BOUNDS CONCERNING THE ALLIANCE NUMBER

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(Received September 26, 2008)

Abstract. P. Kristiansen, S. M. Hedetniemi, and S. T. Hedetniemi, in Alliances in graphs, J. Combin. Math. Combin. Comput. 48 (2004), 157–177, and T. W. Haynes, S. T. Hedetniemi, and M. A. Henning, in Global defensive alliances in graphs, Electron. J. Combin. 10 (2003), introduced the defensive alliance number $a(G)$, strong defensive alliance number $\hat{a}(G)$, and global defensive alliance number $\gamma_a(G)$. In this paper, we consider relationships between these parameters and the domination number $\gamma(G)$. For any positive integers a, b, and c satisfying $a \leqslant c$ and $b \leqslant c$, there is a graph G with $a = a(G)$, $b = \gamma(G)$, and $c = \gamma_a(G)$. For any positive integers a, b, and c, provided $a \leqslant b \leqslant c$ and c is not too much larger than a and b, there is a graph G with $\gamma(G) = a$, $\gamma_a(G) = b$, and $\gamma_{\hat{a}}(G) = c$. Given two connected graphs H_1 and H_2, where $\mathrm{order}(H_1) \leqslant \mathrm{order}(H_2)$, there exists a graph G with a unique minimum defensive alliance isomorphic to H_1 and a unique minimum strong defensive alliance isomorphic to H_2.

Keywords: defensive alliance, global defensive alliance, domination number

MSC 2000: 05C69

1. Introduction

Recall that a *dominating set* of a graph G is a set of vertices $S \subseteq V(G)$ so that for every vertex $v \in V(G)$, either $v \in S$ or v is adjacent to some vertex in S. The minimum order of a dominating set for G is the *domination number* of G, denoted $\gamma(G)$.

In [1] and [2], Kristiansen, Hedetniemi, and Hedetniemi and Haynes, Hedetniemi, and Henning introduced defensive alliances, strong defensive alliances, and global defensive alliances. Their primary motivation was the study of war-time alliances between nations. A set S of vertices in a graph G is a *defensive alliance* if for every $v \in S$, $|N[v] \cap S| \geqslant |N(v) \cap (V - S)|$. Hence, each vertex (nation) in S has at least as many neighboring vertices in its alliance, including itself, as it does neighboring vertices outside its alliance. A defensive alliance S is *strong* if the inequality is strict for every $v \in S$, that is, $|N[v] \cap S| > |N(v) \cap (V - S)|$. An alliance is *global* if S is also a dominating set for the graph G.

A minimum defensive alliance is called an *a-set*, and the order of a minimum defensive alliance in G is denoted $a(G)$. Similarly, a minimum strong defensive alliance is an *$\hat{a}$-set*, with order $\hat{a}(G)$, and a minimum global defensive alliance is an *γ_a-set*, with order $\gamma_a(G)$. The order of a minimum strong global alliance in G is denoted $\gamma_{\hat{a}}(G)$. An a-set or an $\hat{a}$-set always induces a connected subgraph, since any component of a defensive alliance is a defensive alliance.

Several relationships follow naturally from these definitions, including the following:

$$\begin{aligned} &a(G) \leqslant \hat{a}(G),\\ &\gamma(G) \leqslant \gamma_a(G) \leqslant \gamma_{\hat{a}}(G),\\ &a(G) \leqslant \gamma_a(G),\\ &\hat{a}(G) \leqslant \gamma_{\hat{a}}(G). \end{aligned}$$

In this paper, we consider whether there are other, less obvious, relationships between these parameters, and whether any pair of positive integers can be achieved as one of the relationships above by some graph G.

In the first section, we show that a general construction for G is possible for each of the inequalities

$$\begin{aligned} &\gamma(G) \leqslant \gamma_a(G),\\ &a(G) \leqslant \gamma_a(G),\\ &\gamma(G) \leqslant \gamma_a(G) \leqslant \gamma_{\hat{a}}(G), \end{aligned}$$

although for the last inequality, we will need an additional upper bound on the value for $\gamma_{\hat{a}}(G)$. In the second section, we focus on building graphs around arbitrary given subgraphs so that the subgraphs are induced by a-sets, $\hat{a}$-sets, and γ_a-sets. In particular, we show that, given any two connected graphs H_1 and H_2 with order$(H_1) \leqslant$ order(H_2), there is a graph G whose unique a-set induces H_1 as a subgraph and whose unique $\hat{a}$-set induces H_2 as a subgraph. Furthermore, given any connected graph H, there is a graph G whose unique γ_a-set induces a subgraph isomorphic to H.

2. Constructions for inequalities related to alliances

Since every global alliance set is also a dominating set, we know that $\gamma(G) \leqslant \gamma_a(G)$ for any graph G. Every global alliance set is also a defensive alliance set, so $a(G) \leqslant \gamma_a(G)$. In fact, any three positive integers satisfying these inequalities are achievable as the alliance, domination, and global alliance number of some graph G.

Theorem 2.1. *For any positive integers a, b, and c with $a \leqslant c$ and $b \leqslant c$, there exists a connected graph G such that $a(G) = a$, $\gamma(G) = b$, and $\gamma_a(G) = c$.*

Proof. Since the path P_2 has the desired properties when $c = 1$, we assume $c \geqslant 2$.

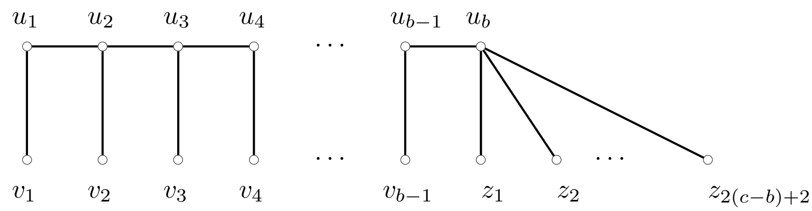
Case I. $b = 1$.

Construct the graph G by starting with K_a and K_{2c-a}. Let U be the vertices of K_a, W be a set of a of the vertices of K_{2c-a}, and X be the remaining vertices of K_{2c-a}. Join every vertex of U to every vertex of W.

It is straightforward to see that the vertices of U form a minimum defensive alliance. Since each vertex of W is adjacent to every other vertex, the domination number is 1. A set consisting of all of the vertices of W and $c - a$ of the vertices of X form a minimum global alliance, so $\gamma_a(G) = c$.

Case II. $a = 1$ and $b \geqslant 2$.

Let $P_b\colon u_1, u_2, \ldots, u_b$ be a path of order b. Then the graph G is obtained from P_b by joining new vertices v_i to u_i for $i \in \{1, \ldots, b-1\}$ and adding $2(c-b)+2$ new vertices $z_1, \ldots, z_{2(c-b)+2}$ to G and joining each z_i to u_b. The graph G is shown below.



Observe that $\{u_1, \ldots, u_b\}$ is the minimum dominating set, so $\gamma(G) = b$. Also, observe that $\{u_1, \ldots, u_b, z_1, \ldots, z_{c-b}\}$ is a dominating set and alliance which realizes the minimum cardinality $\gamma_a(G) = c$. Any one of the end-vertices is a defensive alliance, so $a(G) = 1$.

Case III. $a = b = c = 2$.

The graph C_4 has the desired property.

Case IV. $a \geqslant 2$, $b \geqslant 2$, $c \geqslant 3$, and $b < c$.

Subcase IV(a). $a = c$.

Start with the complete graph K_{2a-1}. Add b new vertices $v_1, v_2, \ldots, v_b$. Join each of the b new vertices to two vertices of K_{2a-1}, so that v_i and v_j have no common neighbor for $i \neq j$, and $\deg(v_i) = 2$ for all i. Any defensive alliance must contain a vertex of K_{2a-1} and, hence, at least a vertices; any a vertices of K_{2a-1} will be a defensive alliance. Any dominating set must contain either v_i or a neighbor of v_i for each i, so $\gamma(G) = b$. A set of a vertices from K_{2a-1}, including a neighbor of each v_i, $1 \leqslant i \leqslant b$, will be a global dominating set. (Note: $b < a$.)

S u b c a s e IV(b). $a < c$.

Construct the graph G as follows. Start with the complete graphs K_a and K_{2c-a-1}. Let U be the vertices of K_a, let W be $a-1$ of the vertices of K_{2c-a-1}, and let X be $V(K_{2c-a-1}) - W$. Notice that X is not empty and has even order. Join every vertex of U to every vertex of W. Add b new vertices $v_1, v_2, v_3, \ldots, v_b$. Join each v_i to either one vertex of U and one vertex of W or to two vertices of X, so that for each i, $\deg v_i = 2$, and for each i and j, $i \neq j$, vertices v_i and v_j have no common neighbors. In particular, v_1 should be joined to two vertices of X and v_2 should be joined to one vertex of U and one vertex of W.

We leave it for the reader to verify that U is a minimum defensive alliance, though possibly not unique.

Since no two v_i and v_j with $i \neq j$ have a common neighbor, any dominating set must contain at least b vertices, including either v_i or a neighbor of v_i for each i. Now, v_2 is adjacent to some $w \in W$ which dominates the rest of the graph, so there is a dominating set with b vertices.

It is straightforward to check that the set consisting of W, one vertex from U, and $c-a$ vertices from X, including at least one neighbor of each v_i, is a minimum global alliance set of order c.

C a s e V. $a \geqslant 2$ and $b = c \geqslant 3$.

Construct G as follows. Start with a complete graph on c vertices $v_1, v_2, \ldots, v_c$, and $\left\lfloor \frac{1}{2}c \right\rfloor$ copies of K_{2a-2}. Join v_1 to $a-1$ of the vertices in the first K_{2a-2} and join v_2 to the other $a-1$ vertices. Similarly, for each i, $2 \leqslant i \leqslant 2\left\lfloor \frac{1}{2}c \right\rfloor$, join v_{2i-1} to $a-1$ of the vertices in the ith copy of K_{2a-2} and join v_{2i} to the other $a-1$ vertices. Also, add $2\left\lfloor \frac{1}{2}c \right\rfloor$ new vertices $u_1, u_2, \ldots, u_{2\left\lfloor \frac{1}{2}c \right\rfloor}$. Join u_i to the same $a-1$ vertices in a copy of K_{2a-1} as v_i, and also join u_i to v_i. If c is odd, add $2\left\lfloor \frac{1}{2}c \right\rfloor$ new vertices $w_1, w_2, \ldots, w_{c-1}$. For each i, $1 \leqslant i \leqslant c-1$, join w_i to v_c and to u_i.

When c is even, the set $N[u_1] - \{v_1\}$, the closed neighborhood of u_1 except for v_1, is a minimum alliance set with a vertices. When c is odd, the set $N[u_1] - \{v_1, w_1\}$ is a minimum alliance set with a vertices.

The set $\{v_1, v_2, \ldots, v_c\}$ is a minimum dominating set with c vertices, and a minimum global alliance, so we have $\gamma(G) = \gamma_a = c = b$. □

Based simply on the definitions, the domination number, global alliance number, and strong global alliance number must satisfy $\gamma(G) \leqslant \gamma_a(G) \leqslant \gamma_{\hat{a}}(G)$ for any graph G. Given any three positive integers $a \leqslant b \leqslant c$, is there a graph G so that $\gamma(G) = a$, $\gamma_a(G) = b$, and $\gamma_{\hat{a}}(G) = c$?

First, suppose $b = 1$. If G is a graph with $\gamma_a(G) = 1$, then there is a single vertex $u \in V(G)$ so that $\{u\}$ is a dominating set and a defensive alliance. Since $\{u\}$ is a dominating set, every other vertex of G is adjacent to u. Since $\{u\}$ is a defensive alliance, there must be at most one vertex adjacent to u. Thus, $G = K_1$ or K_2, and $c = 1$ or $c = 2$.

We will consider the remaining cases in the following proof. First, however, we introduce a useful construction. For any integers i, j, and k with $i \geqslant 1$, $0 \leqslant j \leqslant i-1$, and $j \geqslant 2k-1$, we construct a graph $H(i,j,k)$ with order i, minimum degree j, and containing a clique on k vertices, each of which has degree j in the graph as a whole. Notice that $i \geqslant 2k$. Start with $K_k \cup K_{i-k}$. Then add $k(j-k+1)$ edges between the two complete graphs, distributed as evenly as possible. Thus, each vertex in K_k will have degree $(k-1)+(j-k+1) = j$ and each vertex in K_{i-k} will have degree at least $i-k-1+\lfloor k(j-k+1)/(i-k)\rfloor$. Since $i \geqslant 2k$ and $i > j$, clearly $(i-j-1)(i-2k) \geqslant 0$. With a little arithmetic, this inequality is equivalent to $i-k-1+k(j-k+1)/(i-k) \geqslant j$. Since the right hand side is an integer, we can take the floor function of the left hand side and the inequality will still hold.

Theorem 2.2. *Let a, b, and c be three positive integers with $a \leqslant b \leqslant c$, $2 \leqslant b$, and $c \leqslant \frac{1}{2}(ab+2b-a\lceil b/a\rceil)$. Then there exists a graph G such that $\gamma(G) = a$, $\gamma_a(G) = b$, and $\gamma_{\hat{a}}(G) = c$.*

P r o o f. We construct G as follows. We start with K_b and partition the vertices of K_b into a sets $S_1, S_2, \ldots S_a$ as nearly equal in size as possible, so $|S_i| = \lfloor b/a\rfloor$ or $\lfloor b/a\rfloor + 1$ for each i.

Let $q = \lfloor (c-b)/a\rfloor$. Define a additional graphs $W_1, W_2, \ldots, W_a$ as follows. If $q = 0$, that is, $c - b < a$, then W_i is the graph with no edges on b vertices for $1 \leqslant i \leqslant c-b$ and W_j is the graph with no edges on $b-1$ vertices for $c-b < j \leqslant a$. Otherwise, using the construction described prior to this theorem, define $W_i = H(b, \lceil b/a\rceil + 2q-1, q+1)$, a graph of order b with minimum degree $\lceil b/a\rceil + 2q-1$ and clique size at least $q+1$, for $1 \leqslant i \leqslant c-b-qa$ and $W_j = H(b, \lceil b/a\rceil + 2q-3, q)$ for $c-b-qa < i \leqslant a$.

This is possible provided $\lceil b/a\rceil + 2q - 1 \leqslant b-1$ or, if a divides $c-b$, $\lceil b/a\rceil + 2q-3 \leqslant b-1$. By substituting $\lfloor (c-b)/a\rfloor$ for q and solving for c, we see that the first inequality is satisfied if $c \leqslant \frac{1}{2}(ab+2b-a\lceil b/a\rceil)$. Notice that, due to the floor function in the definition of q, $c \leqslant \frac{1}{2}(ab+2b-a\lceil b/a\rceil)$ implies $\lceil b/a\rceil + 2q-1 \leqslant b-1$ but not vice versa.

Now, join every vertex of W_i to every vertex of S_i for $1 \leqslant i \leqslant a$.

We will show that the set formed by selecting a single entry from each set S_i with $1 \leqslant i \leqslant a$ is a minimum dominating set, that the vertices of K_b form a minimum global alliance set, and that the vertices of K_b along with $\lfloor (c-b)/a \rfloor + 1$ vertices from each W_i, $1 \leqslant i \leqslant c-b-qa$, and $\lfloor (c-b)/a \rfloor$ vertices from each W_j, $c-b-aq < j \leqslant a$, forms a minimum global strong alliance set.

Claim 1. $\gamma(G) = a$.

Notice that the a sets $W_1, W_2, \ldots, W_a$ are disjoint, with the property that for any two vertices $w \in W_i$ and $w' \in W_j$, $i \neq j$, w and w' are not adjacent and have no common neighbor. Thus, any dominating set must contain at least a vertices.

Now, choose one vertex from each set S_i, $1 \leqslant i \leqslant a$. This is a dominating set.

Claim 2. $\gamma_a(G) = b$.

As noted in Claim 1, any dominating set of G must contain either a vertex of W_i or a vertex of S_i for each i, $1 \leqslant i \leqslant a$. Suppose a vertex $u \in S_i$ is in a global alliance set. Since $|N[u]| = 2b$ or $2b-1$, we must have at least b vertices in the set, counting u. Suppose the vertices $w_1, w_2, \ldots, w_r$ from a specific set W_i are in a global alliance set, but no vertex of S_i is in the set. Then each w_i has at least $|S_i|$ enemies and at most r friends, including itself, so $r \geqslant |S_i|$. Thus, if there are no vertices from any S_i in the set, then there must be at least $\sum\limits_{i=1}^{a} |S_i| = b$ vertices from $\bigcup\limits_{i=1}^{a} W_i$ in the set. Either way, $\gamma_a(G) \geqslant b$.

Notice that $\bigcup\limits_{i=1}^{a} S_i$ is a global alliance set of order b.

Claim 3. $\gamma_{\hat{a}}(G) = c$.

Again, any dominating set must contain at least one vertex of $W_i \cup S_i$ for each i. If W_i is an empty graph on b or $b-1$ vertices, then any strong alliance set which contains a vertex of W_i must also contain a vertex of S_i. We may assume, then, that we need at least one vertex u_i from each S_i in this case. We will also need at least $\lceil \frac{1}{2} \deg(u_i) \rceil = b-1$ or b additional vertices from $N[u_i] = W_i \cup K_b$. If $|W_i| = b-1$ and if $\bigcup\limits_{j=1}^{a} S_j = K_b$ is contained in the strong alliance, then no vertex of W_i is needed; each vertex in S_i has b allies and $b-1$ enemies. However, if $|W_i| = b$, then any strong alliance which contains S_i must contain at least one vertex of W_i as well.

For $W_i = H(b, \lceil b/a \rceil + 2q - 1, q+1)$, any strong dominating set which contains a vertex $u \in S_i$ must contain $\lceil \frac{1}{2} \deg(u) \rceil = b$ neighbors of u, including at least one vertex $w \in W_i$. And any strong dominating set which contains $w \in W_i$ must contain at least half of the neighbors of w, at least $\lceil b/a \rceil + q$ vertices, including at least q vertices in W_i, not counting w, or $q+1$ total vertices in W_i. Similarly, for $W_i = H(b, \lceil b/a \rceil + 2q - 3, q)$, any strong dominating set must contain at least q vertices of W_i.

Thus, at a minimum, we will need all b vertices of $\bigcup\limits_{i=1}^{a} S_i$, one vertex from each W_i which is an empty graph on b vertices, $q+1$ vertices from each $H(b, \lfloor b/a \rfloor + 2q - 1,$

$q+1)$, and q vertices from each $H(b, \lfloor b/a \rfloor + 2q - 3, q)$. If we add these, we have at least c vertices. Thus, $\gamma_{\hat{a}}(G) \geqslant c$. Such a set will be a strong global alliance set provided the vertices from each W_i form a clique in that W_i and have the minimum degree in W_i. By our construction of W_i, such a set can be found. $\square$

It is not known whether the condition $c \leqslant \frac{1}{2}(ab+2b-a\lceil b/a\rceil)$ is necessary. However, $\gamma_{\hat{a}}(G)$ can be bounded above by a formula in terms of $\gamma_a(G)$. We mention one such upper bound.

Observation 2.3. For any graph G, $\gamma_{\hat{a}}(G) \leqslant \gamma_a(G)\,(1+\gamma_a(G))$.

To see this bound, suppose that $\gamma_a(G) = b$, and let S be a subgraph of order b which is a global alliance set. Then each vertex of S has at most b neighbors outside of S. Since S is a dominating set, G has at most $b(1+b)$ vertices. Clearly, $V(G)$ is a strong global alliance set.

3. Specified alliance and strong alliance sets

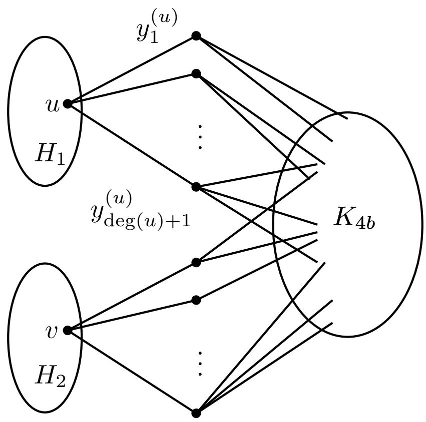
In this section, we specify not only the order of the a-set, $\hat{a}$-set, and/or γ_a-set of the graph but also the subgraphs induced by these sets. If a defensive alliance or strong defensive alliance induces a subgraph that is not connected, then any component of that subgraph would be an alliance of smaller order. Thus, any a-set or $\hat{a}$-set induces a connected subgraph. Provided that two graphs H_1 and H_2 are connected, though, the next theorem shows that there is a graph G whose unique a-set induces a subgraph isomorphic to H_1 and whose unique $\hat{a}$-set induces a subgraph isomorphic to H_2.

Theorem 3.1. *Given $1 \leqslant a \leqslant b$ and any two connected graphs H_1 and H_2 with orders a and b respectively, there exists a connected graph G with the following properties.*

(a) *H_1 is isomorphic to the subgraph induced by the only defensive alliance of G that has minimum cardinality $a(G)$.*
(b) *H_2 is isomorphic to the subgraph induced by the only strong defensive alliance of G that has minimum cardinality $\hat{a}(G)$.*

Proof. Suppose that $1 \leqslant a \leqslant b$ and that H_1 and H_2 are connected graphs such that $a = |V(H_1)|$ and $b = |V(H_2)|$. Since both H_1 and H_2 are connected, $\deg(v) \geqslant 1$ for each vertex v in H_1 or H_2. Modify H_1 and H_2 to get the graph G as follows: (1) For every vertex $u \in V(H_1)$ and $i \in \{1, \ldots, \deg_{H_1}(u)+1\}$, adjoin an end-vertex $y_u^{(i)}$ to u. (2) For every vertex $v \in V(H_2)$ and $i \in \{1, \ldots, \deg_{H_2}(v)\}$, adjoin an end-vertex

$z_v^{(i)}$ to v. (3) Add K_{4b} to the new graph and adjoin each vertex labelled $y_u^{(i)}$ and $z_v^{(i)}$ to each vertex in K_{4b}. The resulting graph is G.



Observe that $V(H_1)(\subseteq V(G))$ is a defensive alliance (with cardinality $a(G)$) and that any defensive alliance which contains a vertex of H_1 must contain every vertex of H_1. Further, one sees that any alliance with vertices in $V(H_2)$ must contain a vertex labelled $z_v^{(i)}$. Also, observe that no alliance of G can contain any vertex labelled $y_u^{(i)}$ or $z_v^{(i)}$ unless it contains at least $1+2b$ vertices. Lastly, notice that any alliance of G that is a subset of $V(K_{4b}) \subseteq V(G)$ must also be an alliance of K_{4b} alone. Any such alliance must have cardinality at least $4b/2 = 2b$. With all these observations, ones sees that $V(H_1)$ must be the only defensive alliance of G with least cardinality. Similarly, $V(H_2)$ is the only strong defensive alliance of G that has minimum cardinality $\hat{a}(G)$. $\square$

Corollary 3.2. *For any $1 \leqslant a \leqslant b$, there exists a connected graph G with $a = a(G) \leqslant b = \hat{a}(G)$.*

Next, we see that any connected graph is the subgraph induced by the unique minimum strong alliance set of some graph. As with a minimum alliance, a minimum strong alliance will always induce a connected subgraph.

Theorem 3.3. *Given a connected graph H, there exists a connected graph G for which H is the subgraph induced by the unique global (respectively, strong global) defensive alliance of G with minimum cardinality $\gamma_a(G)$ (respectively, $\gamma_{\hat{a}}(G)$).*

P r o o f. Adjoin every vertex of $K_{\deg_H(v)+1}$ to each vertex $v \in H$. For proof of the strong global result, adjoin every vertex of $K_{\deg_H(v)}$ to each vertex $v \in H$. $\square$

The next result is a variation on Theorem 3.1. In the construction in Theorem 3.1, the two graphs H_1 and H_2 induced by the a-set and the $\hat{a}$-set, respectively, are disjoint. These two sets could also overlap. We would like to know if we can specify H_1, H_2, and the intersection of the two sets. The next result addresses this question in the case when H_1 is a subgraph of H_2.

First, a comment about notation. For a graph H_2 with subgraph H_1, we will use $H_2 - H_1$ as shorthand for the subgraph induced by the vertices $V(H_2) - V(H_1)$. If u is a vertex in H_1, we will write $\deg_{H_2 - H_1} u$ for the number of edges joining u to vertices in $H_2 - H_1$. Notice that this is a slight abuse of notation, since u is not in $H_2 - H_1$.

Theorem 3.4. *Suppose H_2 is a connected graph with a proper connected subgraph H_1 so that each of the following conditions hold:*

(1) *H_1 is a defensive alliance (not necessarily minimum) in H_2*

(2) *every vertex of H_1 is adjacent to a vertex in $H_2 - H_1$*

(3) *the subgraph of H_2 induced by $V(H_2) - V(H_1)$ is connected*

Then there exists a graph G so that the unique minimum strong defensive alliance of G is isomorphic to H_2 and the unique minimum defensive alliance of G is H_1.

Proof. Assume all of the conditions hold. We will construct G as follows. For each vertex v that is in H_2 and not in H_1, attach $\deg_{H_2} v$ new end-vertices. For each vertex u in H_1, attach $\deg_{H_1} u - \deg_{H_2 - H_1} u + 1$ new end-vertices. (Notice that $\deg_{H_1} u + 1 \geqslant \deg_{H_2 - H_1} u$ since H_1 is a defensive alliance in H_2.) Add a new complete subgraph K_{2n+1}, where n is the order of H_2. Join each of the new end-vertices to each of the vertices in the complete graph.

Claim 1. H_1 is a defensive alliance in G.

Each vertex u in H_1 is defended by itself and $\deg_{H_1} u$ neighbors. It has $\deg_{H_1} u - \deg_{H_2 - H_1} u + 1 + \deg_{H_2 - H_1} u$ enemies. Thus, it is defended.

Claim 2. Any other defensive alliance in G has more than $V(H_1)$ vertices.

Suppose a defensive alliance contains a vertex w in G that is not a vertex of H_2. Then the alliance must also contain at least $\lfloor \frac{1}{2} \deg_G w \rfloor$ of the neighbors of w. Since every vertex w not in H_2 has degree at least $2n$, the alliance must have at least $n + 1 > |V(H_2)| > |V(H_1)|$ vertices. Thus, we may assume without loss of generality that every defensive alliance is a subgraph of H_2.

Suppose a vertex $v \in H_2 - H_1$ is in a defensive alliance. Since it has at least $\deg_{H_2} v$ enemies not in H_2, it must have at least $\deg_{H_2} v - 1$ allies. Thus, all but one of its neighbors in H_2 must also be in the alliance. If the remaining neighbor is not in the alliance, then v has $\deg_{H_2} v + 1$ enemies; so we can conclude that every neighbor of v is in the alliance. Now, since $H_2 - H_1$ is connected, it follows that every vertex in $H_2 - H_1$ is in the alliance; and since every vertex of H_1 is adjacent to a vertex of $H_2 - H_1$, every vertex of H_1 is in the alliance.

Since H_1 is a proper subset of H_2, this alliance is larger than H_1.

Finally, suppose a proper subset of H_1 is a defensive alliance in G. Since H_1 is connected, there must be some $w \in V(H_1)$ which is in the alliance but adjacent to a vertex $u \in V(H_1)$ which is not in the alliance. Then w has at least $\deg_{H_2-H_1}(w) + \deg_{H_1}(w) - \deg_{H_2-H_1}(w) + 1 + 1$ enemies, including u, and at most $\deg_{H_1}(w) - 1 + 1$ allies, counting itself. This is a contradiction.

Claim 3. H_2 is a strong defensive alliance in G.

Consider a vertex v in $H_2 - H_1$. Since v has $\deg_{H_2} v$ allies in H_2 and $\deg_{H_2} v$ enemies outside of H_2, v is strongly defended. A vertex u in H_1 has $\deg_{H_2} u$ allies in H_2 and $\deg_{H_1} u - \deg_{H_2-H_1} u + 1 \leqslant \deg_{H_2} u - 1 + 1$ enemies outside of H_2, so u is also strongly defended.

Claim 4. Any other strong defensive alliance in G has more than $|V(H_2)|$ vertices.

As before, if a vertex $w \notin H_2$ is in a defensive alliance, so are at least half of its neighbors. Since every vertex not in H_2 has degree at least $2n$, this alliance has at least $n + 1$ vertices.

We may assume without loss of generality that any smaller strong defensive alliance is a subgraph of H_2. Any strong alliance is also an alliance, so, as argued in Claim 2, no proper subgraph of H_1 can be a strong alliance. If we consider H_1, then each vertex has one more enemy than ally; thus, H_1 is not a strong defensive alliance.

Suppose a vertex $v \in H_2 - H_1$ is in a strong defensive alliance. Since v has $\deg_{H_2} v$ enemies outside of H_2, every neighbor of v must also be in the alliance. Just as in Claim 2, it follows that every vertex in H_2 must be in the alliance. □

Each of the conditions in the theorem is necessary to the premise of Theorem 3.4.

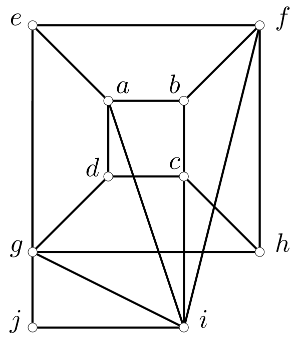
(i) If H_1 is not a defensive alliance in H_2, then it cannot be a defensive alliance in G, since we can only add more enemies. If H_1 is not connected and H_1 is a defensive alliance of G, then any component of H_1 is also a defensive alliance. Similarly, any component of a strong defensive alliance is also a strong defensive alliance.

(ii) We must have every vertex of H_1 adjacent to a vertex of $H_2 - H_1$. Consider the graph H_2 defined by $V(H_2) = \{a, b, c, d, e, f, g, h, i, j\}$ and

$$E(H_2) = \{ab, ad, ae, ai, bc, bf, cd, ch, ci, dg, ef, eg, fh, fi, gh, gi, gj, ij\},$$

with subgraph H_1 induced by $\{g, i, j\}$. Notice that H_1 is a connected subgraph of H_2 and a defensive alliance of H_2, and the graph induced by $H_2 - H_1$ is connected. However, there is no graph G that has H_2 as its minimum strong defensive alliance and H_1 as its minimum defensive alliance. Suppose there were such a graph G. Since

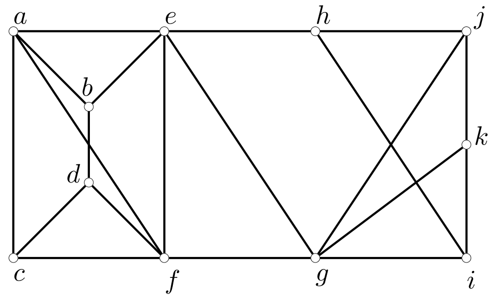
H_1 is a defensive alliance in G, there cannot be any additional vertices adjacent to g or i since they can barely defend themselves against the rest of H_2. Because H_2 is a strong defensive alliance, so is the graph induced by $H_2 - \{j\}$. The only vertices of H_2 defended by j are i and g, but they have no enemies outside of H_2.



(iii) Finally, we must have the subgraph induced by $H_2 - H_1$ connected. Consider the graph H_2 defined by $V(H_2) = \{a, b, c, d, e, f, g, h, i, j, k\}$ and

$$E(H_2) = \{ab, ac, ae, af, bd, be, cd, cf, df, ef, eg, eh, fg, gj, gk, gi, hi, hj, ik, jk\},$$

with subgraph H_1 induced by vertices e, f, g. Then H_1 is connected and a defensive alliance in H_2, and every vertex of H_1 is adjacent to a vertex of $H_2 - H_1$. However, there is no graph G with minimum strong alliance H_2 and minimum alliance H_1. Suppose to the contrary that there is such a G. Since H_1 is a defensive alliance in G, there cannot be any additional vertices adjacent to e, f, or g. However, as before, if H_2 is a strong defensive alliance in G, then so is the graph induced by $\{a, b, c, d, e, f\}$.



References

[1] *P. Kristiansen, S. M. Hedetniemi, S. T. Hedetniemi*: Alliances in graphs. J. Comb. Math. Comb. Comput. *48* (2004), 157–177. zbl

[2] *T. W. Haynes, S. T. Hedetniemi, M. A. Henning*: Global defensive alliances in graphs. Electron. J. Comb. *10* (2003), # R47. zbl

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