n-1}$ be $n-1$ new vertices, colored X_j, and add the edges $xy_1, y_1y_2, y_2y_3, \ldots, y_{n-2}y_{n-1}$. Next for each vertex w of H_2, join w to y_{n-1}. If H_1 is connected, then let G denote the resulting k-stratified graph, otherwise add edges so that the subgraph induced by $\{w_{v,k} \mid v \in V(H_1)\}$ is a path and let G denote the resulting k-stratified graph. Clearly, every vertex v of H_1 has $\mathrm{se}(v) = 1$. Also every vertex w of H_2 is at distance at most $n+1$ from each strata, and in fact, $\varrho_{X_k}(w) = n+1$ so that $\mathrm{se}(w) = n+1$. For each vertex $w_{v,i}$, where $v \in V(H_1)$ and $1 \leqslant i \leqslant k$, note that $\mathrm{se}(w_{v,i}) = 2$ and for $j = 1, 2, \ldots, n-1$, the vertex y_j has $\mathrm{se}(y_i) = j+1$. Thus $\mathrm{SC}(G) = H_1$ while $\mathrm{SP}(G) = H_2$. Furthermore, $d(H_1, H_2) = d(x, w)$ where w is any vertex of H_2 and since $d(x, w) = n$ for each vertex w of H_2, it follows that $d(H_1, H_2) = n$. □

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