ACCENTUATE THE NEGATIVE

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Dedicated to Professor Josip E. Pečarić on the occasion of his 60th birthday

Abstract. A survey of mean inequalities with real weights is given.

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MSC 2000: 26D15

1. Introduction

We will be concerned with inequalities between means that are functions of $n$tuples of real numbers with which are associated some positive weights, a typical example being the geometric-arithmetic mean inequality:

$$\text{(GA)} \qquad \sqrt[W_n]{a_1^{w_1}\dots a_n^{w_n}} \leqslant \frac{w_1a_1+\dots+w_na_n}{W_n},$$

where the weights $w_1,\dots,w_n$ and the variables, $a_1,\dots,a_n$, are positive numbers and $W_n = w_1+\dots+w_n$.[1] There is no real reason for excluding zero values for the weights except that if for instance $w_n = 0$ this effectively means that we are stating or discussing the inequality for a smaller value of n. Equivalently allowing zero weights means that (GA) states the inequality for all values of $k, 1 \leqslant k \leqslant n$. A similar remark can be made about assuming all the variables are distinct.

[1] This notation will be used throughout; given real numbers $q_1,q_2,\dots,q_n$ then $Q_k = \sum\limits_{i=1}^{k} q_i$, $1 \leqslant k \leqslant n$. Also we write $\tilde{Q}_k = Q_n - Q_{k-1} = \sum\limits_{i=k}^{n} q_i$, $1 \leqslant k \leqslant n$.

However it is usual not to allow negative weights even though there is a very good and useful theory that covers this possibility. Classically the first person to consider real weights in detail was Steffensen early in the twentieth century. More recently very significant contributions have been made by Pečarić and his colleagues. The case of real weights has been of interest to Pečarić throughout his career from his student days up to the present. However the results are not generally known and this paper is an attempt to remedy this neglect.

Since almost all the inequalities between means are particular cases of the Jensen inequality for convex functions[2] the paper will concentrate on this result. Applications to particular means will then follow using the lines of the original application of Jensen's inequality.

2. Convex functions

The definitions and properties of convex functions are well known and will not be given in detail here. However the basic inequality of Jensen is equivalent to the definition of convexity and so in this section we will give details that are necessary for later discussion.

Perhaps the simplest analytic definition of a convex function is: let I be an open interval, $I \subseteq \mathbb{R}$,[3] then $f\colon\ I \to \mathbb{R}$ is convex if $\forall x, y \in I$ the function $D\colon\ ]0,1[\to \mathbb{R}$ is non-positive, where:

$$D(t) = D_2(t) = f\big((1-t)x + ty\big) - \big((1-t)f(x) + tf(y)\big) \leqslant 0. \tag{1}$$

It should be noted that if $x, y \in I$ then so is $\overline{x} = (1-t)x + ty$, $\forall\, t$, $0 < t < 1$, so all the terms on the right-hand side are defined.[4] Further note that D is defined for all t such that $\overline{x} \in I$ and use will be made of this in later discussions.

An alternative but equivalent definition is: $\forall\, z \in I$ there is an affine function $S_z\colon\ \mathbb{R} \to \mathbb{R}$ such that:

$$S_z(z) = f(z) \quad \text{and} \quad S_z(x) = f(z) + \lambda(x - z) \leqslant f(x) \ \ \forall x \in I.$$

See [6, p. 27; 8, pp. 70–75, 94–96; 18, p. 5; 20 p. 12].

The geometric interpretations of these definitions are immediate from Figures 1 and 2.

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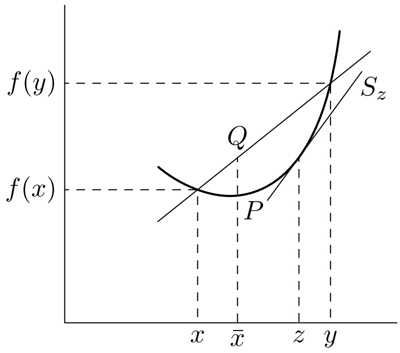
[2] Thus (GA) is just a property of the convexity of the function $f(x) = -\log x$, or the convexity of $g(x) = \mathrm{e}^x$; [4, pp. 6–7; 6, p. 92].

[3] This meaning for I will be used throughout the paper.

[4] More precisely if $x \neq y$, $\overline{x} \in I_0 =]\min\{x, y\}, \max\{x, y\}[$.



$P=(\bar{x}, f(\bar{x}))$ $Q=(\bar{x}, (1-t)f(x)+tf(y))$

Figure 1. Graph of a convex function

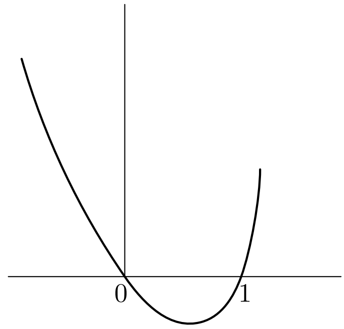


Figure 2. Graph of D

Use will be made of the following properties of convex functions.

(C1) The first divided difference $[x, y; f] = \big(f(x) - f(y)\big)/(x - y)$, $x, y \in I$, $x \neq y$, is increasing in both variables; [6, p. 26; 19, p. 2; 20, p. 6].

(C2) If $x, y, z, u, v \in I$ and $x \leqslant y \leqslant z \leqslant u \leqslant v$ and if $S_z(t) = f(z) + \lambda(t - z)$ then:

$$f(y) - f(x) \leqslant \lambda(y - x), \qquad f(v) - f(u) \geqslant \lambda(v - u).$$

See [16].

(C3) A function convex on I is continuous; [20, p. 4].[5]

(C4) The Hardy-Littlewood-Pólya-Karamata-Fuchs majorization theorem, or just HLPKF, [4, pp. 30–32; 6, pp. 23, 24, 30; 8, pp. 88–91; 10, pp. 64–67; 19, pp. 319–320]: if $\mathbf{a} = (a_1, \ldots, a_n)$,[6] $\mathbf{b} = (b_1, \ldots, b_n)$ are decreasing $n$tuples with entries in the domain of a convex function f and $\mathbf{w} = (w_1, \ldots, w_n)$ a real $n$tuple and if:

$$\sum_{i=1}^{k} w_i a_i \leqslant \sum_{i=1}^{k} w_i b_i, \quad 1 \leqslant k < n \quad \text{and} \quad \sum_{i=1}^{n} w_i a_i = \sum_{i=1}^{n} w_i b_i$$

then:

$$\sum_{i=1}^{n} w_i f(a_i) \leqslant \sum_{i=1}^{n} w_i f(b_i).$$

(C1) and (C2) are rather elementary and have obvious geometric interpretations but (C3) and (C4) are more sophisticated.

Jensen's inequality is an easy deduction from the definition of convexity and in a variety of forms is given in the following theorem.

[5] But not necessarily differentiable; consider $f(x) = |x|$.
[6] This notation for $n$tuples or sequences, will be used throughout.

Theorem 1. *Let $n \in \mathbb{N}, n \geqslant 2, I$ an interval, $f\colon I \to \mathbb{R}$ convex then:*

(a) *$\forall x_i \in I$, $1 \leqslant i \leqslant n$, and $\forall t_i$, $1 \leqslant i \leqslant n$, such that $0 < t_i < 1$, $1 \leqslant i \leqslant n$, and $t_1 = 1 - \sum_2^n t_i$ we have*

$$D(t_2, \dots t_n) = D_n(t_2, \dots t_n) = f\left(\sum_{i=1}^n t_i x_i\right) - \sum_{i=1}^n t_i f(x_i) \leqslant 0;$$

(b) *$\forall a_i \in I$, $1 \leqslant i \leqslant n$, and for all positive weights w_i, $1 \leqslant i \leqslant n$,*

$$f\left(\frac{1}{W_n}\sum_{i=1}^n w_i a_i\right) \leqslant \frac{1}{W_n}\sum_{i=1}^n w_i f(a_i); \tag{J_n}$$

(c) *$\forall\, a_i \in I$, $1 \leqslant i \leqslant n$, and positive weights p_i, $1 \leqslant i \leqslant n$, with $P_n = 1$,*

$$f\left(\sum_{i=1}^n p_i a_i\right) \leqslant \sum_{i=1}^n p_i f(a_i).$$

P r o o f. (i) The most well known proof is by induction, the case $n = 2$, (J_2), being just (1), a definition of convexity; [6, p. 31; 17; 18, pp. 43–44]. □

P r o o f. (ii) Another proof can be based on the support line definition above; [17; 19, pp. 189–190]. □

P r o o f. (iii) A geometric proof can be given as follows.

First note, using (1), that the set bounded by the chord joining $\big(x, f(x)\big)$ to $(y, f(y))$ and the graph of f joining the same points is a convex set. Then by induction show that the point $(\overline{a}, \overline{\alpha})$[7], $\overline{a} = \sum_{i=1}^n p_i a_i, \overline{\alpha} = \sum_{i=1}^n p_i f(a_i)$, lies inside this set and so $\overline{\alpha} \geqslant f(\overline{a})$ which is just (J_n). □

We now turn to the main interest of this paper. What happens if we allow negative weights in (J_n)?

3. The case of two variables

The inequality (J_2) is just $D(t) \leqslant 0$, $0 < t < 1$, and it is immediate from Figures 1 and 2 that if either $t < 0$ or $1 - t < 0$, equivalently $t > 1$, then $D(t) \geqslant 0$, that is the reverse inequality[8] holds. Formally we have the following result where the last of the notations in Theorem 1 is used, [6, p. 33; 9].

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[7] This point is just the weighted centroid of the points $(a_i, f(a_i))$, $1 \leqslant i \leqslant n$, that lie on the graph of f.

[8] The naming of reverse inequalities varies; sometimes the term inverse is used and sometimes converse but reverse seems to be the best usage.

Theorem 2. *If f is convex on the interval I and either $p_1 < 0$ or $p_2 < 0$ then for all a_1, a_2 in I with $\overline{a} = p_1a_1 + p_2a_2 \in I$,*

$$f(p_1a_1 + p_2a_2) \geqslant p_1f(a_1) + p_2f(a_2). \tag{$\sim$ J$_2$}$$

There is no loss in generality in assuming that $a_1 \neq a_2$.

P r o o f. (i) It is an easy exercise to use the second definition of convexity to prove that the function D is convex on its domain. Hence since $D(0) = D(1) = 0$ we must have that $D(t) \leqslant 0, 0 < t < 1$, and $D(t) \geqslant 0$, $t < 0$, $t > 1$, as shown in Figure 2. $\square$

P r o o f. (ii) Assume that $p_2 < 0$ then:

$$a_1 = \frac{\overline{a} - p_2a_2}{p_1} = \frac{\overline{a} - p_2a_2}{1 - p_2}.$$

So, using (J$_2$),

$$f(a_1) = f\Big(\frac{\overline{a} - p_2a_2}{1 - p_2}\Big) \leqslant \frac{f(\overline{a}) - p_2f(a_2)}{1 - p_2} = \frac{f(\overline{a}) - p_2f(a_2)}{p_1}.$$

Rewriting the last line gives ($\sim$ J$_2$). $\square$

P r o o f. (iii) Let us assume that $t < 0$ and, without loss of generality, that $a_1 < a_2$.

Then a_1 lies between $\overline{a}$ and a_2 and $a_1 = (\overline{a} - ta_2)/(1 - t)$. Now let $S = S_{a_1}$ then

$$f(a_1) = S(a_1) = S\Big(\frac{\overline{a} - ta_2}{1 - t}\Big) = \frac{S(\overline{a}) - tS(a_2)}{1 - t} \leqslant \frac{f(\overline{a}) - tf(a_2)}{1 - t},$$

which on rewriting gives ($\sim$ J$_2$).

Note that the condition $\overline{a} \in I$ is necessary as $\overline{a} \notin I_0 =]a_1, a_2[$ and so we must ensure that $f(\overline{a})$ is defined.[9] $\square$

In the case of two variables the situation is completely determined: either the weights are positive when we have Jensen's inequality or one is negative when we have the reverse of inequality.[10] In other terms: for all $x, y \in I, x \neq y$, with $\overline{x} \in I$ the sets $D_+ = \{t;\ t \in \mathbb{R} \wedge D(t) > 0\}$, $D_- = \{t;\ t \in \mathbb{R} \wedge D(t) < 0\}$, $D_0 = \{t;\ t \in \mathbb{R} \wedge D(t) = 0\}$ are partition $\mathbb{R}$ and do not depend on x or y.

This very simple result has been given this much attention as the ideas and methods of proof are used in the more complicated cases we now consider.

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[9] Clearly if $I = \mathbb{R}$ the condition can be omitted.

[10] The geometric-arithmetic mean inequality case of this result was the motivation for one of Pečarić's more interesting collaborations, [9].

This case is very different to the two variable situation discussed above but has its own peculiarities; in addition it introduces ideas needed for the general case. The function D can now be written:

$$D(s,t) = D_3(s,t) = f((1-s-t)x + sy + tz) - ((1-s-t)f(x) + sf(y) + tf(z)).$$

Clearly if x, y, z are distinct D_3 partitions $\mathbb{R}^2$ into three sets[11]: the closed convex 0-level curve D_0, the open convex set D_-, that is the interior of this curve and where (J_3) holds, and the unbounded exterior of the this curve, D_+, where ($\sim \mathrm{J}_3$) holds. However unlike the two variable case these sets depend on the variables x, y, z as we will now see.

The set where Jensen's inequality, (J_3), holds for all $x, y, z \in I$, is the triangle T where the above weights are positive

$$T = \{(s,t);\ 0 < s < 1,\ 0 < t < 1, 0 < s+t < 1\};$$

see Figure 3.

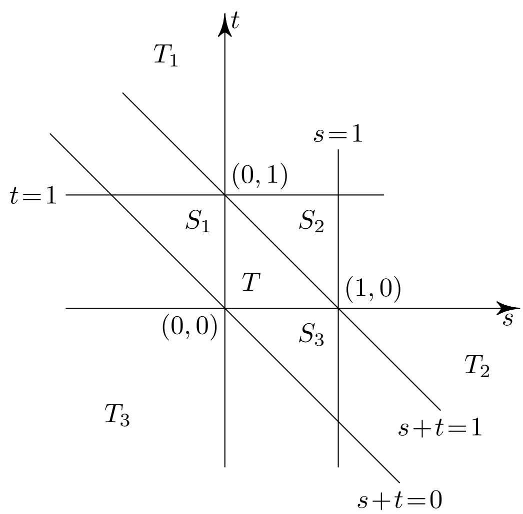

Figure 3.

On the sides of this triangle one of the weights is zero and so we have cases of the two variable situation, as we noted in Section 1, and as a result by (J_2) $D_3 \leqslant 0$ on the

[11] Using the notation of the previous section.

sides of T. Hence by continuity, (C_3), D_3 must be negative on a set larger than the triangle, that is $T \subset D_-$; note that the vertices of T lie on D_0. In any case for some choices of $x, y, z \in I$ (J_3) holds with negative weights and the question is whether there is a larger set than T on which (J_3) holds for a large choice of variables, or for variables satisfying some simple condition: [5; 6, pp. 39–41].

Let us first look at what happens when there are two negative weights.

The next result, due to Pečarić, [14; 15; 22], resolves the case when there is a maximum number of negative weights:[12] [6, p. 43; 19 p. 83].

Theorem 3. *If f is convex on the interval I and only one of p_1, p_2, p_3 is positive and if $a_1, a_2, a_3, \overline{a} = p_1a_1 + p_2a_2 + p_3a_3 \in I$ then*

$$(\sim \mathrm{J}_3) \qquad f(p_1a_1 + p_2a_2 + p_3a_3) \geqslant p_1f(a_1) + p_2f(a_2) + p_3f(a_3).$$

Again there is no loss of generality in assuming the a_1, a_2, a_3 are distinct.

P r o o f. (i) If we consider $D(s,t)$, $(s,t) \in \mathbb{R}^2$, and assume f is differentiable then it can easily be shown that D has no stationary points in its domain. An immediate conclusion is that (J_3) must hold in the triangle T since the maximum and minimum of D must occur on the boundary and it is non-positive there by (J_2). The domains where two of the weights are negative are the three unbounded triangles T_1, T_2, T_3 of Figure 3. By Theorem 2 D is non-negative on the boundaries of these triangles and so it would be reasonable to conclude that D is non-negative on these triangles giving a proof of ($\sim$ J_3). This proof is not quite complete as these are unbounded regions and this simple argument does not work. Let us look at the second proof of Theorem 2. □

P r o o f. (ii) Assume without loss of generality that $p_1 > 0$, $p_2 < 0$, $p_3 < 0$ then

$$a_1 = \frac{\overline{a} - p_2a_2 - p_3a_3}{p_1} = \frac{\overline{a} - p_2a_2 - p_3a_3}{1 - p_2 - p_3}.$$

So, using (J_3),

$$\begin{aligned} f(a_1) &= f\Big(\frac{\overline{a} - p_2a_2 - p_3a_3}{1 - p_2}\Big) \\ &\leqslant \frac{f(\overline{a}) - p_2f(a_2) - p_3f(a_3)}{1 - p_2} = \frac{f(\overline{a}) - p_2f(a_2) - p_3f(a_3)}{p_1}. \end{aligned}$$

Rewriting the last line gives ($\sim$ J_3). □

[12] Clearly three negative weights is the same as three positive weights.

It remains to consider what happens if there is only one negative weight. In order for (J_3) to hold we need $\overline{a} \in \overline{I}_0 = \big[\min\{a_1, a_2, a_3\}, \max\{a_1, a_2, a_3\}\big]$, and for ($\sim \mathrm{J}_3$) to hold $\overline{a} \in I \setminus I_0$. Assume without loss of generality that $a_1 < a_2 < a_3$ and assume that $p_1 < 0$ then

$$\overline{a} = p_1 a_1 + (p_2 + p_3)\frac{a_2 p_2 + a_3 p_3}{p_2 + p_3},$$

The second term on the right of the last term is in the interval $]a_2, a_3[$ and so $\overline{a}$ is to the right of a_2 and can lie either in I_0 or not depending on the value of the negative p_1. Further any condition on p_1 to require one or other of these options would obviously depend on the values of a_1, a_2, a_3.

A similar argument applies if the negative weight is p_3.

However in the case of the middle term a_2 having a negative weight, $p_2 < 0$. Steffensen, [21], obtained a simple condition on the weights that would assure $\overline{a} \in I_0$. Consider

$$\overline{a} = p_3(a_3 - a_2) + (p_3 + p_2)(a_2 - a_1) + a_1 = p_1(a_1 - a_2) + (p_1 + p_2)(a_2 - a_3) + a_3.$$

If we assume that $p_3 + p_2 > 0$ the first expression shows that $\overline{a} > a_1$ and if we require that $p_1 + p_2 > 0$ the second expression shows that $\overline{a} < a_3$. That is: with these two conditions on the weights $\overline{a} \in I_0$ and (J_3) should hold.

The conditions can be put in a simpler form:

$$(\mathrm{S}_3) \qquad 0 < p_1 < 0, \quad 0 < P_2 = p_1 + p_2 < 1.$$

(S_3) is easily seen to be equivalent to

$$(\tilde{\mathrm{S}}_3) \qquad 0 < p_3 < 0, \quad 0 < \tilde{P}_2 = p_3 + p_2 < 1$$

Later Pečarić, [14], gave an alternative form of this condition: the negative weight is dominated by both of the positive weights, that is

$$(\mathrm{P}_3) \qquad P_2 > 0, \quad \tilde{P}_2 > 0.$$

Thus we have the following result of Steffensen, [21]; several proofs are given, in addition to the one sketched above since they extend to give different results when $n > 3$.

Theorem 4. *If f is convex on the interval I and if $0 < p_1 < 1, 0 < P_2 < 1$ and either $a_1 \leqslant a_2 \leqslant a_3$ or $a_1 \geqslant a_2 \geqslant a_3$, with $a_1, a_2, a_3 \in I$, then*

$$f(p_1a_1 + p_2a_2 + p_3a_3) \leqslant p_1f(a_1) + p_2f(a_2) + p_3f(a_3). \tag{J$_3$}$$

There is no loss of generality in assuming no two of a_1, a_2, a_3 are equal.

Proof. (i) [5, p. 39] Assume, as we may, that $a_1 < a_2 < a_3$ and write $\tilde{a} = P_2a_2 + p_3a_3$; note that $a_1 < \overline{a} < a_3$ and $a_2 < \tilde{a} < a_3$. Now

$$\begin{aligned}
p_1f(a_1) &+ p_2f(a_2) + p_3f(a_3) - f(\overline{a}) \\
&= -p_1(f(a_2) - f(a_1)) + P_2f(a_2) + p_3f(a_3) - f(\overline{a}) \\
&\geqslant -p_1(f(a_2) - f(a_1)) + f(\tilde{a}) - f(\overline{a}), \quad \text{by (J}_2\text{)}, \\
&= p_1(a_2 - a_1)\Big(\frac{f(\tilde{a}) - f(\overline{a})}{\tilde{a} - \overline{a}} - \frac{f(a_2) - f(a_1)}{a_2 - a_1}\Big) \\
&\geqslant 0, \quad \text{by (C}_1\text{)}.
\end{aligned}$$

□

Proof. (ii) [14] In this proof we use the condition $p_2 < 0$, a fact that was not used in the first proof.

Again asuming $a_1 < a_2 < a_3$ we have that for some t, $0 < t < 1$ that $a_2 = (1-t)a_1 + ta_3$. Then:

$$\overline{a} = p_1a_1 + p_2(1-t)a_1 + ta_3 + p_3a_3 = (p_1 + (1-t)p_2)a_1 + (p_3 + tp_2)a_3.$$

So by (J$_2$), noting that the coefficients of the last expression are positive and have sum equal to 1,

$$\begin{aligned}
f(\overline{a}) &= f((p_1 + (1-t)p_2)a_1 + (p_3 + tp_2)a_3) \\
&\leqslant (p_1 + (1-t)p_2)f(a_1) + (p_3 + tp_2)f(a_3) \\
&= p_1f(a_1) + p_2((1-t)f(a_1) + tf(a_3)) + p_3f(a_3) \\
&\leqslant p_1f(a_1) + p_2f((1-t)a_1 + ta_3) + p_3f(a_3) \\
&= p_1f(a_1) + p_2f(a_2) + p_3f(a_3),
\end{aligned}$$

□

Proof. (iii) [16; 19, pp. 57–58] Assume without loss in generality that $a_1 < \overline{a} < a_2 < a_3$ and define λ by: $S_{\overline{a}}(x) = f(\overline{a}) + \lambda(x - \overline{a})$.

Using (C2) we get:

$$\begin{aligned} p_1 f(a_1) + p_2 f(a_2) + p_3 f(a_3) - f(\overline{a}) \\ &= p_1\big(f(a_1) - f(\overline{a})\big) + (p_2 + p_3)\big(-f(\overline{a}) + f(a_2)\big) + p_3\big(f(a_3) - f(a_2)\big) \\ &\geqslant p_1\lambda(a_1 - \overline{a}) + (p_2 + p_3)\lambda(a_2 - \overline{a}) + p_3\lambda(a_3 - a_2) = 0. \end{aligned}$$

□

P r o o f. (iv) [17] Without loss of generality assume that $b_1 = a_1 > b_2 = \overline{a} > b_3 = a_2 > b_4 = a_3$. Further define $q_1 = p_1$, $q_2 = -1$, $q_3 = p_2$, $q_4 = p_3$; then if $c_i = \overline{a}, 1 \leqslant i \leqslant 4$:

$$\begin{aligned} &q_1 b_1 = p_1 a_1 \geqslant p_1 \overline{a} = q_1 c_1 \\ &q_1 b_1 + q_2 b_2 = p_1 a_1 - \overline{a} \geqslant q_1 c_1 + q_2 c_2 \\ &q_1 b_1 + q_2 b_2 + q_3 b_3 = p_1 a_1 + p_2 a_2 - \overline{a} \geqslant_1 c_1 + q_2 c_2 + q_3 c_3 \\ &q_1 b_1 + q_2 b_2 + q_3 b_3 + q_4 b_4 = 0 = q_1 c_1 + q_2 c_2 + q_3 c_3 + q_4 c_4 \end{aligned}$$

hence by (C_4), HLPKF:

$$q_1 f(b_1) + q_2 f(b_2) + q_f(b_3) + q_4 f(b_4) \geqslant q_1 f(c_1) + q_2 f(c_2) + q_3 f(c_3) + q_4 f(c_4) = 0,$$

which is just (J_3). □

P r o o f. (v) The Steffensen condition tells us that the point $\big(\overline{a}, p_1 f(a_1) + p_2 f(a_2) + p_3 f(a_3)\big)$ lies in the convex hull of the points $\big(a_i, f(a_i)$, $a \leqslant i \leqslant 3$, and so lies in the convex set $\{(x, y);\ y \geqslant f(x)\}$ and this implies (J_3). □

Using the notation in the definition of D_3 and assuming that $x < y < z$ and $s < 0$ the condition (S_3) is just: $0 \leqslant s + t \leqslant 1$, $0 \leqslant t \leqslant 1$ and so (J_3) holds in the triangle S_1 of Figure 3. Depending on the order of x, y, z and provided the central element has the only negative weight and (S_3) holds then (J_3) will hold in one of S_1, S_2, S_3 of Figure 3.

5. The n variable case

In this section we turn to the general situation and the notations are those of Theorem 1.

Let us first consider the extension of Theorem 3. The second proof of Theorem 3 can easily be adapted to the following result of Pečarić; [6, p. 43; 19, p. 83; 22].

Theorem 5. *If $f\colon I \to \mathbb{R}$ is convex, $n \in \mathbb{N}$, $n \geqslant 2$, $a_i \in I$, $w_i \in \mathbb{R}, w_i \neq 0$, $1 \leqslant i \leqslant n$, further assume that all the weights are negative except one, $W_n \neq 0$, and that $\overline{a} \in I$ then:*

$$(\sim \mathrm{J}_n) \qquad f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i a_i\right) \geqslant \frac{1}{W_n}\sum_{i=1}^{n} w_i f(a_i),$$

or, using an alternative notation,

$$(\sim \mathrm{J}_n) \qquad f\left(\sum_{i=1}^{n} p_i a_i\right) \geqslant \sum_{i=1}^{n} p_i f(a_i).$$

The case $n = 2$ is Theorem 2, and the case $n = 3$ is Theorem 3.

Assume then $n \geqslant 3$ and, without loss of generality, that $p_1 > 0$ and $p_i < 0$, $2 \leqslant i \leqslant n$, then:

$$a_1 = \frac{\overline{a} + \sum\limits_{i=2}^{n}(-p_i)a_i}{p_1} = \frac{\overline{a} + \sum\limits_{i=2}^{n}(-p_i)a_i}{1 + \sum\limits_{i=2}^{n}(-p_i)}.$$

So by (J_n),

$$\begin{aligned} f(a_1) &\leqslant \frac{1}{1 + \sum\limits_{i=2}^{n}(-p_i)}\left(f(\overline{a}) + \sum_{i=2}^{n}(-p_i)f(a_i)\right) \\ &= \frac{1}{p_1}\left(f(\overline{a}) + \sum_{i=2}^{n}(-p_i)f(a_i)\right), \end{aligned}$$

which on rewriting is just $(\sim \mathrm{J}_n)$.

We now turn to the situation where (J_n) holds but there are negative weights, the generalization of Theorem 4 due Steffensen. Note that from Theorem 5 we will need at least two positive weights for (J_n) to hold.

The important conditions put on the weights by Steffensen and Pečarić, (S_3), and (P_3) above, now differ and are as follows, using the alternative notion of Theorem 5.

(S) $\quad 0 < P_i < 1$, $1 \leqslant i \leqslant n-1$; and of course $P_n = 1$.

This implies that $0 < \tilde{P}_k < 1$, $1 < k \leqslant n$, and in particular that $0 < p_1 < 1$ and $0 < p_n < 1$.

For (P) we introduce the following notation:

$$I_+ = \{i; 1 \leqslant i \leqslant n \wedge p_i > 0\} \text{ and } I_- = \{i; 1 \leqslant i \leqslant n \wedge p_i < 0\};$$

obviously $I_+ \cap I_- = \emptyset$ and $I_+ \cup I_- = \{1, 2, \ldots, n\}$.

(P) $\quad p_1, p_n \in I_+$ and $\forall\, i \in I_+,\ p_i + \sum_{j\in I_-} p_j > 0.$

It is easy to see that (P) implies (S). Further we have the following simple result, [6, p. 38; 19, pp. 37–38].

Lemma 6. *If* $\mathbf{a}$ *is monotonic,* $a_1 \neq a_n$ *and* (S) *holds then* $\overline{a} \in I_0 =]\max \mathbf{a}, \min \mathbf{a}[$.

P r o o f. Assume without loss in generality that the $n$tuple is increasing. Since

$$\begin{aligned}\overline{a} &= \sum_{i=1}^{n} p_i a_i = a_n + \sum_{i=1}^{n-1} P_i (a_i - a_{i+1}) \\ &= a_1 + \sum_{i=2}^{n} \tilde{P}_i (a_i - a_{i-1})\end{aligned}$$

the result follows by (S). □

All the proofs of Theorem 4 can be extended to give a proof of the general case.

Theorem 7. *Let* $n \in \mathbb{N}$, $n \geqslant 3$, $I \subseteq \overline{\mathbb{R}}$ *an interval,* $f\colon\ I \to \mathbb{R}$ *convex then for all monotonic* $n$tuples *with terms in* I (J_n) *holds for all non-zero real weights satisfying condition* (S).

P r o o f. (i) The standard proof is by induction starting with then case $n = 3$, Theorem 4; see [6, pp. 37–39]. □

P r o o f. (ii) This proof, due to Pečarić, [14], assumes the stronger condition (P) but in a weaker form that still implies (S):

$(P^\flat)$ $\quad p_1, p_n \in I_+$ and $p_i + \sum_{j\in I_-} p_j > 0,\ i = 1, n.$

We also assume without loss in generality that the $n$tuple is increasing and distinct.

If $i \in I_-$ then $a_1 < a_i < a_n$ and hence for some t_i, $0 < t_i < 1$, $a_i = (1-t_i)a_1 + t_i a_n$ and so

$$\begin{aligned}\overline{a} &= \sum_{i\in I_+} p_i a_i + \sum_{i\in I_-} p_i((1-t_i)a_1 + t_i a_n) \\ &= \Big(p_1 + \sum_{i\in I_-} p_i((1-t_i))\Big) a_1 + \sum_{i\in I_+\setminus\{1,n\}} p_i a_i + \Big(p_n + \sum_{i\in I_-} p_i t_i\Big) a_n\end{aligned}$$

Note that the sum of the weights in this last expression is 1 and that by $(P^\flat)$ they are all positive. Hence by Jensen's inequality

$$\begin{aligned}
f(\overline{a}) &\leqslant \Bigl(p_1 + \sum_{I_-} p_i((1-t_i))\Bigr) f(a_1) + \sum_{I_+\setminus\{1,n\}} p_i f(a_i) + \Bigl(p_n + \sum_{I_-} p_i t_i\Bigr) f(a_n) \\
&= \sum_{i\in I_+} p_i f(a_i) + \sum_{i\in I_-} p_i((1-t_i) f(a_1) + t_i f(a_n)) \\
&\leqslant \sum_{i\in I_+} p_i f(a_i) + \sum_{i\in I_-} p_i f(a_i) \text{ by } (\mathrm{J}_2) \text{ and} \\
&\qquad\qquad\qquad\qquad \text{the negativity of the } p_i \text{ in the last sum,} \\
&= \sum_{i=1}^{n} p_i f(a_i).
\end{aligned}$$

which is (J_n). ☐

P r o o f. (iii) [19, pp. 57–58]. First note that if λ is defined as in Theorem 4 then: $S_{\overline{a}}(x) = f(\overline{a}) + \lambda(x - \overline{a})$; and by (C2):

$$\overline{a} \leqslant u \leqslant v \Longrightarrow f(v) - f(u) \geqslant \lambda(v-u); \quad u \leqslant v \leqslant \overline{a} \Longrightarrow f(v) - f(u) \leqslant \lambda(v-u).$$

By Lemma 6 we have that $a_1 > \overline{a} > a_n$, and let $a_{k+1} \leqslant \overline{a} \leqslant a_k$ for some k, $1 \leqslant k \leqslant n-1$. Then

$$\begin{aligned}
f(\overline{a}) - \sum_{i=1}^{n} p_i f(a_i) &= f(\overline{a}) - \sum_{i=1}^{k} p_i f(a_i) - \sum_{i=1}^{k+1} p_i f(a_i) \\
&= f(\overline{a}) - \sum_{i=1}^{k-1} P_i\bigl(f(a_i) - f(a_{i-1})\bigr) - P_k f(a_k) \\
&\quad - \sum_{i=k}^{n-1} \tilde{P}_i\bigl(f(a_{i+1} - f(a_i))\bigr) - \tilde{P}_{k+1} f(a_k) \\
&= \sum_{i=1}^{k-1} P_i\bigl(f(a_{i-1}) - f(a_i)\bigr) + P_k\bigl(f(\overline{a}) - f(a_k)\bigr) \\
&\quad + \tilde{P}_{k+1}\bigl(f(\overline{a}) - f(a_k)\bigr) + \sum_{i=k}^{n-1} \tilde{P}_i\bigl(f(a_i) - f(a_{i+1})\bigr) \\
&\geqslant \sum_{i=1}^{k-1} \lambda P_i(a_{i-1} - a_i) + \lambda P_k(\overline{a} - a_k) + \lambda \tilde{P}_{k+1}(\overline{a} - a_{k+1}) + \sum_{i=k}^{n-1} \lambda \tilde{P}_i(a_i - a_{i+1}) \\
&= \lambda\Bigl(\overline{a} - \sum_{i=1}^{n} p_i a_i\Bigr) = 0.
\end{aligned}$$

☐

P r o o f. (iv) [14] Using the notations and assumptions of the previous proof define the three $n$tuples $x_1, \ldots, x_{n+1}$, $y_1, \ldots, y_{n+1}$, $q_1, \ldots q_{n+1}$:

$$\begin{aligned} x_i &= a_i,\ q_i = p_i, \quad 1 \leqslant i \leqslant k; \\ x_{k+1} &= \overline{a},\ q_{k+1} = -1; \\ x_i &= a_{i-1},\ q_i = p_{i-1}, \quad k+2 \leqslant i \leqslant n+1; \\ y_i &= \overline{a}, \quad 1 \leqslant i \leqslant n+1. \end{aligned}$$

Simple calculations show that:

$$Q_j = \begin{cases} P_j, & 1 \leqslant j \leqslant k, \\ \tilde{P}_{j-1}, & k+1 \leqslant j \leqslant n, \\ 0, & j = n+1; \end{cases}$$

and,

$$\sum_{i=1}^{j} q_i y_i = \begin{cases} P_j \overline{a}, \quad 1 \leqslant j \leqslant k, \\ = \tilde{P}_{j-1}\overline{a}, \quad k+1 \leqslant j \leqslant n, \\ = 0, \quad j = n+1. \end{cases}$$

$$\sum_{i=1}^{j} q_i x_i = \begin{cases} \sum\limits_{i=1}^{j-1} P_i(x_i - x_{i+1}) + P_j a_j, \quad 1 \leqslant j \leqslant k, \\ = \sum\limits_{i=1}^{j-1} P_i(x_i - x_{i+1}) + \tilde{P}_{j-1}\overline{a}, \quad k+1 \leqslant j \leqslant n, \\ = 0, \quad j = n+1. \end{cases}$$

Hence:

$$\sum_{i=1}^{k} q_i x_i \geqslant \sum_{i=1}^{k} q_i y_i, \quad 1 \leqslant k \leqslant n; \qquad \sum_{i=1}^{n+1} q_i x_i = \sum_{i=1}^{n+1} q_i y_i,$$

and by HLPKF if f is convex then

$$\sum_{i=1}^{n+1} q_i f(x_i) \geqslant \sum_{i=1}^{n+1} q_i f(y_i) = 0,$$

which is just (J_n). □

A variant of this result can be found in [1].

While ($P^\flat$) makes much more demands on the negative weights than does (S) its real advantage in its stronger form (P), as Pečarić pointed out, is that no requirement on monotonicity of the elements of the $n$tuple is needed. This allows an extension of Theorem 7 to convex functions of several variables as we shall now demonstrate; [11].

If $U \subseteq \mathbb{R}^k$, $k \geqslant 1$, where U is a convex set then the definition of convexity is, with a slight change in notation, just that given in (1): for all $\mathbf{x}, \mathbf{y} \in \mathbf{U}$

$$D(t) = D_2(t) = f((1-t)\mathbf{x} + t\mathbf{y}) - ((1-t)f(\mathbf{x}) + tf(\mathbf{y})) \leqslant 0, \quad 0 \leqslant t \leqslant 1,$$

and the convexity of U ensures that $(1-t)\mathbf{x} + t\mathbf{y} \in U$. Further one of the standard proofs of (J_n) can be applied in this situation to obtain Jensen's inequality for such functions f. Of course we cannot hope to extend the Steffensen result, if $k \geqslant 2$, as the concept of increasing order of the points in U is not available but the Pečarić argument can be extended using the same proof as the one given above in the case $k = 1$ and uses the same notations.

Theorem 8. *Let U be an open convex set in $\mathbb{R}^k$, $\mathbf{a}_i \in U$, $1 \leqslant i \leqslant n$, and let p_i, $1 \leqslant i \leqslant n$, be non-zero real numbers with $P_n = 1$ and $I_- = \{i; 1 \leqslant i \leqslant n \wedge p_i < 0\}$, $I_+ = \{i; 1 \leqslant i \leqslant n \wedge p_i > 0\}$. Further assume that $\forall i$, $i \in I_-$, $\mathbf{a}_i$ lies in the convex hull of the set $\{\mathbf{a}_i; i \in I_+\}$ and that $\forall j$, $j \in I_+$, $p_j + \sum\limits_{i \in I_-} p_i \geqslant 0$. If $f\colon U \to \mathbb{R}$ is convex then (J_n) holds.*

Proof. (ii) of Theorem 7 can be applied with almost no change although the notation is a little messier.

If $i \in I_-$ then for some $t_j^{(i)}$, $0 \leqslant t_j^{(i)} \leqslant 1$, $\sum\limits_{j \in I_+} t_j^{(i)} = 1$, $\mathbf{a}_i = \sum\limits_{j \in I_+} t_j^{(i)} \mathbf{a}_j$ and so

$$\begin{aligned}
\overline{\mathbf{a}} = \sum_{i=1}^n p_i \mathbf{a}_i &= \sum_{j \in I_+} p_j \mathbf{a}_j + \sum_{i \in I_-} p_i \biggl(\sum_{j \in I_+} t_j^{(i)} \mathbf{a}_j \biggr) \\
&= \sum_{j \in I_+} \biggl(p_j + \sum_{i \in I_-} p_i t_j^{(i)} \biggr) \mathbf{a}_j \\
&= \sum_{j \in I_+} q_j \mathbf{a}_j.
\end{aligned}$$

where, as in proof (ii) above, $0 < q_j < 1$, $\sum\limits_{j \in I_+} q_j = 1$. In this proof we now use the strong requirement (P) and incidentally provide a needed proof that $\overline{\mathbf{a}} \in U$. The rest of the proof proceeds as in proof (ii) of Theorem 7. $\square$

Note that in the case $k = 1$ the hypotheses imply that the smallest and largest element in the $n$tuple have positive weights each of which dominates the sum of all the negative weights.

We now turn to $(\sim \mathrm{J})$ and note that proof (iv) of Theorem 7 can with a suitable change of hypotheses lead to this inequality; [14; 16].

Theorem 9. *Let n, I, be as in Theorem 7, $p_1, \ldots p_n$ a real $n$tuple with $P_n = 1$, then the reverse Jensen inequality holds for all functions f convex on I and for every monotonic tuple with terms in I if and only if for some $m, 1 \leqslant m \leqslant n$, $P_k \leqslant 0$, $1 \leqslant k < m$, and $\tilde{P}_k \leqslant 0$, $m < k \leqslant n$.*

Proof. Looking at proof (iv) of Theorem 7 we see that the present hypotheses imply that

$$\sum_{i=1}^{k} q_i x_i \leqslant \sum_{i=1}^{k} q_i y_i, \quad 1 \leqslant k \leqslant n; \qquad \sum_{i=1}^{n+1} q_i x_i = \sum_{i=1}^{n+1} q_i y_i,$$

and by HLPKF if f is convex then

$$\sum_{i=1}^{n+1} q_i f(x_i) \leqslant \sum_{i=1}^{n+1} q_i f(y_i) = 0,$$

which is just ($\sim \mathrm{J}_n$). □

6. Applications, cases of equality, integral results

The most obvious application of these extensions and reversals of the Jensen inequality are to mean inequalities. A large variety of means derive from the convexity of a particular function and so we find that these inequalities will now hold with negative weights satisfying the above conditions or will hold reversed.

6.1 An Example. If p_1, p_2, p_3, p_4 are non-zero real numbers with $P_4 = 1$ and a_1, a_2, a_3, a_4 are distinct positive numbers then, using the convexity of the negative of the logarithmic function, the particular case of (GA)

$$a_1^{p_1} a_2^{p_2} a_3^{p_3} a_4^{p_4} \leqslant p_1 a_1 + p_2 a_2 + p_3 a_3 + p_4 a_4$$

can be deduced from Theorem 7 provided one of the following holds:
(i) all the weights are positive;
(ii) $a_1 < a_2 < a_3 < a_4$ or $a_1 > a_2 > a_3 > a_4$ and $0 < p_1 < 1$, $0 < P_2 < 1$, $0 < P_3 < 1$;
(iii) $a_1 < a_2$, $a_3 < a_4$, and $p_1 > 0$, $p_4 > 0$ and $P_3 > 0$, $\tilde{P}_3 > 0$.

The reverse inequality

$$a_1^{p_1} a_2^{p_2} a_3^{p_3} a_4^{p_4} \geqslant p_1 a_1 + p_2 a_2 + p_3 a_3 + p_4 a_4,$$

can be deduced from Theorem 5 or Theorem 9 if one of the following holds:

(i) only one of the weights is positive;

(ii) either $a_1 > a_2 > a_3 > a_4$, or $a_1 < a_2, a_3 < a_4$ and either $0 < p_1 < 1$ and $\tilde{P}_2, \tilde{P}_3, p_4 < 0$, or $0 < p_2 < 1$ and $p_1, \tilde{P}_3 < 0$, $p_4 < 0$, or $0 < p_3 < 1$ and $p_1, P_2, p_4 < 0$ or $0 < p_4 < 1$ and $p_1, P_2, P_3 < 0$.

6.2 The pseudo means of Alzer. A particular case of Theorem 5 has been studied by Alzer under the name of pseudo-means, [3; 6, pp. 171–173].

Corollary 10. *If f is convex on I and p_i, $1 \leqslant i \leqslant n$, are positive weights with $P_n = 1$ then*

$$f\left(\frac{1}{p_1}\left(a_1 - \sum_{i=2}^{n} p_i a_i\right)\right) \geqslant \frac{1}{p_1}\left(f(a_1) - \sum_{i=2}^{n} p_i f(a_i)\right),$$

provided a_i, $1 \leqslant i \leqslant n$, ${p_1}^{-1}\left(a_1 - \sum_{i=2}^{n} p_1 a_1\right) \in I$.

A particular case when $f(x) = x^{s/r}$, $0 < r < s$, $x > 0$, leads to the inequality

$$\left(\frac{1}{p_1}\left(a_1^s - \sum_{i=2}^{n} p_i a_i^s\right)\right)^{1/s} \geqslant \left(\frac{1}{p_1}\left(a_1^r - \sum_{i=2}^{n} p_i a_i^r\right)\right)^{1/r}.$$

A related topic is the Aczél-Lorenz inequalities; [2; 6, pp. 198–199; 19, pp. 124–126].

6.3 The inverse means of Nanjundiah. Nanjundiah devised some very ingenious arguments using his idea of inverse means, [5, pp. 136–137,226; 13]. In the case of $r > 0$ Nanjundiah's inverse r-th power mean of order n is defined as follows: let $\mathbf{a}, \mathbf{w}$, be two sequences of positive numbers then

$$\mathfrak{N}_n^{[r]}(\mathbf{a}; \mathbf{w}) = \left(\frac{W_n}{w_n} a_n^r - \frac{W_{n-1}}{w_n} a_{n-1}^r\right)^{1/r}.$$

An immediate consequence of Theorem 2 with $f(x) = x^{s/r}$, $0 < r < s$, $x > 0$, is the inequality

$$\mathfrak{N}_n^{[r]}(\mathbf{a}; \mathbf{w}) \geqslant \mathfrak{N}_n^{[s]}(\mathbf{a}; \mathbf{w}).$$

6.4 Comparable means. If φ is a strictly increasing function then a quasi-arithmetic mean is defined as follows:

$$\mathfrak{M}_\varphi(\mathbf{a}; \mathbf{w}) = \varphi^{-1}\left(\frac{1}{W_n} \sum_{i=1}^{n} w_i \varphi(a_i)\right).$$

An important question is when two such means are comparable, that is: when is it always true that:

$$\mathfrak{M}_{\varphi}(\mathbf{a};\mathbf{w}) \leqslant \mathfrak{M}_{\psi}(\mathbf{a};\mathbf{w})$$

Writing $\varphi(a_i) = b_i$, $1 \leqslant i \leqslant n$, this last inequality:

$$\psi \circ \varphi^{-1}\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i b_i\right) \leqslant \frac{1}{W_n}\sum_{i=1}^{n} w_i \psi \circ \varphi^{-1}(b_i),$$

showing, from (J_n), that the means are comparable exactly when $\psi\circ\varphi^{-1}$ is convex, [6, pp. 273–277]. Using Theorem 7 we can now allow negative weights in the comparison and by using Theorem 5 or 9 get the opposite comparison; [1].

Daróczy & Páles, [7], have defined a class of general means that they called *L-conjugate means*:

$$L_{\varphi}^{\mathfrak{M}_1,\ldots,\mathfrak{M}_n}(\mathbf{a};\mathbf{u};\mathbf{v}) = L_{\varphi}(\mathbf{a};\mathbf{u};\mathbf{v}) = \varphi^{-1}\left(\sum_{i=1}^{m} u_i \varphi(a_i) - \sum_{j=1}^{n} v_j \varphi \circ \mathfrak{M}_j(\mathbf{a})\right)$$

where $U_m - V_n = 1$, $u_i > 0$, $1 \leqslant i \leqslant m$, $v_j > 0$, $1 \leqslant j \leqslant n$, $\mathfrak{M}_j$, $1 \leqslant j \leqslant n$, are means on $n$tuples and φ is as above.

Now suppose we wish to compare two L-conjugate means:

$$L_{\varphi}(\mathbf{a};\mathbf{u};\mathbf{v}) \leqslant L_{\psi}(\mathbf{a};\mathbf{u};\mathbf{v}),$$

Using the above substitution, $\varphi(a_i) = b_i$, $1 \leqslant i \leqslant m$, and writing $\mathfrak{N}_j = \varphi \circ \mathfrak{M}_j$ this last inequality becomes

$$\psi \circ \varphi^{-1}\left(\sum_{i=1}^{m} u_i b_i - \sum_{j=1}^{n} v_j \mathfrak{N}_j(\mathbf{b})\right) \leqslant \sum_{i=1}^{m} u_i \psi \circ \varphi^{-1}(b_i) - \sum_{j=1}^{n} v_j \psi \circ \varphi^{-1} \circ \mathfrak{N}_j(\mathbf{b})$$

which, from Theorem 8 in the case $k = 1$, holds if $\psi \circ \varphi^{-1}$ is convex, as for the quasi-arithmetic means; [11].

In this sense this result of Pečarić gives a property of convex functions analogous to that of Jensen's inequality but useful for these means whereas Jensen's inequality is useful for the classical quasi-arithmetic means.

It should be remarked that extensions of this comparison result can be obtained allowing the weights $\mathbf{u}, \mathbf{v}$ to be real and using Theorem 7; see [1].

6.5 Cases of equality. Clearly the function D of (1) is zero if either $t = 0$, $t = 1$ or $x = y$; if otherwise $D < 0$ then f is said to be strictly convex. If this is the case then Jensen's inequality, (J_n), is strict unless $a_1 = \ldots = a_n$.

It follows easily from the proof of Theorem 5 that ($\sim$ J$_n$) holds strictly for strictly convex functions under the conditions of that theorem unless $a_1 = \ldots = a_n$.

In Theorem 7, Steffensen's extension of Jensen's inequality, the same is true by a consideration of proof (ii); see [1].

6.6 Integral results. Most if not all of the above results have integral analogues but a discussion of these would take us beyond the bounds of this paper; [6, p. 371; 15; 19, pp.45–47, 84–87].

References

[1] *Abramovich, S., Klaričić Bakula, M., Matić, M., Pečarić, J.*: A variant of Jensen-Steffensen's inequality and quasi-arithmetic means. J. Math. Anal. Appl. *307* (2005), 370–386.

[2] *Aczél, J.*: Some general methods in the theory of functional equations in one variable. New applications of functional equations. Uspehi Mat. Nauk (N.S.) *11* (1956), 3–68.

[3] *Alzer, Horst*: Inequalities for pseudo arithmetic and geometric means. General Inequalities *6* (Oberwolfach 1990), 5–16.

[4] *Beckenbach, Edwin F., Bellman, Richard*: Inequalities. Springer, Berlin, 1961.

[5] *Bullen, P. S.*: The Jensen-Steffensen inequality. Math. Ineq. Appl. *1* (1998), 391–401.

[6] *Bullen, P. S.*: Handbook of Means and Their Inequalities. Kluwer Academic Publishers, Dordrecht, 2003.

[7] *Daróczy, Zoltán, Páles, Zsolt*: On a class of means of several variables. Math. Ineq. Appl. *4* (2001), 331–334.

[8] *Hardy, G. H., Littlewood, J. E., Pólya, G.*: Inequalities. Cambridge University Press, Cambridge, 1934.

[9] *Landsberg, P. T., Pečarić, J. E.*: Thermodynamics, inequalities and negative heat capacities. Phys. Rev. *A35* (1987), 4397–4403.

[10] *Marshall, Albert W., Olkin, Ingram*: Inequalities: Theory of Majorization and Its Applications. Academic Press, New York, 1979.

[11] *Matković, Anita, Pečarić, Josip*: A variant of Jensen's inequality for convex functions of several variables. J. Math. Ineq. *1* (2007), 45–51.

[12] *Mitrinović, D. S., Pečarić, J. E., Fink, A. M.*: Classical and New Inequalities in Analysis. Reidel, Dordrecht, 1993.

[13] *Nanjundiah, T. S.*: Sharpening some classical inequalities. Math. Student *20* (1952), 24–25.

[14] *Pečarić, J. E.*: Inverse of Jensen-Steffensen inequality. Glasnik Mat. *16* (1981), 229–233.

[15] *Pečarić, Josip E.*: A new proof of the Jensen-Steffensen inequality. Mathematica Rév. Anal. Num. Théorie Appr. *23* (1981), 73–77.

[16] *Pečarić, J. E.*: Inverse of Jensen-Steffensen inequality. (II), Glasnik Mat. *19* (1984), 235–238.

[17] *Pečarić, Josip E.*: A simple proof of the Jensen-Steffensen inequality. Amer. Math. Monthly *91* (1984), 195–196.

[18] *Pečarić, Josip E.*: Notes on Jensen's inequality. General Inequalities 6 (Oberwolfach, 1990), 449–454; (1992). Internat. Ser. Numer. Math., 103, Birkhäuser, Basel.

[19] *Pečarić, Josip E., Proschan, Frank, Tong, Y. L.*: Convex Functions, Partial Orderings and Statistical Applications. Academic Press, New York, 1992.

[20] *Roberts, A. Wayne, Varberg, Dale E.*: Convex Functions. Academic Press, New York, 1973.
[21] *Steffensen, J. F.*: On certain inequalities and methods of approximation. J. Inst. Actuar. *51* (1919), 274–297.
[22] *Vasić, Petar, M., Pečarić, Josip E.*: On the Jensen inequality. Univ. Beograd Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 634–No. 677 (1979), 50–54.

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