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ON SOME COHOMOLOGICAL PROPERTIES OF THE LIE ALGEBRA OF EUCLIDEAN MOTIONS

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Abstract. The external derivative d on differential manifolds inspires graded operators on complexes of spaces $\Lambda^r g^*$, $\Lambda^r g^* \otimes g$, $\Lambda^r g^* \otimes g^*$ stated by g^* dual to a Lie algebra g. Cohomological properties of these operators are studied in the case of the Lie algebra $g = se(3)$ of the Lie group of Euclidean motions.

Keywords: Lie group, Lie algebra, dual space, twist, wrench, cohomology

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1. Introduction

In robotics a basic theoretical tool is the Lie group $SE(3)$ of Euclidean motions (rotations, translations, helical motions) in the Euclidean space E_3. Then every property of this group, its Lie algebra $se(3)$ and its dual space $se^*(3)$ has useful applications in robotics. Throughout this paper we prefer the matrix form of investigation. It means that the elements of $se(3)$ are considered as couples of two vectors called twists (this notion is often used in robotic literature). Analogously the elements of $se^*(3)$ are couples of two vectors called wrenches.

In the second chapter of this paper we recall some basic notions of the Lie algebra $se(3)$ such as the representation $Ad\colon SE(3) \to GL(se(3))$ of the group $SE(3)$ in the vector space $se(3)$, the representation $ad\colon se(3) \to \mathrm{end}(se(3))$ of the Lie algebra $se(3)$ in the vector space $se(3)$, Klein's and Killing's bilinear forms in $se(3)$. The third chapter is devoted to the space $se^*(3)$. We recall robotic interpretations of the wrench such as pure forces, pure torques, the internal map $i^{Kl}\colon se(3) \to se^*(3)$, (its inversion) determined by Klein's form Kl, the representation of $se(3)$ in $se^*(3)$ which is dual to ad and their properties. The main goal of this paper is to investigate some cohomological properties of the Lie algebra $se(3)$. In the fourth chapter we deal with some graded operators on the complexes of spaces $\Lambda^r g^*$, $\Lambda^r g^* \otimes g$, $\Lambda^r g^* \otimes g^*$ inspired by the external derivative d on differential manifolds and by the 0-representation of $se(3)$ in $\mathbb{R}$, by the representations ad and ad^*. We compute the first cohomological groups of these operators. The basic literature we refer to is [1], [2], [3], [5], [6], [7], [8], especially [9] for the matrix twist and wrench calculus in robotics and [4] for the cohomological considerations and its technical applications.

2. Some properties of the Lie algebra $se(3)$

The Lie group $SE(3)$ of Euclidean motions (rotations, translations, helical motions) in the Euclidean space E_3 and its Lie algebra $se(3)$ are the basic means for the description of robot activities. In this chapter we briefly recall some basic notions of $SE(3)$ and first of all $se(3)$ which we will need. For details we refer to [1], [9].

Let $\mathcal{S}_0$ be a coordinate system in E_3. If we use homogeneous coordinates $(x_1, x_2, x_3, 1)^T \equiv \begin{pmatrix} \bar{x} \\ 1 \end{pmatrix} \in E_3$, where $\bar{x} = (x_1, x_2, x_3)^T$ are the coordinates of the position vector $\overline{OL}$ in $\mathcal{S}_0$ then the left action $L' = HL$ of $SE(3)$ in E_3, $H \in SE(3)$, has the matrix form

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ 1 \end{pmatrix} = \begin{pmatrix} A & \bar{p} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ 1 \end{pmatrix}, \quad H = \begin{pmatrix} A & \bar{p} \\ 0 & 1 \end{pmatrix}$$

where A is an orthogonal 3×3 matrix, $\det A = 1$ and $\overline{OP} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \bar{p}$ is the position vector of the point P at which the origin O goes in the action of the element $H \in SE(3)$. It is easy to see that the coordinate system $\mathcal{S}_0$ determines the isomorphism $SE(3) \simeq SO(3) \rtimes \mathbb{R}^3$ where $SO(3)$ denotes the Lie group of all orthogonal matrices A, $\det A = 1$, which represents the Lie group of all spherical motions around O, $\mathbb{R}^3$ means the Lie group of all translations in E_3 and $\rtimes$ denotes the semidirect product of these groups. In this paper we deal only with structural properties of the group $SO(3) \rtimes \mathbb{R}^3$ and its Lie algebra with the dual space. Taking into account the isomorphism $SE(3) \simeq SO(3) \rtimes \mathbb{R}^3$ all our assertions about these properties are true for the group $SE(3)$ and its Lie algebra $se(3)$ with the dual space $se^*(3)$.

A Euclidean motion $\kappa(t)$ can be written in the form $L(t) = H(t)L_0$, where $H(0) = E$ is the unit matrix. Differentiation of the matrix $H(t)$ at $t = 0$ gives $\dot{H}(0) = \begin{pmatrix} C^{\overline{\omega}} & \bar{b} \\ 0 & 0 \end{pmatrix}$, where $C^{\overline{\omega}} = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$ is skewsymmetric and $\bar{b} =$

$(b_1, b_2, b_3)^T$, $\overline{\omega} = (\omega_1, \omega_2, \omega_3)^T$ are vectors where $\bar{b}$ is the instantaneous velocity of the origin O and $\overline{\omega}$ is the angular velocity of the instantaneous helical motion ϱ around the axis o through the point C, $\overline{OC} = \overline{\omega} \times \bar{b}/\overline{\omega}^2$, with the direction vector $\overline{\omega}$. If $\overline{\omega} \cdot \bar{b} = 0$, $\overline{\omega} \neq \bar{0}$, then ϱ is a rotation. If $\overline{\omega} = \bar{0}$ the ϱ is a translation with the vector $\bar{b}$. Recall that the velocities of any point L_0 at the motion ϱ and $\kappa(t)$ at $t = 0$ are equal. Throughout this paper we use the column coordinate form of vectors, $\overline{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = (v_1, v_2, v_3)^T$ where T denotes the transpose of a matrix. Let us recall that $C^{\overline{\omega}}\overline{v} = (\overline{\omega} \times \overline{v})$ where $\overline{\omega} \times \overline{v}$ denotes the cross product of the vectors $\overline{\omega}$ and $\overline{v}$.

In robotics the "twist" form $X = \begin{pmatrix} \overline{\omega} \\ \bar{b} \end{pmatrix} := \dot{H}(0)$ or $(\overline{\omega}, \bar{b})^T = \begin{pmatrix} \overline{\omega} \\ \bar{b} \end{pmatrix}$ is often used. All twists form the Lie algebra $se(3)$ in which the Lie bracket is

$$[X_1, X_2] = \begin{pmatrix} \overline{\omega}_1 \times \overline{\omega}_2 \\ \overline{\omega}_1 \times \bar{b}_2 + \bar{b}_1 \times \overline{\omega}_2 \end{pmatrix} \approx \dot{H}_1 \dot{H}_2 - \dot{H}_2 \dot{H}_1. \tag{1}$$

Let us recall two representations.

1. The adjoint representation $Ad\colon\ SE(3) \to GL(se(3))$ of the group $SE(3)$ in the vector space $se(3)$ where Ad_H is determined by the tangential prolongation of the internal automorphism $\mathbf{H} \mapsto H\mathbf{H}H^{-1}$ at the unit $e \in SE(3)$, $\mathbf{H} \in SE(3)$ and it has the matrix form (see [9])

$$Ad_H(X) = \begin{pmatrix} A & 0 \\ C^{\bar{p}}A & A \end{pmatrix} \begin{pmatrix} \overline{\omega} \\ \bar{b} \end{pmatrix} = \begin{pmatrix} A\overline{\omega} \\ C^{\bar{p}}A\overline{\omega} + A\bar{b} \end{pmatrix}. \tag{2}$$

2. The representation ad of the Lie algebra $se(3)$ in the vector space $se(3)$ is deduced from Ad and its matrix form is (see [9])

$$\begin{aligned} ad_{X_1} X_2 &= \begin{pmatrix} C^{\overline{\omega}_1} & 0 \\ C^{\bar{b}_1} & C^{\overline{\omega}_1} \end{pmatrix} \begin{pmatrix} \overline{\omega}_2 \\ \bar{b}_2 \end{pmatrix} = \begin{pmatrix} C^{\overline{\omega}_1}\overline{\omega}_2 \\ C^{\bar{b}_1}\overline{\omega}_2 + C^{\overline{\omega}_1}\bar{b}_2 \end{pmatrix} \\ &= \begin{pmatrix} \overline{\omega}_1 \times \overline{\omega}_2 \\ \bar{b}_1 \times \overline{\omega}_2 + \overline{\omega}_1 \times \bar{b}_2 \end{pmatrix} = [X_1, X_2]. \end{aligned} \tag{3}$$

Let us recall the well known relations which we will use:

$$Ad_H[X_1, X_2] = [Ad_H X_1, Ad_H X_2], \tag{4}$$

$$ad_X[X_1, X_2] = [ad_X X_1, X_2] + [X_1, ad_X X_2], \tag{5}$$

$$Ad_{\exp X} = \exp ad_X, \tag{6}$$

where exp denotes the exponential map $\exp\colon\ g \to G$ from any Lie algebra g into its Lie group G.

We will use two bilinear forms defined in $se(3)$.

1. Klein's form Kl is defined by the rule

$$Kl(X_1, X_2) = \overline{\omega}_1 \cdot \bar{b}_2 + \bar{b}_1 \cdot \overline{\omega}_2, \quad X_i = \begin{pmatrix} \overline{\omega}_i \\ \bar{b}_i \end{pmatrix}, \quad i = 1, 2, \tag{7}$$

where dot denotes the scalar product of a vector in the Euclidian space.

2. Killing's form K fulfils

$$K(X_1, X_2) = \overline{\omega}_1 \cdot \overline{\omega}_2. \tag{8}$$

It is well known that the forms Kl and K are Ad-invariant and thus their values do not depend on the choice of the coordinate system $\mathcal{S}_0$. In the case of the Lie algebra g of a general Lie group G, Killing's form is defined by the prescription $\widetilde{K}(X_1, X_2) = \operatorname{tr}(ad_{X_1} ad_{X_2})$ where on the right hand side there is the trace of the linear map $ad_{X_1} ad_{X_2} \in \operatorname{end}(g)$, where $\operatorname{end}(g)$ denotes the space of all linear maps the on the vector space g. Using (3) in the case of $g = se(3)$ we have

$$\operatorname{tr}(ad_{X_1} ad_{X_2}) = \operatorname{tr}\left(\begin{pmatrix} C^{\overline{\omega}_1} & 0 \\ C^{\bar{b}_1} & C^{\overline{\omega}_1} \end{pmatrix} \begin{pmatrix} C^{\overline{\omega}_2} & 0 \\ C^{\bar{b}_2} & C^{\overline{\omega}_2} \end{pmatrix} \right) = 2 \operatorname{tr} C^{\overline{\omega}_1} C^{\overline{\omega}_2} = -4 \overline{\omega}_1 \cdot \overline{\omega}_2.$$

So we have

$$\widetilde{K}(X_1, X_2) = -4K(X_1, X_2). \tag{9}$$

Killing's form is evidently singular since $K(X, X) = 0$ for any translating twist $X = \begin{pmatrix} \bar{0} \\ \bar{b} \end{pmatrix}$. Recall that a twist $X = \begin{pmatrix} \overline{\omega} \\ \bar{b} \end{pmatrix}$ is translating or rotational or helical if $K(X, X) = 0$, i.e. $\overline{\omega} = \bar{0}$ or $Kl(X, X) = 2\overline{\omega} \cdot \bar{b} = 0$, $\overline{\omega} \neq \bar{0}$ or $\overline{\omega} \neq \bar{0}$, $\overline{\omega} \cdot \bar{b} \neq 0$ respectively. The maps Ad preserve the kind of twists (for example if X is rotational then $Ad_H(X)$ is also rotational). The maps ad preserve only translating twists.

3. On the space $se^*(3)$ dual to $se(3)$

The dual space $se^*(3)$ to the vector space $se(3)$ is the vector space of all linear functions (1-forms) $\xi\colon se(3) \to \mathbb{R}$. We consider $se(3)$ as the space of twists (of couples $X = \begin{pmatrix} \overline{\omega} \\ \bar{b} \end{pmatrix}$ of vectors); then an element of $se^*(3)$ is also a couple $\xi = \begin{pmatrix} \overline{m} \\ \overline{f} \end{pmatrix}$ of vectors called the wrench (see [9], [3]), where the value of ξ on X (the evaluation of ξ on X) can be expressed in the form

$$\xi \circ X = \begin{pmatrix} \overline{m} \\ \overline{f} \end{pmatrix} \circ \begin{pmatrix} \overline{\omega} \\ \bar{b} \end{pmatrix} := (\overline{m}, \overline{f}) \begin{pmatrix} \overline{\omega} \\ \bar{b} \end{pmatrix} = \overline{m} \cdot \overline{\omega} + \overline{f} \cdot \bar{b}. \tag{10}$$

Evidently it does not depend on the choice on $\mathcal{S}_0$.

R e m a r k 1. A wrench $(\overline{m}, \overline{f})^T$ can be interpreted by momenta and force:

(a) $(\overline{m} = \bar{r} \times \overline{f}, \overline{f})^T$, $\overline{f}$ is the force and $\overline{m} = \bar{r} \times \overline{f}$ is the moment of force $\overline{f}$ at the point with the position vector $\bar{r}$. In general the wrench $(\overline{m}, \overline{f})^T$, $\overline{m} \cdot \overline{f} = 0$, $\overline{f} \neq \bar{0}$, is called the pure force.

(b) $\xi = (\overline{m}, \bar{0})^T$ is the so-called pure torque and represents a double force.

(c) Every wrench $\xi = (\overline{m}, \overline{f})^T$ is a linear combination of the pure force and the pure torque.

The evaluation $\xi \circ X$ we interpret as the work of ξ on X.

R e m a r k 2. Klein's form Kl determines a (1,1)-correspondence $i^{Kl}\colon se(3) \to se^*(3)$ by the rule $i^{Kl}(X) \equiv i_X Kl \in se^*(3)$, where $i_X Kl(Y) = Kl(X, Y)$, i.e. $i_X Kl = Kl(X, \cdot)$. If $X = (\overline{\omega}_X, \bar{b}_X)^T$, $Y = (\overline{\omega}_Y, \bar{b}_Y)^T$ then $i^{Kl}(X) = (\bar{b}_X, \overline{\omega}_X)^T$ as $i_X Kl(Y) = Kl(X, Y) = \overline{\omega}_X \cdot \bar{b}_Y + \bar{b}_X \cdot \overline{\omega}_Y$. In the matrix form $i^{Kl} = \begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix}$ as $\begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix} \begin{pmatrix} \overline{\omega}_X \\ \bar{b}_X \end{pmatrix} = \begin{pmatrix} \bar{b}_X \\ \overline{\omega}_X \end{pmatrix}$. The inverse matrix is the same, i.e. $\begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix}$.

R e m a r k 3. A twist $X = (\overline{\omega}, \bar{b})^T$, $\overline{\omega} \neq \bar{0}$ determines a line p (axis of X) through the point C with the position vector $\overline{OC} = (\overline{\omega} \times \bar{b})/\overline{\omega}^2$ and with the direction vector $\overline{\omega}$. Analogously the line of a wrench $\xi = \begin{pmatrix} \overline{m} \\ \overline{f} \end{pmatrix}$, $\overline{f} \neq \bar{0}$, goes through the point C, $\overline{OC} = (\overline{f} \times \overline{m})/\overline{f}^2$ and $\overline{f}$ is its direction vector. Then the axis of X and the line of $i_X Kl$ coincide.

From the relation (2) it is clear that by the rule $H \mapsto (Ad_{H^{-1}})^*$ dual to $Ad_{H^{-1}}$ determines a representation ϱ of the group $SE(3)$ in the vector space $se^*(3)$. Then the map $X \mapsto (ad_{-X})^*$ dual to ad_{-X} determines the so-called from ad deduced representation of the Lie algebra $se(3)$ in the vector space $se^*(3)$, i.e. the homomorphism $ad^*\colon se(3) \to \mathrm{end}(se^*(3))$ where $\mathrm{end}(se^*(3))$ is the Lie algebra of all linear maps on $se^*(3)$ with the Lie bracket $[\alpha, \beta] = \alpha\beta - \beta\alpha \in \mathrm{end}(se^*(3))$. The relation (3) implies that the matrix of the map $ad^*(X) = (ad_{-X})^*$ is

$$\begin{pmatrix} C^{\overline{\omega}} & C^{\bar{b}} \\ 0 & C^{\overline{\omega}} \end{pmatrix} = \begin{pmatrix} C^{-\overline{\omega}} & 0 \\ C^{-\bar{b}} & C^{-\overline{\omega}} \end{pmatrix}^T .$$

Let us denote (see [9])

$$\{X, \xi\} := (ad_{-X})^* \xi, \quad \xi \in se^*(3), \quad X \in se(3). \tag{11}$$

In the matrix form we have for $X = (\overline{\omega}, \bar{b})^T$, $\xi = (\overline{m}, \overline{f})^T$

$$\{X, \xi\} = \begin{pmatrix} C^{\overline{\omega}} & C^{\bar{b}} \\ 0 & C^{\overline{\omega}} \end{pmatrix} \begin{pmatrix} \overline{m} \\ \overline{f} \end{pmatrix} = \begin{pmatrix} C^{\overline{\omega}} \overline{m} + C^{\bar{b}} \overline{f} \\ C^{\overline{\omega}} \overline{f} \end{pmatrix} = \begin{pmatrix} \overline{\omega} \times \overline{m} + \bar{b} \times \overline{f} \\ \overline{\omega} \times \overline{f} \end{pmatrix} . \tag{11'}$$

Recall that the space $\Lambda^r se^*(3)$ is the vector space of all scalar skewsymmetric forms of degree r (shortly of r-*forms* on $se(3)$). In general, $\Lambda^r se^*(3)(V) \equiv \Lambda^r se^*(3) \otimes V$ denotes the space of all skewsymmetric forms of degree r with values in a vector space V. In this spirit, $\Lambda^r se^*(3) \equiv \Lambda^r se^*(3)(\mathbb{R})$ and $\Lambda se^*(3)$ denotes the graded algebra of all skewsymmetric scalar forms with external product of scalar forms which is in the case of 1-forms of the form $\alpha \wedge \beta(X,Y) = \alpha(X)\beta(Y) - \alpha(Y)\beta(X)$. Analogously we use the notation $\Lambda^r g^*$, $\Lambda^r g^*(V) = \Lambda^r g^* \otimes V$ for any Lie algebra g.

4. Operators $\hat{d}, \tilde{d}$ and $\tilde{d}^*$. Cohomological properties

First we recall the operator $d\colon \Lambda^r g \otimes V \to \Lambda^{r+1} g \otimes V$ which is inspired by the external differentiation on manifolds, see for example [4]. Let ϱ be a representation of a Lie algebra g in a vector space V, i.e. $\varrho\colon g \to \operatorname{end}(V)$ is a homomorphism of Lie algebras. Let $\alpha \in \Lambda^r g^* \otimes V$. Then the operator d is defined by the rule

$$d\alpha(X_1,\dots,X_{r+1}) = \sum_{j=1}^{r+1} (-1)^{j+1} \varrho(X_j)\alpha(X_1,\dots,\widehat{X}_j,\dots,X_{r+1}) + \sum_{i<j} (-1)^{i+j} \alpha([X_i,X_j],\dots,\widehat{X}_i,\dots,\widehat{X}_j,\dots,X_{r+1}), \quad X_1,\dots,X_{r+1} \in g, \tag{12}$$

where $\widehat{X}$ denotes the omission of X. For $r = 0,1,2$ this gives

$$\bar{v} \in V \Rightarrow d\bar{v}(X) = \varrho(X)\bar{v}, \tag{12_0}$$

$$\alpha \in g^* \otimes V \Rightarrow d\alpha(X,Y) = \varrho(X)\alpha(Y) - \varrho(Y)\alpha(X) - \alpha([X,Y]), \tag{12_1}$$

$$\alpha \in \Lambda^2 g^* \otimes V \Rightarrow d\alpha(X,Y,Z) = \varrho(X)\alpha(Y,Z) - \varrho(Y)\alpha(X,Z) + \varrho(Z)\alpha(X,Y) - \alpha([X,Y],Z) + \alpha([X,Z],Y) - \alpha([Y,Z],X). \tag{12_2}$$

It is clear that $d^2 = dd = 0$ and we get the cohomological complex

$$V \xrightarrow{d} g^* \otimes V \xrightarrow{d} \Lambda^2 g^* \otimes V \xrightarrow{d} \dots \xrightarrow{d} \Lambda^n g^* \otimes V \xrightarrow{d} 0, \qquad n = \dim g.$$

We use the standard notation:

$$B^r = d(\Lambda^{r-1} g^* \otimes V) \subset \Lambda^r g^* \otimes V\text{—the } r\text{th co-boundary of } d,$$
$$Z^r = \{\alpha \in \Lambda^r g^* \otimes V, d\alpha = 0 \in \Lambda^{r+1} g^* \otimes V\}\text{—the } r\text{th co-cycle of } d,$$
$$H^r = Z^r/B^r\text{—the } r\text{th cohomological group of } d.$$

We will treat three cases:

(a) $V = \mathbb{R}$, with the trivial zero-representation $\varrho = 0$,
(b) $V = g$, with the representation $\varrho = ad$,
(c) $V = g^*$, with the representation $\varrho = ad^*$.

(a) Let $V = \mathbb{R}$, $\varrho = 0$ and let the operator d be rewritten as $\hat{d}$. We have

$(\widehat{12_0})$ $c \in \mathbb{R}, \hat{d}c(X) = 0$ and thus $B^1(\hat{d}) = 0$,

$(\widehat{12_1})$ $\alpha \in g^*, \hat{d}\alpha(X, Y) = -\alpha([X, Y])$,

$(\widehat{12_2})$ $\alpha \in \Lambda^2 g^*, \hat{d}\alpha(X, Y, Z) = -\alpha([X, Y], Z) + \alpha([X, Z], Y) - \alpha([Y, Z], X)$,

$$\mathbb{R} \xrightarrow{\hat{d}} g^* \xrightarrow{\hat{d}} \Lambda^2 g^* \xrightarrow{\hat{d}} \ldots \xrightarrow{\hat{d}} \Lambda^n g^* \xrightarrow{\hat{d}} 0.$$

Proposition 1. *Let $A \subset g$ be a subspace. Let $A^\perp = \{\alpha \in g^*, \alpha(A) = 0\}$ be the subspace of all 1-forms $\alpha \in g^*$ for which $\alpha(X) = 0$ for all $X \in A$. Then A is a subalgebra of g iff $\hat{d}\alpha|_A = 0$, i.e. iff $\hat{d}\alpha(X, Y) = 0$ for all $X, Y \in A$ and any $\alpha \in A^\perp$.*

Proof. The proof follows from (12_1) as $\hat{d}\alpha(X, Y) = -\alpha([X, Y])$ is zero for all $X, Y \in A$ and any $\alpha \in A^\perp$ iff $[X, Y] \in A$. □

Corollary 1. *As in $g = se(3)$ there is no 5-dimensional subalgebra (see [6]) therefore the restriction $\hat{d}\alpha$, $\alpha \in se^*(3)$, $\alpha \neq 0$ to the space $\ker \alpha = \{X \in se(3), \alpha(X) = 0\}$ cannot be zero.*

Proof. In the case $\alpha \neq 0$ we have $\dim(\ker \alpha) = 5$. If $\hat{d}\alpha|_{\ker \alpha} = 0$ then by Proposition 1 $\ker \alpha$ is a subalgebra but this is impossible. □

Remark 4. Recall that the Jacobian of an n-parametric robot (robot with n joints) is a map $J\colon \mathbb{R}_n \to se(3)$, $J(\dot{u}_1, \ldots, \dot{u}_n) = \dot{u}_1 Y_1 + \ldots + \dot{u}_n Y_n$ where $\dot{u}_1(t), \ldots, \dot{u}_n(t)$ are the joint velocities and $Y_i(t)$ is the twist determined by the position of the i-th joint at time t. The map $J^*\colon se^*(3) \to \mathbb{R}_n$ dual to J maps wrenches into joint moments such that, if $X = J(\dot{u} = (\dot{u}_1, \ldots, \dot{u}_n))$ and $\alpha \in se^*(3)$ then $\alpha(X) = J^*\alpha(\dot{u})$. So if $\alpha \in \ker J^*$ and $X = J(\dot{u})$ then $\alpha(X) = 0$. Therefore $(J(\mathbb{R}_n))^\perp = \ker J^*$. Therefore $J(\mathbb{R}_n)$ is a subalgebra of $se(3)$ iff $\hat{d}\alpha|_{J(\mathbb{R}_n)} = 0$ for all $\alpha \in \ker J^*$.

Recall that the Lie bracket $[\,,\,]$ in a Lie algebra g is a skew bilinear map $[\,,\,]\colon g \times g \to g$. Let $\mathrm{Im}[\,,\,]$ denote the set of all images of the map $[\,,\,]$. Evidently we have: if $\alpha \in g^*$ then $\hat{d}\alpha = 0$ iff $\mathrm{Im}[\,,\,] \subset \ker \alpha$.

In what follows we will use the fact that $se(3) = so(3) \mathbin{\bar{\oplus}} \mathbb{R}_3$ is a semi-direct sum where $so(3) = \left\{X = \begin{pmatrix} \overline{\omega} \\ \bar{b} \end{pmatrix}, \bar{b} = \bar{0}\right\}$, $\mathbb{R}_3 = \left\{X = \begin{pmatrix} \overline{\omega} \\ \bar{b} \end{pmatrix}, \overline{\omega} = \bar{0}\right\}$ and thus $\left[\begin{pmatrix} \overline{\omega}_1 \\ \bar{0} \end{pmatrix}, \begin{pmatrix} \overline{\omega}_2 \\ \bar{0} \end{pmatrix}\right] = \begin{pmatrix} \overline{\omega}_1 \times \overline{\omega}_2 \\ \bar{0} \end{pmatrix}$, $\left[\begin{pmatrix} \bar{0} \\ \bar{b}_1 \end{pmatrix}, \begin{pmatrix} \bar{0} \\ \bar{b}_2 \end{pmatrix}\right] = \begin{pmatrix} \bar{0} \\ \bar{0} \end{pmatrix}$.

Lemma 1. *Let $\alpha \in se^*(3)$. Then $\hat{d}\alpha = 0$ iff $\alpha = 0$.*

P r o o f. It is sufficient to show that $\mathrm{Im}[\,,\,] = g$. Let $X = \begin{pmatrix} \varpi \\ \bar{b} \end{pmatrix} \in g$. Then there are such vectors ϖ_1, ϖ_2, $\bar{b}_2$ that $\varpi = \varpi_1 \times \varpi_2$ and $\bar{b} = \varpi_1 \times \bar{b}_2$. In detail, if ϖ, $\bar{b}$ are collinear then ϖ_1, ϖ_2, $\bar{b}_2$ are complanar with a plane orthogonal to ϖ. If $\varpi, \bar{b}$ are not collinear then ϖ_1 is collinear to the intersection of two planes when one of them is orthogonal to ϖ and the other to $\bar{b}$. We have $\begin{pmatrix} \varpi \\ \bar{b} \end{pmatrix} = \left[\begin{pmatrix} \varpi_1 \\ \bar{0} \end{pmatrix}, \begin{pmatrix} \varpi_2 \\ \bar{b}_2 \end{pmatrix} \right]$. $\square$

Corollary 2. *The co-cycle Z^1 of $\hat{d}$ is $Z^1(\hat{d}) = 0$ and so $H^1 = Z^1(\hat{d})/B^1(\hat{d}) = 0$.*

Proposition 2. *The second co-cycle of $\hat{d}$ is isomorphic to $se(3)^*$, i.e. $Z^2(\hat{d}) \approx se(3)^*$.*

P r o o f. By the relation $(\widehat{12_1})$ the second co-boundary of $\hat{d}$ is isomorphic to $se(3)^*$, $B^2(\hat{d}) \approx se(3)^*$. Therefore it is sufficient to show that $\dim Z^2(\hat{d}) = \dim se(3)^*$. We choose basis vectors $E_1 = (1,0,\ldots,0)^T = \begin{pmatrix} \bar{e}_1 \\ \bar{0} \end{pmatrix}$, $E_2 = \begin{pmatrix} \bar{e}_2 \\ \bar{0} \end{pmatrix}$, $E_3 = \begin{pmatrix} \bar{e}_3 \\ \bar{0} \end{pmatrix}$, …, $E_6 = \begin{pmatrix} \bar{0} \\ \bar{e}_3 \end{pmatrix}$ in $se(3)$ and the dual basis $E^1 = \begin{pmatrix} \bar{e}_1 \\ \bar{0} \end{pmatrix}$, …, $E^6 = \begin{pmatrix} \bar{0} \\ \bar{e}_3 \end{pmatrix}$ in $se(3)^*$, (i.e. $E^i(E_j) = E^i \circ E_j = \delta^i_j = 1$ for $i = j$ or $\delta^i_j = 0$ for $i \neq j$ and $\bar{e}_1, \bar{e}_2, \bar{e}_3$ is an orthonormal basis in the Euclidian vector space $\mathbb{E}_3$). Any 2-form $\alpha \in \Lambda^2 se^*(3)$ is of the coordinate form $\alpha = \sum\limits_{i<j}^{6} \alpha_{ik} E^i \wedge E^k, \alpha_{ik} = -\alpha_{ki}$. We have $E^i \wedge E^k(E_j, E_h) = (E^i \circ E_j)(E^k \circ E_h) - (E^i \circ E_h)(E^k \circ E_j) = \delta^i_j \delta^k_h - \delta^i_h \delta^k_j$. If $(i,k) \neq (j,h)$ then $E^i \wedge E^k(E_j, E_h) = 0$, $E^i \wedge E^k(E_i, E_k) = 1$. Evidently $\dim \Lambda^2 se^*(3) = 15$. The condition $\hat{d}\alpha = 0$ is satisfied iff $\hat{d}\alpha(E_i, E_j, E_k) = 0$ for $i,j,k = 1,\ldots,6$, $i < j < k$. Using the relation $(\widehat{12_2})$ we obtain 9 independent linear equations for α_{ik}. For example: $0 = \hat{d}\alpha(E_1, E_3, E_6) = -\alpha([E_1, E_3], E_6) + \alpha([E_1, E_6], E_3) - \alpha([E_3, E_6], E_1) = -\alpha(-E_2, E_6) + \alpha(-E_5, E_3) - \alpha(\bar{0}, E_1) = \alpha_{26} + \alpha_{35}$. Therefore all 2-forms α fulfilling $\hat{d}\alpha = 0$ form a $15 - 9 = 6$ dimensional vector space. Therefore dim $Z^2(\hat{d}) = 6$ and $Z^2(\hat{d}) \approx se^*(3)$. $\square$

Corollary 3. *The second cohomological group of $\hat{d}$ is zero: $H^2(\hat{d}) = Z^2(\hat{d})/B^2(\hat{d}) = se^*(3)/se^*(3) = 0$.*

R e m a r k 5. Recall that if G is a semi-simple group (its Killing's form is regular) then $H^1(\hat{d}) = 0$, $H^2(\hat{d}) = 0$. Killing's form of the group $SE(3)$ is singular. Also in this case $H^1(\hat{d}) = 0$, $H^2(\hat{d}) = 0$.

We will show the connections of the bracket $\{X,\alpha\}$ to the operator $\hat{d}$. Our considerations will be general for any Lie group G, its Lie algebra g and g^*. Recall that the map $(Ad_{\exp(-X)})^*\colon g^* \to g^*$ is dual to the map $Ad_{\exp(-X)} = (Ad_{\exp X})^{-1}$. In general, if $f\colon V_1 \to V_2$ is a linear map from a vector space V_1 into another vector space V_2 then the dual map $f^*\colon V_2^* \to V_1^*$ to f is defined by the relation $f^*(\alpha)\circ X = \alpha\circ f(X)$, $X\in V_1$, $\alpha\in V_2^*$. This relation for $V_1 = V_2$ and for a regular f gives $\alpha\circ X = (f^*)^{-1}(\alpha)\circ f(X) = (f^{-1})^*(\alpha)\circ f(X)$. As

$$\frac{\mathrm{d}}{\mathrm{dt}}(Ad_{\exp tX})_{t=0}Y = ad_X Y = [X,Y],$$
$$\frac{\mathrm{d}}{\mathrm{dt}}(Ad_{\exp(-tX)})^*_{t=0}(\alpha) = (ad_{-X})^*(\alpha) = \{X,\alpha\},$$

the differentiation of the relation

$$(Ad_{\exp(-tX)})^*(\alpha)\circ Ad_{\exp tX}(Y) = \alpha\circ Y$$

with respect to t at $t=0$ gives $(ad_{-X})^*(\alpha)\circ Y + \alpha\circ ad_X Y = 0$, i.e.

$$\{X,\alpha\}\circ Y = -\alpha\circ[X,Y]. \tag{13}$$

Proposition 3. *If $\alpha\in g^*, X,Y\in g$ then $\hat{d}\alpha(X,Y) = \{X,\alpha\}\circ Y$.*

Corollary 4. $i_X\hat{d}\alpha = \{X,\alpha\}$.

Remark 6. The relation $L_X = i_X d + d i_X$ well known for the Lie derivation on differentiable manifolds can be thought of as the definition of L_X in g^*. Then for $\alpha\in g^*$ we get $L_X\alpha = i_X\hat{d}\alpha + \hat{d}i_X\alpha = i_X\hat{d}\alpha = \{X,\alpha\}$.

(b) We turn to the case when $V = se(3)$ and $\varrho = ad$, $\varrho(X) = ad_X$. The operator d will be denoted by $\tilde{d}$. So we have

$$\tilde{d}\alpha(X_1,\ldots,X_{r+1}) = \sum_{j=1}^{r+1}(-1)^{j+1}[X_j,\alpha(X_1,\ldots,\widehat{X}_j,\ldots,X_{r+1})]$$
$$+\sum_{i<j}(-1)^{i+j}\alpha([X_i,X_j],\ldots,\widehat{X}_i,\ldots,\widehat{X}_j,\ldots,X_{r+1}),$$

$(\widetilde{12_0})$ $X\in se(3)\Rightarrow \tilde{d}X(Y) = ad_X Y = [X,Y]$,

$(\widetilde{12_1})$ $\alpha\in L(se(3),se(3))\equiv se^*(3)\otimes se(3)$

$$\Rightarrow \tilde{d}\alpha(X,Y) = [X,\alpha(Y)] - [Y,\alpha(X)] - \alpha([X,Y]),$$

$$se(3)\xrightarrow{\tilde{d}} se^*(3)\otimes se(3)\xrightarrow{\tilde{d}}\Lambda^2 se^*(3)\otimes se(3)\xrightarrow{\tilde{d}}\ldots\xrightarrow{\tilde{d}}\Lambda^n se^*(3)\otimes se(3)\xrightarrow{\tilde{d}} 0.$$

The relation $(\widetilde{12_0})$ gives $\tilde{d}X = ad_X$ and so the first co-boundary $B^1(\tilde{d})$ is isomorphic to $se(3)$. Let $\alpha \in L(se(3), se(3)) = se^*(3) \otimes se(3)$. Then by $(\widetilde{12_1})$, $\alpha \in Z^1(\tilde{d}) = \{\beta \in se^*(3) \otimes se(3), \tilde{d}\beta = 0\}$ iff $\alpha[X,Y] = [\alpha(X), Y] + [X, \alpha(Y)]$, i.e. iff α is a derivation on the Lie algebra $se(3)$. The equation (5) gives that $ad_X \in Z^1(\tilde{d})$. It is well known, see for example [4], that in the case of the Lie algebra $so(3)$ all derivations on $so(3)$ are of type ad_X. This immediately follows from the property that $so(3)$ is isomorphic to $\mathbb{R}^3$ with the Lie bracket $[\overline{y}, \overline{z}] = \overline{y} \times \overline{z}$ and that the only matrices of type 3×3 for which $A(\overline{y} \times \overline{z}) = A\overline{y} \times \overline{z} + \overline{y} \times \bar{A}\overline{z}$ are skewsymmetric matrices. We find all derivations on $se(3)$. We will use again the fact that $se(3) = so(3) \bar{\oplus} \mathbb{R}_3$ where the bracket on $\mathbb{R}_3$ is trivial, i.e. $[\overline{v}_1, \overline{v}_2] = \bar{0}$.

The matrix form of any linear map on $se(3)$ is $X' = \mathcal{H}X$, i.e.

$$\begin{pmatrix} \overline{\omega}' \\ \bar{b}' \end{pmatrix} = \begin{pmatrix} \mathcal{H}_1 & \mathcal{H}_2 \\ \mathcal{H}_3 & \mathcal{H}_4 \end{pmatrix} \begin{pmatrix} \overline{\omega} \\ \bar{b} \end{pmatrix} = \begin{pmatrix} \mathcal{H}_1\overline{\omega} + \mathcal{H}_2\bar{b} \\ \mathcal{H}_3\overline{\omega} + \mathcal{H}_4\bar{b} \end{pmatrix},$$

where $\mathcal{H}_1, \ldots, \mathcal{H}_4$ are 3×3 matrices. We find the conditions for $\mathcal{H}_1, \ldots, \mathcal{H}_4$ to satisfy the relation

(14) $\mathcal{H}[X,Y] = [\mathcal{H}X, Y] + [X, \mathcal{H}Y]$ for all $X = (\overline{\omega}_X, \bar{b}_X)^T, X = (\overline{\omega}_Y, \bar{b}_Y)^T \in se(3)$.

The restriction of (14) to $so(3) \bar{\oplus} \bar{0}$, $\bar{b}_X = \bar{0}$, $\bar{b}_Y = \bar{0}$ gives $\mathcal{H}_1(\overline{\omega}_X \times \overline{\omega}_Y) = \mathcal{H}_1\overline{\omega}_X \times \overline{\omega}_Y + \overline{\omega}_X \times \mathcal{H}_1\overline{\omega}_Y$, $\mathcal{H}_3(\overline{\omega}_X \times \overline{\omega}_Y) = \mathcal{H}_3\overline{\omega}_X \times \overline{\omega}_Y + \overline{\omega}_X \times \mathcal{H}_3\overline{\omega}_Y$. Therefore the matrices $\mathcal{H}_1$, $\mathcal{H}_3$ are skewsymmetric. Restricting (14) to the subalgebra $0 \bar{\oplus} \mathbb{R}_3(\overline{\omega}_X = \bar{0}, \overline{\omega}_Y = \bar{0})$ we get $\bar{0} = \mathcal{H}_2\bar{b}_X \times \bar{b}_Y + \bar{b}_X \times \mathcal{H}_2\bar{b}_Y$ for all $\bar{b}_X, \bar{b}_Y \in \mathbb{R}_3$. This is possible iff $\mathcal{H}_2 = 0$. If $X = (\overline{\omega}_X, \bar{0})^T$, $Y = (\bar{0}, \bar{b}_Y)^T$ then $\mathcal{H}_4(\overline{\omega}_X \times \bar{b}_Y) = \mathcal{H}_1\overline{\omega}_X \times \bar{b}_Y + \overline{\omega}_X \times \mathcal{H}_4\bar{b}_Y$. As $\mathcal{H}_1$ is skewsymetric therefore $\mathcal{H}_1\overline{\omega}_X \times \bar{b}_Y = \mathcal{H}_1(\overline{\omega}_X \times \bar{b}_Y) - \overline{\omega}_X \times \mathcal{H}_1\bar{b}_Y$. Then $(\mathcal{H}_4 - \mathcal{H}_1)(\overline{\omega}_X \times \bar{b}_Y) = \overline{\omega}_X \times (\mathcal{H}_4 - \mathcal{H}_1)\bar{b}_Y$. This is true iff $\mathcal{H}_4 - \mathcal{H}_1 = kE$, where E is the 3×3 unit matrix and $k \in \mathbb{R}$. We conclude: A linear map $se(3) \to se(3)$ is a derivation iff it is of the form

$$\begin{aligned} \begin{pmatrix} \overline{\omega}' \\ \bar{b}' \end{pmatrix} &= \begin{pmatrix} C^{\overline{v}}, & 0 \\ C^{\overline{z}}, & C^{\overline{v}} + kE \end{pmatrix} \begin{pmatrix} \overline{\omega} \\ \bar{b} \end{pmatrix} \\ &= \begin{pmatrix} C^{\overline{v}}\overline{\omega} \\ C^{\overline{z}}\overline{\omega} + C^{\overline{v}}\bar{b} + k\bar{b} \end{pmatrix} = ad_{(\overline{v},\overline{z})^T} \begin{pmatrix} \overline{\omega} \\ \bar{b} \end{pmatrix} + k \begin{pmatrix} \bar{0} \\ \bar{b} \end{pmatrix}. \end{aligned}$$

Let $pr_2\colon (\overline{\omega}, \bar{b})^T \to (\bar{0}, \bar{b})^T$ be the projection $se(3) = so(3) \bar{\oplus} \mathbb{R}_3 \to \mathbb{R}_3$ onto the second factor. We have proved

Proposition 4. *A linear map d on $se(3)$ is a derivation on $se(3)$ iff it is of the form $ad_X + k\,pr_2$.*

Corollary 5. *The co-cycle Z^1 of the operator $\tilde{d}$ is isomorphic to $se(3) \bar{\oplus} \mathbb{R}$ and thus the first cohomology group of $\tilde{d}$ is isomorphic to $\mathbb{R}$, $H^1 \approx \mathbb{R}$.*

(c) Let $\tilde{d}^*$ denote the operator d when $V = g^*$ and $\varrho = ad^*$, $ad^*(X) = (ad_{-X})^*$, $(ad_{-X})^*(\alpha) = \{X, \alpha\}$, $X \in g$, $\alpha \in g^*$. Now for $\alpha \in \Lambda^r g^* \otimes g^*$ we have

$$\tilde{d}^*\alpha(X_1, \dots, X_{r+1}) = \sum_{j=1}^{r+1} (-1)^{j+1}\{X_j, \alpha(X_1, \dots, \widehat{X}_j, \dots, X_{r+1})\}$$
$$+ \sum_{i<j} (-1)^{i+j}\alpha([X_i, X_j], \dots, \widehat{X}_i, \dots, \widehat{X}_j, \dots, X_{r+1}),$$

(12_0^*) $\qquad \alpha \in g^* \Rightarrow \tilde{d}^*\alpha(X) = \{X, \alpha\}, \tilde{d}^*\alpha \in g^* \otimes g^*,$

(12_1^*) $\qquad \lambda \in g^* \otimes g^* \Rightarrow \tilde{d}^*\lambda(X, Y) = \{X, \lambda(Y)\} - \{Y, \lambda(X)\} - \lambda([X, Y]),$

$$g^* \xrightarrow{\tilde{d}^*} g^* \otimes g^* \xrightarrow{\tilde{d}^*} \Lambda^2 g^* \otimes g^* \xrightarrow{\tilde{d}^*} \dots \xrightarrow{\tilde{d}^*} \Lambda^n g^* \otimes g^* \xrightarrow{\tilde{d}} 0.$$

From (12_0^*) it is clear that for $\alpha \in g^*$ we have $\tilde{d}^*\alpha = 0$ iff $\alpha = 0$. Then $B^1(\tilde{d}^*) \approx g^*$. We are interested in $Z^1(\tilde{d}^*)$ in the case of $g = se(3), g^* = se^*(3)$. We find all $\lambda \in se^*(3) \otimes se^*(3)$, i.e. the linear maps $\lambda\colon se(3) \to se^*(3)$ for which $\tilde{d}^*\lambda = 0$. If $\alpha \in se^*(3)$ then $\tilde{d}^*\alpha \in se^*(3) \otimes se^*(3)$ determines a linear map $\lambda_\alpha\colon se(3) \to se^*(3), \lambda_\alpha(X) = \{X, \alpha\}$. Equation $(11')$ implies that the matrix of λ_α is $\begin{pmatrix} -C^{\overline{m}} & -C^{\overline{f}} \\ -C^{\overline{f}} & 0 \end{pmatrix}$ for $\alpha = \begin{pmatrix} \overline{m} \\ \overline{f} \end{pmatrix}$. Indeed, if $X = \begin{pmatrix} \overline{\omega} \\ \overline{b} \end{pmatrix}$ then

$$\begin{pmatrix} -C^{\overline{m}}, & -C^{\overline{f}} \\ -C^{\overline{f}}, & 0 \end{pmatrix} \begin{pmatrix} \overline{\omega} \\ \overline{b} \end{pmatrix} = \begin{pmatrix} -C^{\overline{m}}\overline{\omega} - C^{\overline{f}}\overline{b} \\ -C^{\overline{f}}\overline{\omega} \end{pmatrix} = \begin{pmatrix} \overline{\omega} \times \overline{m} + \overline{b} \times \overline{f} \\ \overline{\omega} \times \overline{f} \end{pmatrix} = \{X, \alpha\}.$$

The map $i^{Kl}\colon se(3) \to se^*(3)$ is regular and the matrix of its inversion is again $\begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix}$. Denote $(i^{Kl})^{-1}\alpha \equiv X_\alpha$, $\alpha \in se^*(3)$. Using (3) we get $\lambda_\alpha = -i^{Kl}ad_{X_\alpha}$. Then $\{X, \alpha\} = \lambda_\alpha(X) = -i^{Kl}ad_{X_\alpha}X = -i^{Kl}[X_\alpha, X]$. Every λ can be expressed in the form $\lambda = i^{Kl}\mathcal{H}$, $\mathcal{H}\colon se(3) \to se(3)$. We have $\{X, \lambda(Y)\} = -i^{Kl}[X_{\lambda(Y)}, X] = -i^{Kl}[X_{i^{Kl}\mathcal{H}(Y)}, X] = -i^{Kl}[\mathcal{H}Y, X] = i^{Kl}[X, \mathcal{H}Y]$. Then (12_1^*) is of the form $\tilde{d}^*\lambda(X, Y) = i^{Kl}([X, \mathcal{H}Y] + [\mathcal{H}X, Y] - \mathcal{H}[X, Y])$.

Proposition 5. *A linear map $\lambda\colon se(3) \to se^*(3)$ has the property $\tilde{d}^*\lambda = 0$ iff it is of the form $\lambda = \lambda_\alpha + k\, i^{Kl}\, pr_2$.*

Proof. $\tilde{d}^*\lambda(X, Y) = 0$ iff $[X, \mathcal{H}Y] + [\mathcal{H}X, Y] = \mathcal{H}[X, Y]$. By Proposition 4 this is possible iff $\mathcal{H} = ad_X + k\, pr_2$, i.e. iff $\lambda = i^{Kl}(ad_{-X} + k\, pr_2) = \lambda_\alpha + k\, i^{Kl}\, pr_2$, $X = X_{-\alpha}$. $\square$

Corollary 6. $Z^1(\tilde{d}^*) \approx se(3)^* \oplus \mathbb{R}$, $B^1(\tilde{d}^*) \approx se(3)$ *and* $H^1(\tilde{d}^*) \approx \mathbb{R}$

Example 1. Recall the linear map $i^{Kl}\colon se(3) \to se^*(3)$, $i^{Kl}(X) = i_X Kl$ introduced in Remark 2. We have $\tilde{d}^* i^{Kl}(X,Y) = \{X, i^{Kl}Y\} - \{Y, i^{Kl}X\} - i^{Kl}[X,Y] = \left\{\begin{pmatrix}\overline{\omega}_X\\ \bar{b}_X\end{pmatrix}, \begin{pmatrix}\bar{b}_Y\\ \overline{\omega}_Y\end{pmatrix}\right\} - \left\{\begin{pmatrix}\overline{\omega}_Y\\ \bar{b}_Y\end{pmatrix}, \begin{pmatrix}\bar{b}_X\\ \overline{\omega}_X\end{pmatrix}\right\} - \begin{pmatrix}\overline{\omega}_X \times \bar{b}_Y + \bar{b}_X \times \overline{\omega}_Y\\ \overline{\omega}_X \times \overline{\omega}_Y\end{pmatrix} = \begin{pmatrix}\overline{\omega}_X \times \bar{b}_Y + \bar{b}_X \times \overline{\omega}_Y - \overline{\omega}_Y \times \bar{b}_X + \bar{b}_Y \times \overline{\omega}_X - \overline{\omega}_X \times \bar{b}_Y - \bar{b}_X \times \overline{\omega}_Y\\ \overline{\omega}_X \times \overline{\omega}_Y - \bar{b}_Y \times \overline{\omega}_X - \overline{\omega}_X \times \overline{\omega}_Y\end{pmatrix} = \begin{pmatrix}\overline{\omega}_X \times \bar{b}_Y + \bar{b}_X \times \overline{\omega}_Y\\ \overline{\omega}_X \times \bar{b}_Y\end{pmatrix} = \{X, i^{Kl}Y\}$. We get $\tilde{d}^* i^{Kl}(X,Y) = \{X, i^{Kl}Y\}$.

Example 2. Let N be the general inertion bilinear form on $se(3)$ connected with a solid body with mass $\overline{m}$ and with the position vector $\bar{r}$ of its centre of mass. Its 6×6 matrix is $N = \begin{pmatrix} I & \overline{m}C^{\bar{r}}\\ -\overline{m}C^{\bar{r}} & \overline{m}E\end{pmatrix}$, where I is the inertia tensor in $\mathbb{E}_3$, SE is the 3×3 identity matrix, see [9]. Recall that $SE_K = \frac{1}{2}N(X,X)$ is the kinetic energy of the body at the motion $\exp tX$. It determines a map $i^N\colon se(3) \to se^*(3)$, $i^N(X) = i_X N, i_X N(Y) = N(X,Y)$. By direct calculation we get that the values of the form $\tilde{d}^* i^N \in \Lambda^2 se^*(3) \otimes se^*(3)$ are pure torques.

Remark 7. Remark 4 and Example 2 show the possibilities of some applications of our considerations in robotics. We intend to direct our further investigations to deeper applications of cohomological properties of the spaces $se(3)$ and $se^*(3)$ in dynamic and statics in the spirit of the papers [2], [5], [8].

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