CONSTRUCTIONS PRESERVING n-WEAK AMENABILITY OF BANACH ALGEBRAS

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Abstract. A surjective bounded homomorphism fails to preserve n-weak amenability, in general. We however show that it preserves the property if the involved homomorphism enjoys a right inverse. We examine this fact for certain homomorphisms on several Banach algebras.

Keywords: weak amenability, n-weak amenability, derivation, second dual, direct sum, Banach algebra, Arens product

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1. Introduction

Let $\mathcal{A}$ be a Banach algebra and let $\mathcal{X}$ be a Banach $\mathcal{A}$-module or simply an $\mathcal{A}$-bimodule. A bounded linear mapping $D\colon \mathcal{A} \to \mathcal{X}$ is said to be a derivation if $D(ab) = a \cdot D(b) + D(a) \cdot b$ for all $a, b \in \mathcal{A}$. A derivation $D\colon \mathcal{A} \to \mathcal{X}$ is said to be inner if there exists $x \in \mathcal{X}$ such that $D(a) = a \cdot x - x \cdot a$ for all $a \in \mathcal{A}$; in this case we say the inner derivation D is implemented by x. For a Banach $\mathcal{A}$-module $\mathcal{X}$, the dual $\mathcal{X}^*$ of $\mathcal{X}$ equipped with the actions $(f \cdot a)(x) = f(ax)$ and $(a \cdot f)(x) = f(xa)$ is a Banach $\mathcal{A}$-module. Similarly, the n-th dual $\mathcal{X}^{(n)}$ of $\mathcal{X}$ is a Banach $\mathcal{A}$-module. In particular, $\mathcal{A}^{(n)}$ is a Banach $\mathcal{A}$-module.

For terminology and background materials see [3], [10], [14].

The notion of weak amenability was first introduced by Bade, Curtis and Dales [2] for commutative Banach algebras and then by Johnson [11] for arbitrary Banach algebras. A Banach algebra $\mathcal{A}$ is called weakly amenable if every bounded derivation from $\mathcal{A}$ into the dual Banach module $\mathcal{A}^*$ is inner. The concept of n-weak amenability was initiated by Dales, Ghahramani and Gronbæk [4]. A Banach algebra $\mathcal{A}$ is said to be n-weakly amenable ($n \in \mathbb{N}$) if every bounded derivation from $\mathcal{A}$ into $\mathcal{A}^{(n)}$ is inner. Trivially, 1-weak amenability is nothing else than weak amenability. In [4] the authors presented many substantial properties of this variety of Banach algebras. For instance, they showed that C^*-algebras are n-weakly amenable for all $n \in \mathbb{N}$ as well as $L^1(G)$ is $(2n-1)$-weakly amenable for all $n \in \mathbb{N}$ and for any locally compact group G.

One of the basic tools for making a new amenable Banach algebra from an old one is the fact that: a continuous homomorphic image of an amenable Banach algebra is again amenable. The same is true for weak amenability in the commutative case, but is false in general. An interesting counterexample is given by Gronbæk [7]. (Indeed, the tensor algebra $X\widehat{\otimes}X^*$ is weakly amenable for any Banach space X. However, the quotient $\mathcal{N}(X)$, the nuclear operators, is weakly amenable if and only if the kernel has dimension less than or equal to 1. Taking $X = E \oplus E$ with a Banach space E failing the approximation property, we get an example of a weak amenable Banach algebra whose quotient by a closed ideal is not weakly amenable.) The interested reader is referred to [8] for a sufficient condition for weak amenability of the homomorphic images.

In this paper we give a sufficient condition for the n-weak amenability of the homomorphic images. More precisely, we show that the homomorphic image of an n-weakly amenable Banach algebra is n-weakly amenable if the involved homomorphism enjoys a right inverse. We use this fact for obtaining some n- weakly amenable Banach algebras from old ones. Moreover, our method in turn provides a unified approach from which some results of Ghahramani and Laali [6] and also Lau and Loy [13] can be directly derived.

2. Right invertible homomorphisms preserve n-weak amenability

For a Banach algebra $\mathcal{A}$, we denote the natural $\mathcal{A}$-module actions on $\mathcal{A}^*$ by

$$\langle a \cdot a^*, b\rangle = \langle a^*, ba\rangle, \qquad \langle a^* \cdot a, b\rangle = \langle a^*, ab\rangle \qquad (a, b \in \mathcal{A}, a^* \in \mathcal{A}^*).$$

In the case when $\Theta\colon \mathcal{A} \to \mathcal{B}$ is a bounded homomorphism, $\mathcal{B}^{(n)}$ can be regarded as an $\mathcal{A}$-module under the module actions

$$a \cdot b^{(n)} = \Theta(a) \cdot b^{(n)}, \quad b^{(n)} \cdot a = b^{(n)} \cdot \Theta(a) \quad (a \in \mathcal{A}, b^{(n)} \in \mathcal{B}^{(n)}).$$

A direct verification reveals that the adjoint mappings $\Theta^*\colon \mathcal{B}^* \to \mathcal{A}^*$ and $\Theta^{**}\colon \mathcal{A}^{**} \to \mathcal{B}^{**}$ are $\mathcal{B}$-module morphisms. The same fact is true for the higher adjoint mappings $\Theta^{(2n-1)}\colon \mathcal{B}^{(2n-1)} \to \mathcal{A}^{(2n-1)}$ and $\Theta^{(2n)}\colon \mathcal{A}^{(2n)} \to \mathcal{A}^{(2n)}$.

We commence with the next lemma which plays a key role in the sequel.

Lemma 2.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras and let $\Theta\colon \mathcal{A}\to\mathcal{B}$ and $\Phi\colon \mathcal{B}\to\mathcal{A}$ be bounded homomorphisms such that $\Theta\circ\Phi = I_{\mathcal{B}}$ and $n\in\mathbb{N}$.*

(i) *If $D\colon \mathcal{B}\to\mathcal{B}^{(2n-1)}$ is a derivation then so is $D_1 = (\Theta^{(2n-1)}\circ D\circ\Theta)\colon \mathcal{A}\to\mathcal{A}^{(2n-1)}$. Moreover, if D_1 is inner then so is D.*

(ii) *If $D\colon \mathcal{B}\to\mathcal{B}^{(2n)}$ is a derivation then so is $D_2 = (\Phi^{(2n)}\circ D\circ\Theta)\colon \mathcal{A}\to\mathcal{A}^{(2n)}$. Moreover, if D_2 is inner then so is D.*

(iii) *If $\mathcal{A}$ is n-weakly amenable then so is $\mathcal{B}$.*

Proof. (i) As we have mentioned before, $\Theta^{(2n-1)}\colon \mathcal{B}^{(2n-1)}\to\mathcal{A}^{(2n-1)}$ is an $\mathcal{A}$-module morphism. Now let $D\colon \mathcal{B}\to\mathcal{B}^{(2n-1)}$ be a derivation. Then for every $a,c\in\mathcal{A}$,

$$\begin{aligned}
D_1(ac) = (\Theta^{(2n-1)}\circ D\circ\Theta)(ac) &= \Theta^{(2n-1)}\circ D(\Theta(a)\Theta(c))\\
&= \Theta^{(2n-1)}(a\cdot D(\Theta(c)) + D(\Theta(a))\cdot c)\\
&= a\cdot\Theta^{(2n-1)}(D(\Theta(c))) + \Theta^{(2n-1)}(D(\Theta(a)))\cdot c\\
&= a\cdot D_1(c) + D_1(a)\cdot c,
\end{aligned}$$

which means that $D_1\colon \mathcal{A}\to\mathcal{A}^{(2n-1)}$ is a derivation.

Now assume that D_1 is inner which is implemented by $F\in\mathcal{A}^{(2n-1)}$. As $\Theta\circ\Phi = I_{\mathcal{B}}$, we have $\Theta^{(2n-2)}\circ\Phi^{(2n-2)} = I_{\mathcal{B}^{2n-2}}$, and so for every $b\in\mathcal{B}$ and $G\in B^{2n-2}$ (with $\mathcal{B}^{(0)}=\mathcal{B}$ in mind) we have

$$\begin{aligned}
\langle D(b), G\rangle &= \langle D(\Theta\circ\Phi(b)), (\Theta^{(2n-2)}\circ\Phi^{(2n-2)})(G)\rangle\\
&= \langle D_1(\Phi(b)), \Phi^{(2n-2)}(G)\rangle\\
&= \langle \Phi(b)\cdot F - F\cdot\Phi(b), \Phi^{(2n-2)}(G)\rangle\\
&= \langle \Phi^{(2n-1)}(b\cdot F - F\cdot b), G\rangle\\
&= \langle b\cdot\Phi^{(2n-1)}(F) - \Phi^{(2n-1)}(F)\cdot b, G\rangle.
\end{aligned}$$

(Note that in the penultimate identity we used the fact that $\Phi^{(2n-1)}\colon \mathcal{A}^{(2n-1)}\to\mathcal{B}^{(2n-1)}$ is actually a $\mathcal{B}$-module morphism.) Therefore, D is an inner derivation implemented by $\Phi^{(2n-1)}(F)$. The proof of (ii) is similar to that of (i). Part (iii) follows trivially from (i) and (ii). $\square$

Remark 2.2. It is worthwhile to mention that the hypothesis imposed in Lemma 2.1 is equivalent to the fact that $\mathcal{A}$ is the topological direct sum of Banach spaces $\ker(\Theta)$ and B. Indeed, a direct verification reveals that the mapping $\omega\colon \mathcal{A}\to\ker(\Theta)\times\mathcal{B}$ defined by $\omega(a) = (a-\Phi(\Theta(a)),\Theta(a))$ is a bicontinuous isomorphism, whose inverse is $\varpi\colon \ker(\Theta)\times\mathcal{B}\to\mathcal{A}$ defined by $\varpi(a,b) = a+\Phi(b)$.

As an application of Lemma 2.1 we have the following result which shows that n-weak amenability inherits by those closed subalgebras which are complemented by a closed ideal.

Theorem 2.3. *Let $\mathcal{A}$ be a Banach algebra such that $\mathcal{A} = \mathcal{B} \oplus \mathcal{I}$ for some closed ideal $\mathcal{I}$ and closed subalgebra $\mathcal{B}$. Then n-weak amenability of $\mathcal{A}$ implies that of $\mathcal{B}$.*

Proof. Let $\Theta\colon \mathcal{A} \to \mathcal{B}$ be the natural projection onto B and let $\Phi\colon \mathcal{B} \to \mathcal{A}$ be the natural injection into $\mathcal{A}$. Trivially Θ and Φ are (bounded) homomorphisms with $\Theta \circ \Phi = I_{\mathcal{B}}$. Now the conclusion follows from Lemma 2.1. $\square$

The next result is a restatement of the above result in the language of quotient algebras and split short exact sequences. Recall that a short exact sequence $0 \to \mathcal{C} \to \mathcal{A} \to \mathcal{B} \to 0$ in the category of Banach algebras and bounded homomorphisms splits if the morphism $\mathcal{A} \to \mathcal{B}$ enjoys a right inverse morphism; cf. [9].

Corollary 2.4. *If $\mathcal{A}$ is n-weakly amenable and $\mathcal{I}$ is a closed ideal of $\mathcal{A}$ then the quotient algebra $\mathcal{A}/\mathcal{I}$ is n-weakly amenable provided the natural short exact sequence $0 \to \mathcal{A} \to \mathcal{A} \to \mathcal{A}/\mathcal{I} \to 0$ splits.*

Let $\mathcal{X}$ be a Banach $\mathcal{A}$-module. Then the l^1-direct sum $\mathcal{A} \oplus \mathcal{X}$ is a Banach algebra under

$$(a, x) \cdot (b, y) = (ab, ay + xb) \qquad (a, b \in \mathcal{A}, x, y \in \mathcal{X}),$$

which is known as a module extension Banach algebra. Some properties of algebras of this form have been studied in [4]. In particular, the n-weakly amenability of this kind of Banach algebras has been extensively investigated in [15, Theorems 2.1 and 2.2]. Since $\mathcal{X}$ and $\mathcal{A}$ are ideal and closed subalgebra of $\mathcal{A} \oplus \mathcal{X}$, respectively, as a consequence of Theorem 2.3 we have

Corollary 2.5. *Let $\mathcal{X}$ be a Banach $\mathcal{A}$-module. If the module extension Banach algebra $\mathcal{A} \oplus \mathcal{X}$ is n-weakly amenable then so is $\mathcal{A}$.*

3. n-weak amenability of second dual

Recall that for a Banach algebra $\mathcal{A}$, the second dual $\mathcal{A}^{**}$ of $\mathcal{A}$ can be made into a Banach algebra supplied with either the first Arens product $\square$ or the second Arens product $\lozenge$. For $a \in \mathcal{A}$, $a^* \in \mathcal{A}^*$ and $a^{**}, b^{**} \in \mathcal{A}^{**}$, the elements $a^{**} \cdot a^*$ and $a^* \cdot a^{**}$ of $\mathcal{A}^*$ are defined by the formulas

$$\langle a^{**} \cdot a^*, a\rangle = \langle a^{**}, a^* \cdot a\rangle, \qquad \langle a^* \cdot a^{**}, a\rangle = \langle a^{**}, a \cdot a^*\rangle.$$

Next, $a^{**}\Box\, b^{**}$ and $a^{**}\Diamond\, b^{**}$ are defined in $\mathcal{A}^{**}$ by the formulas

$$\langle a^{**}\Box\, b^{**}, a^*\rangle = \langle a^{**}, b^{**}\cdot a^*\rangle, \qquad \langle a^{**}\Diamond\, b^{**}, a^*\rangle = \langle b^{**}, a^*\cdot a^{**}\rangle.$$

Ample information about Arens products may be found in Arens' original paper [1] (see also [3]).

For a Banach space $\mathcal{X}$, when there is no risk of confusion, we usually identify $\mathcal{X}$ ($x \in \mathcal{X}$) with its canonical image in $\mathcal{X}^{**}$.

A Banach algebra $\mathcal{A}$ is said to be a dual Banach algebra if there is a closed submodule $\mathcal{A}_*$ of $\mathcal{A}^*$ for which $\mathcal{A} = (\mathcal{A}_*)^*$ ($\mathcal{A}_*$ is called the predual of $\mathcal{A}$). As a consequence of Theorem 2.3 we shall see that in the case when $\mathcal{A}$ is a dual Banach algebra the n-weak amenability of $\mathcal{A}^{**}$ implies that of $\mathcal{A}$; see [6, Theorem 2.2] for $n = 1$.

Proposition 3.1. *For a dual Banach algebra $\mathcal{A}$, if $\mathcal{A}^{**}$ (supplied with either of the Arens products) is n-weakly amenable, then so is $\mathcal{A}$.*

Proof. We present a proof for the first Arens product $\Box$. Let $\mathcal{A}$ be a dual Banach algebra with respect to $\mathcal{A}_*$. Suppose that $\Theta\colon \mathcal{A}^{**} \to \mathcal{A}$ is the adjoint of the canonical embedding $J\colon \mathcal{A}_* \to (\mathcal{A}_*)^{**} = \mathcal{A}^*$ and let $\Phi\colon \mathcal{A} \to \mathcal{A}^{**}$ be also the canonical embedding. Then trivially $\Theta\circ\Phi = I_{\mathcal{A}}$ and also $\Phi\colon \mathcal{A} \to (\mathcal{A}^{**}, \Box)$ is a homomorphism. Moreover, $\Theta\colon (\mathcal{A}^{**}, \Box) \to \mathcal{A}$ is a homomorphism. Indeed, let $a_* \in \mathcal{A}_*$, $a^{**}, b^{**} \in \mathcal{A}^{**}$ and let $\{a_\alpha\}$, $\{b_\beta\}$ be two nets in $\mathcal{A}$ converging to a^{**} and b^{**} in the w^*-topology on $\mathcal{A}^{**}$, respectively. Then

$$\begin{aligned}\langle\Theta(a^{**}\Box\, b^{**}), a_*\rangle &= \langle a^{**}\Box\, b^{**}, J(a_*)\rangle = \lim_\alpha\lim_\beta\langle a_\alpha b_\beta, J(a_*)\rangle\\ &= \lim_\alpha\lim_\beta\langle\Theta(a_\alpha)\Theta(b_\beta), a_*\rangle = \lim_\alpha\langle J(a_*\cdot\Theta(a_\alpha)), b^{**}\rangle\\ &= \lim_\alpha\langle a_\alpha, J(\Theta(b^{**})\cdot a_*)\rangle = \langle a^{**}, J(\Theta(b^{**})\cdot a_*)\rangle\\ &= \langle\Theta(a^{**})\Theta(b^{**}), a_*\rangle.\end{aligned}$$

Using Theorem 2.3 we conclude the result. $\Box$

For a Banach algebra $\mathcal{A}$ let $\mathcal{A}^{\mathrm{op}}$ be the Banach algebra endowed with the reversed product of $\mathcal{A}$, whose underlying Banach space is $\mathcal{A}$ itself. The n-weak amenability of $\mathcal{A}$ is trivially equivalent to that of $\mathcal{A}^{\mathrm{op}}$. As another application of Theorem 2.3 we have the following result; see [6, Theorem 2.3] for $n = 1$.

Proposition 3.2. *Let $\mathcal{A}$ be a Banach algebra admitting a continuous antihomomorphism λ such that $\lambda^2 = I_{\mathcal{A}}$. Then $(\mathcal{A}^{**}, \Box)$ is n-weakly amenable if and only if $(\mathcal{A}^{**}, \Diamond)$ is n-weakly amenable. A similar conclusion holds if λ is a continuous involution on $\mathcal{A}$.*

Proof. As $\lambda^2 = I_{\mathcal{A}}$, for the second adjoint λ^{**} of λ we have $(\lambda^{**})^2 = I_{\mathcal{A}^{**}}$. Using a limit process similar to what we used in the proof of Proposition 3.1, one can show that $\lambda^{**}\colon (\mathcal{A}^{**}, \Box) \to (\mathcal{A}^{**}, \Diamond)^{\mathrm{op}}$ and $\lambda^{**}\colon (\mathcal{A}^{**}, \Diamond)^{\mathrm{op}} \to (\mathcal{A}^{**}, \Box)$ are homomorphisms. The conclusion follows from Theorem 2.3. In the case when λ is an involution on $\mathcal{A}$ a similar proof may apply. $\Box$

4. Introverted subspaces of $\mathcal{A}^*$

Let $\mathcal{A}$ be a Banach algebra. A closed subspace $\mathcal{X}$ of $\mathcal{A}^*$ is said to be left invariant if $\mathcal{X} \cdot \mathcal{A} \subseteq \mathcal{X}$ (or equivalently, $\mathcal{X}$ is a right Banach $\mathcal{A}$-submodule of $\mathcal{A}^*$). A left invariant subspace $\mathcal{X}$ of $\mathcal{A}^*$ is said to be left introverted if $\mathcal{X}^* \cdot \mathcal{X} \subseteq \mathcal{X}$. The dual $\mathcal{X}^*$ of a left introverted subspace $\mathcal{X}$ of $\mathcal{A}^*$ has a natural (first Arens type) multiplication $\Box$ on it in the same manner as $\mathcal{A}^{**}$ does. Right invariance and right introversion for a subspace $\mathcal{X}$ of $\mathcal{A}^*$ can be defined similarly. In the case when $\mathcal{X}$ is right introverted then $\mathcal{X}^*$ can be supplied with a natural (second Arens type) multiplication $\Diamond$. A trivial example of a (left and right) introverted subspace is $\mathcal{A}^*$ itself. More illuminating examples of introverted subspaces of $\mathcal{A}^*$ are $WAP(\mathcal{A})$ (= weakly almost periodic elements of $\mathcal{A}^*$) and $AP(\mathcal{A})$ (=almost periodic elements of $\mathcal{A}^*$). It can be shown that every w^*-closed invariant subspace of $\mathcal{A}^*$ is introverted. The same is valid for every (norm) closed subspace of $WAP(\mathcal{A})$; see [13, Lemma 1.2]. In the case when $\mathcal{A}$ enjoys a bounded right (left) approximate identity, the Cohen-Hewitt Factorization Theorem [10, Theorem 32.22] shows that $\mathcal{A}^* \cdot \mathcal{A}$ ($\mathcal{A} \cdot \mathcal{A}^*$) is a closed subspace of $\mathcal{A}^*$. Moreover, $\mathcal{A}^* \cdot \mathcal{A}$ is left introverted while $\mathcal{A} \cdot \mathcal{A}^*$ is right introverted.

For every two left introverted subspaces $\mathcal{Y} \subseteq \mathcal{X}$ of $\mathcal{A}^*$ it can be readily verified that the restriction map $P\colon \mathcal{X}^* \to \mathcal{Y}^*$ is a continuous homomorphism onto $\mathcal{Y}^*$ whose kernel is the w^*-closed ideal $\mathcal{Y}^\perp = \{x^* \in \mathcal{X}^*\colon \langle x^*, y\rangle = 0$ for all $y \in \mathcal{Y}\}$ of $\mathcal{X}^*$; see [13, Lemma 1.1]. This provides the direct sum decomposition $\mathcal{X}^* = \mathcal{Y}^* \oplus \mathcal{Y}^\perp$. A problem which is of interest in this section is: for two left introverted subspaces $\mathcal{Y} \subseteq \mathcal{X}$ of $\mathcal{A}^*$, under what conditions the n-weak amenability of $\mathcal{X}^*$ implies that of $\mathcal{Y}^*$? According to Theorem 2.3 it is the case if $\mathcal{Y}^*$ is isometrically isomorphic to a closed subalgebra of $\mathcal{X}^*$. In this case we say the pair $(\mathcal{Y}^*, \mathcal{X}^*)$ is admissible. We summarize these observations in the next result.

Theorem 4.1. *Let $\mathcal{Y}, \mathcal{X}$ be two left introverted subspaces of $\mathcal{A}^*$ such that $\mathcal{Y} \subseteq \mathcal{X}$. If the pair $(\mathcal{Y}^*, \mathcal{X}^*)$ is admissible then n-weak amenability of $\mathcal{X}^*$ implies that of $\mathcal{Y}^*$.*

Corollary 4.2. *Let A have a right approximate identity bounded by 1. Then the n-weak amenability of $\mathcal{A}^{**}$ implies that of $(\mathcal{A}^* \cdot \mathcal{A})^*$.*

Proof. Using the latter result it is enough to show that the pair $((\mathcal{A}^* \cdot \mathcal{A})^*, \mathcal{A}^{**})$ is admissible. To see this we shall show that $(\mathcal{A}^* \cdot \mathcal{A})^*$ is isometrically isomorphic to the closed subalgebra $e^{**}\mathcal{A}^{**}$ of $\mathcal{A}^{**}$, in which e^{**} is a w^*-cluster point of the involved bounded approximate identity in $\mathcal{A}^{**}$ which in turn is a right identity for $\mathcal{A}^{**}$. It can be readily verified that the restriction map $\pi\colon e^{**}\mathcal{A}^{**} \to (\mathcal{A}^* \cdot \mathcal{A})^*$ is a norm decreasing (Banach algebra) isomorphism, but not necessarily isometric. As the involved right approximate identity is bounded by 1, π actually is an isometry. In fact, for every $a^{**} \in e^{**}\mathcal{A}^{**}$ we have trivially $e^{**}\Box\, a^{**} = a^{**}$ and so for each $a^* \in \mathcal{A}^*$,

$$\begin{aligned}|\langle a^{**}, a^*\rangle| = |\langle e^{**}\Box\, a^{**}, a^*\rangle| &= \lim_{\alpha} |\langle \pi(a^{**}), a^* \cdot e_\alpha\rangle| \\ &\leqslant \lim_{\alpha} \|\pi(a^{**})\| \|a^*\| \|e_\alpha\| \leqslant \|\pi(a^{**})\| \|a^*\|,\end{aligned}$$

which means that $(\mathcal{A}^* \cdot \mathcal{A})^*$ is isometrically isomorphic to the closed subalgebra $e^{**}\mathcal{A}^{**}$ of $\mathcal{A}^{**}$. $\square$

Since $\mathcal{A} = L^1(G)$, where G is a locally compact group, always enjoys an approximate identity of norm 1, using the fact that $\mathcal{A}^* \cdot \mathcal{A} = L^\infty(G) \cdot L^1(G) = LUC(G)$, as a rapid application of the latter result we have the following corollary. A special version of it (for the case $n = 1$) is given in [13, Proposition 4.14].

Corollary 4.3. *If $L^1(G)^{**}$ is n-weakly amenable then so is $LUC(G)^*$.*

A result of Leptin [12] asserts that if G is amenable then the Fourier algebra $\mathcal{A} = A(G)$ has a bounded (by 1) approximate identity (and vice versa). Also in this case, $\mathcal{A}^* \cdot \mathcal{A} = UC(\widehat{G})^*$. Applying Corollary 4.2 we have the following result, whose special case $n = 1$ is given in [13, Proposition 6.3].

Corollary 4.4. *If G is an amenable group then n-weak amenability of $A(G)^{**}$ implies the same for $UC(\widehat{G})^*$.*

Remark 4.5. (i) Note that in the case when $\mathcal{A}$ is a dual Banach algebra (with predual $\mathcal{A}_*$) then trivially $\mathcal{A}_*$ is an introverted subspace of $\mathcal{A}^*$ (which in turn supports the decomposition $\mathcal{A}^{**} = \mathcal{A} \oplus {A_*}^{\perp}$). As $(\mathcal{A}_*)^* = \mathcal{A}$ is (isometrically isomorphic to) a closed subalgebra of $\mathcal{A}^{**}$, the pair $(\mathcal{A}, \mathcal{A}^{**})$ is an admissible pair, and this provides a short proof for Proposition 3.1 (by virtue of Theorem 4.1).

(ii) Lau and Loy in their extensive work [13] gave a train of admissible pairs, in particular in the group algebras $L^\infty(G)$, $VN(G)$ and $PM_p(G)$. For instance, we quote some of them which are related to $L^\infty(G)$, whose details can be found in [13, Sections 3 and 4].

• $(M(G/N), M(G))$, where N is a compact normal subgroup of G.

• $(M(G), \mathcal{X}^*)$, for every introverted subspace $\mathcal{X}$ of $L^\infty(G)$ with $C_0(G) \subseteq \mathcal{X} \subseteq C(G)$. In particular, the pairs $(M(G), LUC(G)^*)$ and $(M(G), WAP(G)^*)$.

• $(AP(G)^*, WAP(G)^*)$.

• $(WAP(G/N)^*, WAP(G)^*)$, where N is a closed normal subgroup of G.

• The pairs $(\mathcal{X}^*, L^1(G)^{**})$ and $(L^1(G/N)^{**}, L^1(G)^{**})$, in which G is amenable, N is a closed normal subgroup of G and $\mathcal{X}$ is a non-zero w^*-closed self-adjoint translation invariant subalgebra of $L^\infty(G)$.

One may also find some other admissible pairs related to $VN(G)$ and $PM_p(G)$ in Sections 6 and 7 of [13].

According to Theorem 4.1, in each of the above mentioned pairs, the n-weak amenability of the second component implies that of the first.

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