Function Spaces: Past, Present, Future
(A personal view)

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Riemann (1826-1866), Ueber die Darstellbarkeit einer Function durch eine trigonometrische Reihe [On the representability of a function in terms of a trigonometric series] (Habilitationsschrift, Göttingen, 1854, published by Dedekind, 1867).

$$f(x) \sim a_0 + \sum_{k=1}^{\infty} (a_k \sin kx + b_k \cos kx), \qquad 0 \leq x \leq 2\pi,$$

Two proposals: 1. Take second differences,

$$\Delta_h^2 f(x) = f(x+2h) - 2f(x+h) + f(x), \qquad \frac{\Delta_h^2 f(x)}{h}, \quad \frac{\Delta_h^2 f(x)}{h^2}.$$

2. If one is not sure whether $F(x)$ is twice differentiable take

$$\int F(x) \frac{d^2 \varrho}{dx^2}(x)\, dx, \qquad \varrho \text{ sufficiently smooth, vanishing at 0 and } 2\pi.$$

This means: take distributional derivatives.

Zygmund, Smooth functions, 1945. In: Trigonometric series, 2. ed. 1977: $\mathcal{C}^1(\mathbb{R})$,

$$\|f \,|\mathcal{C}^1(\mathbb{R})\| = \sup_{x\in\mathbb{R}} |f(x)| + \sup_{x\in\mathbb{R}, 0<|h|\le 1} \frac{|(\Delta_h^2 f)(x)|}{|h|}.$$

Then

$$|f(x+h) - f(x)| \le c\, h\, |\log h|\, \|f \,|\mathcal{C}^1(\mathbb{R})\|, \quad x \in \mathbb{R}, \quad 0 < h < 1/2.$$

Optimal: If $g(h) > 0$, $0 < h < 1/2$, such that

$$|f(x+h) - f(x)| \le h\, g(h)\, \|f \,|\mathcal{C}^1(\mathbb{R})\| \quad \text{then} \quad g(h) \ge c\, |\log h|, \text{ some } c > 0.$$

Nowadays: Special case of continuity envelope.

Sobolev, 1936-38, book 1950: Ω domain in $\mathbb{R}^n$. As usual: $D(\Omega)$ compactly supported C^∞-functions, $D'(\Omega)$ distributions.

$$\int_\Omega f(x)\, D^\alpha \varphi \, \mathrm{d}x = (-1)^{|\alpha|} \int_\Omega f_\alpha(x)\, \varphi(x)\, \mathrm{d}x, \qquad \varphi \in D(\Omega).$$

$f_\alpha = D^\alpha f$, distributional derivative, unique. Classical Sobolev spaces $W^k_p(\Omega)$, $1 < p < \infty$, $k \in \mathbb{N}_0$, normed by

$$\|f \,|W^k_p(\Omega)\| = \sum_{|\alpha| \le k} \|D^\alpha f \,|L_p(\Omega)\|.$$

L. Schwartz, $\sim$ 1948, Fields medal 1950: $S(\mathbb{R}^n)$: collects all $\varphi \in C^\infty(\mathbb{R}^n)$,

$$\|\varphi\|_k = \sup_{x \in \mathbb{R}^n; |\alpha| \le k} (1 + |x|^2)^{k/2} |D^\alpha \varphi(x)| < \infty, \qquad \text{for all } k \in \mathbb{N}.$$

$S'(\mathbb{R}^n)$ (tempered distributions): All linear bounded functionals f: For some $c > 0$, some $k \in \mathbb{N}$,

$$|f(\varphi)| \le c\, \|\varphi\|_k \qquad \text{for all } \varphi \in S(\mathbb{R}^n).$$

Fourier transform: $\varphi \in S(\mathbb{R}^n)$,

$$(Ff)(\xi) = \widehat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\xi} \varphi(x) \, \mathrm{d}x, \qquad \xi \in \mathbb{R}^n.$$

Inverse Fourier transform $F^{-1}\varphi(\xi) = \varphi^\vee(\xi)$: replace $-i$ by i. Extension to $S'(\mathbb{R}^n)$ by duality. Sobolev spaces $H^s_p(\mathbb{R}^n)$, $1 < p < \infty$, $s \in \mathbb{R}$: All $f \in S'(\mathbb{R}^n)$ such that

$$\|f \,| H^s_p(\mathbb{R}^n)\| = \big\| \big((1 + |\xi|^2)^{s/2} \widehat{f}\big)^\vee | L_p(\mathbb{R}^n)\big\| < \infty.$$

A. P. Calderón, $\sim$ 1960, Aronszajn, Smith, Taibleson (1964).

Proposition 1. If $s = k \in \mathbb{N}_0$ and $1 < p < \infty$ then

$$H^s_p(\mathbb{R}^n) = W^k_p(\mathbb{R}^n).$$

Differences $\Delta_h^m f(x)$, $m \in \mathbb{N}$, $x \in \mathbb{R}^n$, $h \in \mathbb{R}^n$,

$$\Delta_h^1 f(x) = f(x+h) - f(x), \qquad \Delta_h^{m+1} f = \Delta_h^1 \Delta_h^m f, \qquad m \in \mathbb{N}.$$

Classical Besov spaces $B^s_{p,q}(\mathbb{R}^n)$, $s > 0$, $1 \le p, q \le \infty$: Collects all $f \in L_p(\mathbb{R}^n) \subset S'(\mathbb{R}^n)$ such that for $s < M \in \mathbb{N}$,

$$\|f \,|B^s_{p,q}(\mathbb{R}^n)\|_M = \|f \,|L_p(\mathbb{R}^n)\| + \Big(\int_{|h|\le 1} |h|^{-sq} \big\|\Delta_h^M f \,|L_p(\mathbb{R}^n)\big\|^q \frac{\mathrm{d}h}{|h|^n} \Big)^{1/q} < \infty.$$

Nikol'skij ~ 1951, $q = \infty$ (related to approximation theory), Besov ~ 1960. Independent of M (equivalent norms). Usual modification if $q = \infty$. In particular

$$\mathcal{C}^1(\mathbb{R}^n) = B^1_{\infty,\infty}(\mathbb{R}^n),$$

n-dimensional version of the above Zygmund space $\mathcal{C}^1(\mathbb{R})$.

$A_0, A_1 \subset A$, two complex Banach spaces. Real interpolation, J. L. Lions, Peetre, 1964,

$$(A_0, A_1)_{\theta,q}, \qquad 0 < \theta < 1, \quad 1 \le q \le \infty,$$

Complex interpolation, A.P. Calderón, 1964,

$$[A_0, A_1]_\theta, \qquad 0 < \theta < 1.$$

Several forerunners. Lions-Peetre paper 'old' methods. Most powerful real interpolation: K-method (Peetre $\sim$ 1963), $a \in A = A_0 + A_1$,

$$K(t,a) = \inf_{a=a_0+a_1} \big(\|a_0 \,|A_0\| + t\|a_1 \,|A_1\| \big), \qquad t > 0.$$

For $0 < \theta < 1$, $1 \le q \le \infty$ Then $(A_0, A_1)_{\theta,q}$ collects all $a \in A = A_0 + A_1$ such that

$$\|a \,|(A_0, A_1)_{\theta,q}\| = \Big(\int_0^\infty t^{-\theta q} K(t,a)^q \frac{\mathrm{d}t}{t} \Big)^{1/q} < \infty.$$

Theorem 2. (Peetre) $1 < p < \infty$, $0 < \theta < 1$, $1 \le q \le \infty$, $s_0 \ne s_1$. Then

$$B^s_{p,q}(\mathbb{R}^n) = \big(H^{s_0}_p(\mathbb{R}^n), H^{s_1}_p(\mathbb{R}^n) \big)_{\theta,q}, \qquad s = (1-\theta)s_0 + \theta s_1.$$

$$\varphi_0 \in D(\mathbb{R}^n), \quad \varphi_0(x) = 1 \text{ if } |x| \le 1, \quad \varphi_0(y) = 0 \text{ if } |y| \ge 3/2.$$

$$\varphi_k(x) = \varphi_0(2^{-k}x) - \varphi_0(2^{-k+1}x), \qquad x \in \mathbb{R}^n, \quad k \in \mathbb{N}.$$

Then

$$\sum_{j=0}^{\infty} \varphi_j(x) = 1, \qquad \text{dyadic resolution of unity.}$$

Recall that $(\varphi_j \widehat{f})^\vee(x)$ is an entire analytic function for any $f \in S'(\mathbb{R}^n)$.
Plancherel-Pólya (1937), Nikol'skij (1951):

$$\text{If } (\varphi_j \widehat{f})^\vee \in L_p(\mathbb{R}^n) \quad \text{then} \quad (\varphi_j \widehat{f})^\vee \in L_q(\mathbb{R}^n) \text{ for all } 0 < p \le q \le \infty.$$

Definition 3. $\varphi = \{\varphi_j\}_{j=0}^{\infty}$ as above, $s \in \mathbb{R}$, $0 < p, q \le \infty$ ($p < \infty$ for F-spaces). Then $B^s_{p,q}(\mathbb{R}^n)$, resp. $F^s_{p,q}(\mathbb{R}^n)$, collects all $f \in S'(\mathbb{R}^n)$ such that

$$\|f \,|B^s_{p,q}(\mathbb{R}^n)\|_\varphi = \Big(\sum_{j=0}^{\infty} 2^{jsq} \|(\varphi_j \widehat{f})^\vee | L_p(\mathbb{R}^n)\|^q \Big)^{1/q} < \infty,$$

$$\|f \,|F^s_{p,q}(\mathbb{R}^n)\|_\varphi = \Big\| \Big(\sum_{j=0}^{\infty} 2^{jsq} |(\varphi_j \widehat{f})^\vee(\cdot)|^q \Big)^{1/q} \,| L_p(\mathbb{R}^n) \Big\| < \infty.$$

Remark 4. Quasi-Banach spaces, independent of φ. Peetre 1967-75, T 1973, based on Hardy-Littlewood $\sim$ 1930, Flett 1971, $B^s_{p,q}$, $p > 0$ on circle, Fefferman-Stein 1972 (H_p, $p < 1$).

Remark 5. $B^s_{p,q}(\mathbb{R}^n)$, $s > 0$, $1 \le p, q \le \infty$ classical Besov spaces.

$$F^s_{p,2}(\mathbb{R}^n) = H^s_p(\mathbb{R}^n), \qquad s \in \mathbb{R}, \quad 1 < p < \infty.$$

$W^k_p(\mathbb{R}^n) = H^k_p(\mathbb{R}^n)$ classical Sobolev spaces.

Theorem 6. (i) If $0 < p, q, q_0, q_1 \le \infty$, $0 < \theta < 1$,

$$-\infty < s_0 < s_1 < \infty, \qquad s = (1-\theta)s_0 + \theta s_1,$$

then for $A \in \{B, F\}$ ($p < \infty$ for F-spaces),

$$\left(A^{s_0}_{p,q_0}(\mathbb{R}^n), A^{s_1}_{p,q_1}(\mathbb{R}^n)\right)_{\theta,q} = B^s_{p,q}(\mathbb{R}^n).$$

(ii) If $p_0, p_1, q_0, q_1 \in (0, \infty)$, $s_0 \in \mathbb{R}$, $s_1 \in \mathbb{R}$, $0 < \theta < 1$,

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad s = (1-\theta)s_0 + \theta s_1,$$

then

$$\left[F^{s_0}_{p_0,q_0}(\mathbb{R}^n), F^{s_1}_{p_1,q_1}(\mathbb{R}^n)\right]_\theta = F^s_{p,q}(\mathbb{R}^n).$$

Same with B in place of F.

Remark 7. Real method: Extension from Banach spaces to quasi-Banach spaces: technical matter.

Complex method: Banach spaces, A.P. Calderón. Several attempts to extend to quasi-Banach spaces. Not possible in general. N. Kalton and co-workers (Mendez, Mitrea and others), 2000, 2007: Method for a sub-class of all quasi-Banach spaces which includes (almost) all spaces spaces $B^s_{p,q}(\mathbb{R}^n)$ and $F^s_{p,q}(\mathbb{R}^n)$. Above theorem due to this method.

Q_{jm}, $j \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$, left corner $2^{-j}m$, side length 2^{-j+1}. (s,p)-atoms a_{jm} where $s \in \mathbb{R}$, $0 < p \leq \infty$:

$$\operatorname{supp} a_{jm} \subset Q_{jm}, \quad j \in \mathbb{N}_0, \quad m \in \mathbb{Z}^n,$$

$$|D^\alpha a_{jm}(x)| \leq 2^{-j(s-\frac{n}{p})+j|\alpha|}, \qquad |\alpha| \leq K \quad \text{for some } K \in \mathbb{N}_0 \text{ with } s < K,$$

$$\int_{\mathbb{R}^n} x^\beta a_{jm}(x)\,\mathrm{d}x = 0, \quad |\beta| < L, \quad j \in \mathbb{N}, \quad m \in \mathbb{Z}^n,$$

where $L \in \mathbb{N}_0$ with $L > -s + \sigma_p$ where $\sigma_p = n\big(\max(\frac{1}{p}, 1) - 1\big)$. Sequence spaces

$$\lambda = \big\{\lambda_{jm} \in \mathbb{C} : \ j \in \mathbb{N}_0, \ m \in \mathbb{Z}^n\big\},$$

$$\|\lambda \,|b_{pq}\| = \Big(\sum_{j=0}^{\infty}\Big(\sum_{m \in \mathbb{Z}^n} |\lambda_{jm}|^p\Big)^{q/p}\Big)^{1/q} < \infty.$$

Theorem 8. $f \in S'(\mathbb{R}^n)$. Then $f \in B^s_{p,q}(\mathbb{R}^n)$ if, and only if,

$$f = \sum_{j,m} \lambda_{jm}\, a_{jm}(x), \quad a_{jm} \text{ are } (s,p)\text{-atoms}, \quad \lambda \in b_{pq},$$

$$\|f \,|B^s_{p,q}(\mathbb{R}^n)\| = \inf \|\lambda \,|b_{pq}\|.$$

Remark 9. Similar for $F^s_{p,q}(\mathbb{R}^n)$, including Sobolev spaces. Some history: H_p Coifman, 1974 ($n = 1$), extension to $n \in \mathbb{N}$: Latter, 1978. B-spaces: PhD Susan Sands, 1981, Maryland, F-spaces: Netrusov 1987-89. Frazier-Jawerth, 1985 B-spaces, 1990 F-spaces.

Wavelets $\psi_F \in C^u(\mathbb{R})$, $\psi_M \in C^u(\mathbb{R})$, $u \in \mathbb{N}$, real,

$$\int_{\mathbb{R}} \psi_M(x) x^v \, \mathrm{d}x = 0, \qquad v \in \mathbb{N}_0, \quad v < u.$$

$$\Psi^j_{G,m}(x) = 2^{jn/2} \prod_{w=1}^{n} \psi_{G_w}(2^j x_w - m_w), \qquad G \in G^j = \{F, M\}^{n*}, \quad m \in \mathbb{Z}^n.$$

G^0 has 2^n elements, G^j, $j \in \mathbb{N}$, has $2^n - 1$ elements (at least on M). Q_{jm} cube in $\mathbb{R}^n$, $2^{-j}m$ left corner, side-length 2^{-j+1}, $j \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$,

$$\operatorname{supp} \Psi^j_{G,m} \subset Q_{jm}.$$

Wavelet expansion:

$$f = \sum_{j=0}^{\infty} \sum_{G \in G^j} \sum_{m \in \mathbb{Z}^n} \lambda^{j,G}_m \, 2^{-jn/2} \Psi^j_{G,m},$$

with

$$\lambda^{j,G}_m = \lambda^{j,G}_m(f) = 2^{jn/2} \int_{\mathbb{R}^n} f(x) \Psi^j_{G,m}(x) \, \mathrm{d}x.$$

Proposition 10. $\{\Psi^j_{G,m}\}$ orthonormal basis in $L_2(\mathbb{R}^n)$.

Extension to $B^s_{p,q}(\mathbb{R}^n)$ and $F^s_{p,q}(\mathbb{R}^n)$. Sequence spaces, adapted, $b^s_{p,q}$ (similarly $f^s_{p,q}$),

$$\|\lambda\,|b^s_{p,q}\| = \Big(\sum_{j=0}^{\infty} 2^{j(s-\frac{n}{p})q} \sum_{G\in G^j} \Big(\sum_{m\in\mathbb{Z}^n} |\lambda^{j,G}_m|^p \Big)^{q/p} \Big)^{1/q}.$$

Theorem 11. $0 < p, q \le \infty$, $s \in \mathbb{R}$, $u > \max(s, \sigma_p - s)$. $f \in S'(\mathbb{R}^n)$. Then $f \in B^s_{p,q}(\mathbb{R}^n)$ if, and only if,

$$f = \sum_{j=0}^{\infty} \sum_{G\in G^j} \sum_{m\in\mathbb{Z}^n} \lambda^{j,G}_m \, 2^{-jn/2} \Psi^j_{G,m}, \qquad \lambda \in b^s_{p,q}.$$

Unique, basis if $p, q < \infty$,

$$\|f\,|B^s_{p,q}(\mathbb{R}^n)\| \sim \|\lambda\,|b^s_{p,q}\|.$$

Similar for F-spaces.

Remark 12. Total reduction to sequence spaces. $\lambda^{j,G}_m(f)$ as before, interpreted as dual pairing in $S(\mathbb{R}^n), S'(\mathbb{R}^n)$.

Isotropic distributional (or Lebesgue-integrable) spaces on $\mathbb{R}^n$.

The Sobolev-Nikol'skij-Besov-Peetre line.
Sobolev: 1936-38, $W_p^k(\mathbb{R}^n)$ (and, mainly, in domains), $D^\alpha f$.
Nikol'skij-Besov: 1951, 1960, $B^s_{p,q}(\mathbb{R}^n) = B^s_q L_p(\mathbb{R}^n)$, $p \geq 1, s > 0$, $\Delta^m_h f$.
Peetre: 1967-75, $A^s_{p,q}(\mathbb{R}^n)$, $0 < p, q \leq \infty, s \in \mathbb{R}$, $\varphi_j(D)f = (\varphi_j \widehat{f})^\vee$.

The Morrey-Campanato-Brudnyi line.
Morrey: 1938, $\mathcal{L}^r_p(\mathbb{R}^n)$, $p \geq 1$, $-n/p \leq r < 0$.
Campanato: 1963-65, $\mathcal{L}^r_p(\mathbb{R}^n)$, $-n/p \leq r < \infty$, $p \geq 1$,
Brudnyi: 1965-70, $\mathcal{L}^r_p(\mathbb{R}^n)$, $0 < p < \infty$.

Attempts to merge.
Kozono-Yamazaki: 1994, $B^s_q \mathcal{L}^r_p(\mathbb{R}^n)$, $s \in \mathbb{R}$, $p > 1$, $-n/p \leq r < 0$
Tang-Xu: 2005, $A^s_q \mathcal{L}^r_p(\mathbb{R}^n)$, $s \in \mathbb{R}$, $0 < p \leq \infty$, $-n/p \leq r < 0$.
Dachun Yang: 2008, $A^{s,\tau}_{p,q}(\mathbb{R}^n)$, $s, \tau \in \mathbb{R}$, $0 < p, q \leq \infty$,
T: 2012, $\mathcal{L}^r A^s_{p,q}(\mathbb{R}^n)$, $s \in \mathbb{R}$, $0 < p, q \leq \infty$, $-n/p \leq r < \infty$.

Comments: Sometimes domain $\Omega \subset \mathbb{R}^n$ instead of $\mathbb{R}^n$ (Sobolev, Brudnyi). Recall $A \in \{B, F\}$. Here preference to $A = B$. Goal is to convince that merging is more than generalising: It is part of the future of the Theory of Function Spaces.

Q_{JM}, $J \in \mathbb{N}_0$, $M \in \mathbb{Z}^n$, cube in $\mathbb{R}^n$, left corner $2^{-J}M$, sides parallel to the axes, length 2^{-J+1}.

Notation:

uniform space: $\sup_{M \in \mathbb{Z}^n} \cdots Q_{0,M}$, local space: $\sup_{J \in \mathbb{N}_0, M \in \mathbb{Z}^n} \cdots Q_{JM}$.

$\mathcal{L}_p(\mathbb{R}^n)$, $0 < p \le \infty$, uniform L_p-space:

$$\|f \,|\mathcal{L}_p(\mathbb{R}^n)\| = \sup_{M \in \mathbb{Z}^n} \|f \,|L_p(Q_{0,M})\|.$$

$\mathcal{L}_\infty(\mathbb{R}^n) = L_\infty(\mathbb{R}^n)$. $\mathbb{N}_{-1} = \mathbb{N}_0 \cup \{-1\}$.
$\mathcal{P}_k$: all polynomials of degree $\le k$ with $k \in \mathbb{N}_{-1}$, where $\mathcal{P}_{-1} = \{0\}$.

Definition 13. $0 < p \le \infty$, $-n/p \le r < \infty$, $k \in \mathbb{N}_{-1}$, $k+1 > r$. Then $\mathcal{L}^r_p(\mathbb{R}^n)$: all measurable functions in $\mathbb{R}^n$, such that

$$\|f \,|\mathcal{L}^r_p(\mathbb{R}^n)\|_k = \|f \,|\mathcal{L}_p(\mathbb{R}^n)\| + \sup_{J \in \mathbb{N}, M \in \mathbb{Z}^n} 2^{J(\frac{n}{p}+r)} \inf_{P \in \mathcal{P}_k} \|f - P \,|L_p(Q_{JM})\|$$

is finite.

Theorem 14. (i) $\mathcal{L}_p^r(\mathbb{R}^n)$ independent of k (equivalent quasi-norms).

(ii) $0 < p \leq \infty$, $r > 0$. Then $\mathcal{L}_p^r(\mathbb{R}^n) = \mathcal{C}^r(\mathbb{R}^n)$.

(iii) $0 < p < \infty$. Then $\mathcal{L}_p^{-n/p}(\mathbb{R}^n) = \mathcal{L}_p(\mathbb{R}^n)$.

(iv)

$$\mathcal{L}_\infty^0 = \mathcal{L}_\infty(\mathbb{R}^n) = L_\infty(\mathbb{R}^n)$$

$$\mathcal{L}_p^0(\mathbb{R}^n) = bmo(\mathbb{R}^n), \qquad 0 < p < \infty.$$

Remark 15. $r < 0, k = -1$: Morrey 1938. $r = 0$: John-Nirenberg, 1961, $r > 0$: Campanato 1964, Brudnyi, 1965-69, 1971.

Wavelets as above,

$$\Psi^u = \{\Psi^j_{G,m} : \ j \in \mathbb{N}_0, \ G \in G^j, \ m \in \mathbb{Z}^n\}, \quad \text{based on } \psi_{F,u},$$

$$\text{supp } \Psi^j_{G,m} \subset Q_{jm}.$$

$$\mathbb{P}_{JM} = \{j \in \mathbb{N}_0, \ G \in G^j, \ m \in \mathbb{Z}^n : \ Q_{jm} \subset Q_{JM}\}, \qquad J \in \mathbb{N}_0, \quad M \in \mathbb{Z}^n.$$

Sequence spaces $\mathcal{L}^r b^s_{p,q}(\mathbb{R}^n)$:

$$\lambda = \{\lambda^{j,G}_m \in \mathbb{C} : \ j \in \mathbb{N}_0, \ G \in G^j, \ m \in \mathbb{Z}^n\}$$

quasi-normed by

$$\|\lambda \,|\mathcal{L}^r b^s_{p,q}(\mathbb{R}^n)\| = \sup_{J \in \mathbb{N}_0, M \in \mathbb{Z}^n} 2^{J(\frac{n}{p}+r)} \Big(\sum_{j=J}^{\infty} 2^{j(s-\frac{n}{p})q} \Big(\sum_{m,G:(j,G,m)\in \mathbb{P}_{JM}} |\lambda^{j,G}_m|^p \Big)^{q/p} \Big)^{1/q}.$$

Similarly $\mathcal{L}^r f^s_{p,q}(\mathbb{R}^n)$.

Recall that

$$f = \sum_{j=0}^{\infty} \sum_{G \in G^j} \sum_{m \in \mathbb{Z}^n} \lambda_m^{j,G} \, 2^{-jn/2} \Psi^j_{G,m},$$

with

$$\lambda_m^{j,G} = \lambda_m^{j,G}(f) = 2^{jn/2} \int_{\mathbb{R}^n} f(x) \Psi^j_{G,m}(x) \, \mathrm{d}x.$$

Definition 16. $0 < p, q \leq \infty$, $s \in \mathbb{R}$, $-n/p \leq r < \infty$,

$$u > \max(s + r^+, \sigma_p - s).$$

Then $\mathcal{L}^r B^s_{p,q}(\mathbb{R}^n)$ collects all $f \in S'(\mathbb{R}^n)$ for which

$$\begin{aligned} \|f \,|\mathcal{L}^r B^s_{p,q}(\mathbb{R}^n)\|_{\Psi^u} &= \sup_{J,M} 2^{J(\frac{n}{p}+r)} \Big\| \sum_{(j,G,m) \in \mathbb{P}_{JM}} \lambda_m^{j,G}(f) \, 2^{-jn/2} \Psi^j_{G,m} \,|B^s_{p,q}(\mathbb{R}^n) \Big\| \\ &\sim \|\lambda(f) \,|\mathcal{L}^r b^s_{p,q}(\mathbb{R}^n)\| < \infty. \end{aligned}$$

Recall $\sigma_p = n\big(\max(1/p, 1) - 1\big)$ and $r^+ = \max(r, 0)$. Similarly $\mathcal{L}^r F^s_{p,q}(\mathbb{R}^n)$.

Theorem 17. (i) $\mathcal{L}^r B^s_{p,q}(\mathbb{R}^n)$, $\mathcal{L}^r F^s_{p,q}(\mathbb{R}^n)$ independent of Ψ^u.

(ii) (Morrey) If $1 < p < \infty$, $-\frac{n}{p} \le r < 0$ then

$$\mathcal{L}^r F^0_{p,2}(\mathbb{R}^n) = \mathcal{L}^r L_p(\mathbb{R}^n) = \mathcal{L}^r_p(\mathbb{R}^n).$$

(iii) (John-Nirenberg) If $2 \le p < \infty$ then $\mathcal{L}^0 L_p(\mathbb{R}^n) = bmo(\mathbb{R}^n)$.

(iv) (Campanato) If $r > 0$ then

$$\mathcal{L}^r B^s_{p,q}(\mathbb{R}^n) = \mathcal{L}^r F^s_{p,q}(\mathbb{R}^n) = \mathcal{C}^{s+r}(\mathbb{R}^n).$$

Remark 18. [Future]: Raise at the same level as $B^s_{p,q}(\mathbb{R}^n)$, $F^s_{p,q}(\mathbb{R}^n)$: Embeddings, traces, equivalent norms (in terms of differences Δ^M_h).

Spaces on domains: Compact embeddings etc.

Applications: heat equations, Navier-Stokes equations.