A NOTE ON THE FUNDAMENTAL MATRIX OF VARIATIONAL EQUATIONS IN $\mathbb{R}^3$

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Abstract. The paper is devoted to the question whether some kind of additional information makes it possible to determine the fundamental matrix of variational equations in $\mathbb{R}^3$. An application concerning computation of a derivative of a scalar Poincaré mapping is given.

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1. Introduction

In [3], [4] C. Chicone developed a very powerful geometric technique dealing with the problem of persistence of a periodic solution of a small parameter differential system $\dot{x} = F(t, x, \varepsilon)$, usually written as

$$\dot{x} = f(x) + \varepsilon g(x, \varepsilon) \quad \text{or} \quad \dot{x} = f(x) + \varepsilon g(t, x, \varepsilon). \tag{1}$$

Chicone's method not only gives a clear view of the geometry of the problem, but at the same time yields explicit formulae for calculation of bifurcation functions and Melnikov-like integrals. On the very bottom of the method is Diliberto's theorem [5, Theorem 5.5] capable to give the principal fundamental matrix Y of the first variational equation of planar autonomous systems. Even though Chicone was able to work with some 3-dimensional systems (obtained by conversion of (1) with $x \in \mathbb{R}^2$ and a time-periodic function g to an autonomous system) using the concept of a *periodic manifold*, or even n-dimensional systems (using the fact that for $\varepsilon = 0$ the unperturbed system $\dot{x} = f(x)$ uncouples into systems suitable for Diliberto's theorem), he apparently never considered the possibility of using some extra information about an *arbitrary* invariant manifold M in order to obtain some coefficients of the matrix Y.

The existence of this possibility, at least in $\mathbb{R}^3$ is evident, so it is not the question whether but how to do it as we can not use the information about M for any kind of lowering the order of our system because any such reduction would spoil our view of the geometry of the problem considered. One of the possible solutions is presented in this work.

2. Hypotheses and auxiliary results

Throughout the paper we will consider the autonomous ordinary differential equation

$$\dot{x} = f(x), \tag{2}$$

in which $x,\ f(x) \in \mathbb{R}^3$ and $t \in \mathbb{R}$. We use the symbol $\cdot$ for differentiation with respect to the time t, similarly by D we denote differentiation with respect to the space variable x, so e.g. $Df(x)$ means the differential of the mapping f at x or the Jacobian matrix of f. By $\langle\cdot|\cdot\rangle$ and $\|\cdot\|$ we denote the Euclidean inner product and norm, respectively, $\times$ denotes the usual cross-product of vectors in $\mathbb{R}^3$. By $I_{n\times n}$ we denote the n-dimensional identity matrix. We will also assume

(H1) $f(x)$ is a $\mathrm{C}^1(\mathbb{R}^3, \mathbb{R}^3)$ function.

This hypothesis ensures that the initial problem (2), $x(0) = x$ has a unique solution $\varphi(t,x)$ which is at least C^1 in (t,x).

(H2) all solutions of (2) are defined for all $t \in \mathbb{R}$.

There is a locally invariant 2-dimensional submanifold M for (2). Because our considerations are of local nature, it is possible to suppose

(H3) there is a C^2 function $g\colon\ \mathbb{R}^3 \to \mathbb{R}$, $Dg(x) \neq 0$ if $x \in M := \{x \in \mathbb{R}^3 :\ g(x) = 0\}$, and M is a locally invariant manifold of (2).

Up to the end of this section the dimension of the phase space is arbitrary, not only 3. In [7] the following characteristic of the invariant set M in $\mathbb{R}^n$ is given: M is an $(n-1)$-dimensional locally invariant manifold of (2) if and only if there is a continuous function $N(x)$ such that $Dg(x)f(x) = N(x)g(x)$.

If $x \in M$, $u \in T_x\mathbb{R}^n$ and $h(x) := N(x)g(x)$, then $h(x+u) - h(x) = N(x+u) \times g(x+u) = [N(x) + o(1)]\,[Dg(x)u + o(\|u\|)] = N(x)Dg(x)u + o(\|u\|)$, therefore

$$D(N(x)g(x)) = N(x)Dg(x) \qquad \text{for} \qquad x \in M. \tag{3}$$

It is possible to show that in our case $N(x)$ is unique and easily computable, e.g. for $x \in M$, which is the only interesting case here, it follows easily from (3) that

$$N(x) = \frac{D\left(Dg(x) \cdot f(x)\right) \cdot Dg(x)}{\|Dg(x)\|^2}.$$

Moreover, $N(x) \equiv 0$ iff $g(x)$ is a first integral of (2) as this is equivalent to $Dg(x)f(x) \equiv 0$ ([6, p. 114]).

Lemma 2.1. *Suppose that* (H1), (H2), (H3) *hold and* $x \in M$. *Then the vector function* $y(t) = Dg(\varphi(t,x))$ *is a nontrivial solution of the equation*

$$\dot{y} = -y\left[Df(\varphi(t,x)) - I_{n\times n}N(\varphi(t,x))\right], \tag{4}$$

where y *is a row* n*-vector.*

Proof. For all $x \in M$ we have $N(x)Dg(x) = D(Dg(x))f(x) + Dg(x)Df(x)$, hence

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}Dg(\varphi(t,x)) &= D(Dg(\varphi(t,x)))f(\varphi(t,x)) \\ &= N(\varphi(t,x))Dg(\varphi(t,x)) - Dg(\varphi(t,x))Df(\varphi(t,x)) \\ &= -Dg(\varphi(t,x))\left[Df(\varphi(t,x)) - I_{n\times n}N(\varphi(t,x))\right].\end{aligned}$$

□

If $g(x)$ is a first integral of (2) with the zero-level set M, then the equation (4) is the usual adjoint equation of the *first variational equation*

$$\dot{y} = -Df(\varphi(t,x))y. \tag{5}$$

3. The main result

Let the hypotheses (H1), (H2), (H3) be fulfilled. Then at any $x \in M \setminus \partial M$ there is a C^1 orthogonal basis $\mathcal{B}(x) = \{f(x),\ f_\perp(x),\ n(x)\}$, where $f_\perp(x) := n(x) \times f(x)$ and $n(x) := \frac{\tilde{n}(x)}{\|\tilde{n}(x)\|}$ is the unit normal vector ($\tilde{n}(x) \sim Dg(x)$). Clearly the tangent space of M at x is $T_xM = \operatorname{span}\{f(x),\ f_\perp(x)\}$.

For any $t \geqslant 0$ we have $\varphi(t,\cdot)\colon\ \mathbb{R}^3 \to \mathbb{R}^3$, so $D_x\varphi(t,x)\colon\ T_x\mathbb{R}^3 \to T_{\varphi(t,x)}\mathbb{R}^3$ and similarly, because $\varphi(t,\cdot)\colon\ M \to M$, even $D_x\varphi(t,x)\colon\ T_xM \to T_{\varphi(t,x)}M$. Moreover, $Y(t) := D_x\varphi(t,x)$ is the solution of the matrix initial problem

$$\dot{Y} = Df(\varphi(t,x))Y, \qquad Y(0) = I_{3\times 3},$$

so $Y(t)$ is the *principal fundamental matrix* of the first variational equation (5) at 0 and $y(t)$ is a solution of (5) fulfilling $y(0) = \xi$ iff $y(t) = Y(t)\xi$. In particular, $f(\varphi(t,x)) = D_x\varphi(t,x)f(x)$ is the solution fulfilling the initial condition $y(0) = f(x)$. Denoting the trajectory of φ by $\{\varphi\}$ we clearly have $\varphi(t,\cdot)\colon \{\varphi\} \to \{\varphi\}$, hence $D_x\varphi(t,x)\colon T_x\{\varphi\} \to T_{\varphi(t,x)}\{\varphi\}$. Summing this up we obtain that the principal fundamental matrix of (5) at 0 relative to the basis $\mathcal{B}(t) = \mathcal{B}(\varphi(t,x)) = \{f(\varphi(t,x)),\ f_\perp(\varphi(t,x)),\ n(\varphi(t,x))\}$ is given by

$$Y(t) = D_x\varphi(t,x) = \begin{bmatrix} 1, & a, & \alpha \\ 0, & b, & \beta \\ 0, & 0, & \gamma \end{bmatrix},$$

where $a(t) = a(t,\varphi(t,x))$, $b(t) = b(t,\varphi(t,x))$, $\alpha(t) = \alpha(t,\varphi(t,x))$, $\beta(t) = \beta(t,\varphi(t,x))$ and $\gamma(t) = \gamma(t,\varphi(t,x))$ are C^1 functions in t. Clearly

$$a(0) = \alpha(0) = \beta(0) = 0, \qquad b(0) = \gamma(0) = 1. \tag{6}$$

The functions $a(t)$, $b(t)$ are determined by the following version of Diliberto's theorem ([1] and references given there):

Theorem 3.1. *Suppose that* (H1), (H2), (H3) *hold,* $\varphi(t,x)$ *is the solution of the differential equation* (2), $\varphi(0,x) = x$ *and* $x \in M$. *If* $f(x) \neq 0$, *then the principal fundamental matrix* $Y(t)$ *at* $t = 0$ *of the variational equation* (5) *is such that* $Y(t)f(x) = f(\varphi(t,x))$, $Y(t)f_\perp(x) = \tilde{a}(t,x)f(\varphi(t,x)) + \tilde{b}(t,x)f_\perp(\varphi(t,x))$ *and*

$$\tilde{b}(t,x) = \frac{\|f(x)\|^2}{\|f(\varphi(t,x))\|^2}\exp\int_0^t \{\langle f|Df\cdot f\rangle + \langle f_\perp|Df\cdot f_\perp\rangle\}(\varphi(s,x))\,\mathrm{d}s, \tag{7}$$

$$\tilde{a}(t,x) = \int_0^t \{(2\kappa_g\|f\| + \langle Df\cdot f_\perp|f\rangle - \langle Df\cdot f|f_\perp\rangle)\tilde{b}\}(\varphi(s,x))\,\mathrm{d}s, \tag{8}$$

where κ_g *is the geodesic curvature of* $\varphi(\cdot,x)$ *equipped with natural parametrization and the other functions are evaluated at* $\varphi(s,x)$.

Clearly $a(t) = \tilde{a}(t,x)$ and $b(t) = \tilde{b}(t,x)$, so it remains to identify $\alpha(t)$, $\beta(t)$ and $\gamma(t)$. To this end we express $Y(t)\tilde{n}(x)$ in the basis $\mathcal{B}(\varphi(t,x)) = \{f(\varphi(t,x)),\ f_\perp(\varphi(t,x)),\ n(\varphi(t,x))\}$ (henceforth omitting arguments t and x to keep formulae clearly arranged) as

$$Y\tilde{n} = \alpha f + \beta f_\perp + \gamma n = \alpha f + \beta f_\perp + \tilde{\gamma}\tilde{n}, \quad \tilde{\gamma}\|\tilde{n}\| = \gamma,$$

and use the fact that $y(t) = Y(t)\tilde{n}(x)$ is the solution of (5), $y(0) = \tilde{n}(x)$. Therefore

$$Df\cdot[\alpha f + \beta f_\perp + \tilde{\gamma}\tilde{n}] = \dot{\alpha}f + \alpha Df\cdot f + \dot{\beta}f_\perp + \beta Df_\perp\cdot f + \dot{\tilde{\gamma}}\tilde{n} + \tilde{\gamma}\dot{\tilde{n}}.$$

By applying Lemma 2.1 to this relation we obtain

$$\beta\left(Df\cdot f_\perp - Df_\perp\cdot f\right) + \tilde\gamma Df\tilde n = \dot\alpha f + \dot\beta f_\perp + \dot{\tilde\gamma}\tilde n - \tilde\gamma\left(Df - I_{3\times3}N\right)^*\tilde n,$$

where the asterisk denotes the transpose, or

$$\beta\left(Df\cdot f_\perp - Df_\perp\cdot f\right) + \tilde\gamma\left(Df + Df^*\right)\tilde n = \dot\alpha f + \dot\beta f_\perp + \dot{\tilde\gamma}\tilde n + \tilde\gamma N\tilde n. \tag{9}$$

Multiplying (9) by $\tilde n$ we obtain

$$\beta\langle[f, f_\perp]\,|\tilde n\rangle + \tilde\gamma\langle\left(Df + Df^*\right)\tilde n|\tilde n\rangle = \dot{\tilde\gamma}\|\tilde n\|^2 + \tilde\gamma N\|\tilde n\|^2.$$

It follows from the Frobenius theorem (e.g. [2, p. 113]) that for any two vector fields $f_1, f_2\colon M \to \mathbb{R}^3$ if both f_1 and f_2 are tangent vector fields on M, then the Lie bracket $[f_1, f_2]$ is a tangent field on M. Hence $\langle[f, f_\perp]\,|\tilde n\rangle = 0$, so we have obtained the scalar equation

$$\dot{\tilde\gamma} = \left\{2\frac{\langle Df\cdot\tilde n|\tilde n\rangle}{\|\tilde n\|^2} - N\right\}\tilde\gamma.$$

If $x \in M$, then $\langle Df\cdot\tilde n|\tilde n\rangle = D\langle f|\tilde n\rangle\tilde n - \langle f|D\tilde n\cdot\tilde n\rangle = D(Ng)\tilde n - \langle D\tilde n\cdot f|\tilde n\rangle = N\|\tilde n\|^2 - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde n\|^2$, hence

$$\dot{\tilde\gamma} = \left\{N - \frac{\mathrm{d}}{\mathrm{d}t}\log\|\tilde n\|^2\right\}\tilde\gamma,$$

and simple calculation, by virtue of (6), gives

$$\gamma(t) = \frac{\|Dg(x)\|}{\|Dg(\varphi(t,x))\|}\exp\left\{\int_0^t N(\varphi(s,x))\,\mathrm{d}s\right\}. \tag{10}$$

Similarly, multiplying (9) by f and $f_\perp$ and using the obvious fact that $\langle f|f\rangle = \langle f_\perp|f_\perp\rangle$, we obtain a two-dimensional system

$$\begin{aligned}\dot\beta &= \frac{\langle Df\cdot f_\perp - Df_\perp\cdot f|f_\perp\rangle}{\|f\|^2}\beta + \frac{\langle\left(Df + Df^*\right)\tilde n|f_\perp\rangle}{\|f\|^2}\tilde\gamma,\\ \dot\alpha &= \frac{\langle Df\cdot f_\perp - Df_\perp\cdot f|f\rangle}{\|f\|^2}\beta + \frac{\langle\left(Df + Df^*\right)\tilde n|f\rangle}{\|f\|^2}\tilde\gamma.\end{aligned}$$

The solution of this system fulfilling the initial condition (6) is

$$\beta(t) = \int_0^t \frac{b(t)}{b(s)}\left\{\frac{\langle\left(Df + Df^*\right)n|f_\perp\rangle}{\|f\|^2}\gamma\right\}(\varphi(s,x))\;\mathrm{d}s, \tag{11}$$

$$\begin{aligned}\alpha(t) = &\int_0^t \left\{\left[2\kappa_g\|f\| + \langle Df\cdot f_\perp|f\rangle - \langle Df\cdot f|f_\perp\rangle\right]\beta\right\}(\varphi(s,x))\,\mathrm{d}s\\ &+ \int_0^t\left\{\frac{\langle\left(Df + Df^*\right)n|f\rangle}{\|f\|^2}\gamma\right\}(\varphi(s,x))\;\mathrm{d}s,\end{aligned} \tag{12}$$

where κ_g is the geodesic curvature of $\varphi(\cdot, x)$ equipped with natural parametrization, see [2, p. 173].

Summing this up we have proved the following theorem:

Theorem 3.2. *Let the hypotheses* (H1), (H2), (H3) *be fulfilled,* $x \in M$, $f(x) \neq 0$ *and let* $\varphi(t, x)$ *denote the solution of* (2), $\varphi(0, x) = x$. *Then the principal fundamental matrix of* (5) *at* 0 *relative to the basis* $\{f(\varphi(t, x)),\ f_\perp(\varphi(t, x)),\ n(\varphi(t, x))\}$ *is given by*

$$Y(t) = D_x\varphi(t, x) = \begin{bmatrix} 1, & a(t), & \alpha(t) \\ 0, & b(t), & \beta(t) \\ 0, & 0, & \gamma(t) \end{bmatrix}, \tag{13}$$

where the functions $a(t), b(t), \alpha(t), \beta(t), \gamma(t)$ *are given by* (8), (7), (12), (11), (10).

4. Applications

As an elementary example of a typical application of Theorem 3.2 we will repeat Chicone's calculation ([3, Section 3]) of the partial derivatives with respect to ε of a displacement function but this time for a 3-dimensional system with a small parameter ε

$$\dot{x} = f(x) + \varepsilon g(x), \tag{14}$$

which has for $\varepsilon_0 = 0$ a p-periodic solution $\varphi(t, x_0, \varepsilon_0)$. We will suppose that all hypotheses (H1), (H2), (H3) are fulfilled and that the invariant manifold M does not depend on ε. The case when M depends on ε will be postponed to another article.

Surprisingly even this easier case occurs frequently in applications—e.g. in chemical kinetics the differential system is determined only approximately by the Guldberg and Waage law of mass action, whereas the invariant manifold M, given by the law of conservation of mass, is known exactly.

Such setting enables us to replace the concept of the displacement function F for the concept of the scalar displacement function or, which is almost the same, for the concept of the scalar return map h.

Let $0 \leqslant \varepsilon < \Delta$, $\delta > 0$ and let $\sigma\colon (-\delta, \delta) \to \{\sigma\} \subseteq M$ be a parametrization of the transversal $\{\sigma\}$ to (14) with respect to M, $\sigma(0) = x_0$ and $\sigma'(s) \neq 0$ for $s \in (-\delta, \delta)$. Let $P\colon \{\sigma\} \times [0, \Delta) \to \{\sigma\}$ be a *Poincaré mapping* of (14), $P(x, \varepsilon) := \varphi(\tau(x, \varepsilon), x, \varepsilon)$, where $\tau\colon \{\sigma\} \times [0, \Delta) \to (0, \infty)$, $\tau(x_0, 0) = p$ is the *first return time function.* The *scalar return map* $h\colon (-\delta, \delta) \times [0, \Delta) \to \mathbb{R}$ is defined as $h(s, \varepsilon) := (\sigma^{-1} \circ P \circ (\sigma \times$

$Id_{\mathbb{R}}))(s,\varepsilon)$, and our aim is to compute $h'_\varepsilon(0,0)$. To this end, differentiate the relation $(\sigma\circ h)(x,\varepsilon) = (P\circ(\sigma\times Id_{\mathbb{R}}))(s,\varepsilon)$ with respect to ε at $(s,\varepsilon)=(0,0)$ to obtain $\sigma'(0)h'_\varepsilon(0,0) = f(x_0) + D_3\varphi(p,x_0,0)$. Multiplying this relation by $f_\perp(x_0)$ we obtain

$$h'_\varepsilon(0,0) = \frac{\langle D_3\varphi(p,x_0,0)|f_\perp(x_0)\rangle}{\langle\sigma'(0)|f_\perp(x_0)\rangle}. \tag{15}$$

It is well-known that $D_3\varphi(t,x_0,0)$ is the solution of the initial problem

$$\dot y = Df(\varphi(t,x_0,0))y + g(\varphi(t,x_0,0)),\quad y(0)=0,$$

so

$$D_3\varphi(t,x_0,0) = \int_0^t Y(t)Y^{-1}(s)g(D_3\varphi(s,x_0,0))\,\mathrm{d}s. \tag{16}$$

However, the principal fundamental matrix $Y(t)$ is in the basis $\{f(\varphi(t,x)),$ $f_\perp(\varphi(t,x)),\ n(\varphi(t,x))\}$ given by (13). From elementary linear algebra we have

$$Y^{-1}(s) = \begin{bmatrix} 1, & -a(s)b^{-1}(s), & -\alpha(s)\gamma^{-1}(s) + a(s)\beta(s)b^{-1}(s)\gamma^{-1}(s) \\ 0, & b^{-1}(s), & -\beta(s)b^{-1}(s)\gamma^{-1}(s) \\ 0, & 0, & \gamma^{-1}(s) \end{bmatrix}.$$

Similarly, in this basis $g(\varphi(t,x_0,0))$ has the coordinate representation $(\langle g(\varphi(t,x_0,0))$ $|\ f(\varphi(t,x_0,0))\ \|f(\varphi(t,\ x_0,\ 0))\|^{-1}\rangle$, $\langle g(\varphi(t,x_0,0))\ |\ f_\perp(\varphi(t,x_0,0))\ \|f(\varphi(t,x_0,0))\|^{-1}\rangle$, $\langle g(\varphi(t,x_0,0))\ |\ n(\varphi(t,x_0,0))\rangle)$, and the second coordinate of the left-hand side of (16) multiplied by $\|f(\varphi(t,x_0,0))\|$ is for $t=p$ exactly $\langle D_3\varphi(p,x_0,0)|f_\perp(x_0)\rangle$. Summing this up we obtain an explicit formula for $h'_\varepsilon(0,0)$:

$$\begin{aligned} h'_\varepsilon(0,0) = \frac{\|f(x_0)\|}{\langle\sigma'(0)|f_\perp(x_0)\rangle}\int_0^p \Bigg\{ & \frac{b(p)}{b(s)}\frac{\langle g(\varphi(s,x_0,0))|f_\perp(\varphi(s,x_0,0))\rangle}{\|f(\varphi(s,x_0,0))\|} \\ &+ \frac{(\beta(p) - b(p)\beta(s)b^{-1}(s))}{\gamma(s)}\langle g(\varphi(s,x_0,0))|n(\varphi(s,x_0,0))\rangle\Bigg\}\,\mathrm{d}s. \end{aligned}$$

References

[1] *L. Adamec*: A note on a generalization of Diliberto's theorem for certain differential equations of higher dimension. To appear in Appl. Math.
[2] *I. Agricola, T. Friedrich*: Global Analysis. American Mathematical Society, Rode Island, 2002.
[3] *C. Chicone*: Bifurcation of nonlinear oscillations and frequency entrainment near resonance. SIAM J. Math. Anal. *23* (1992), 1577–1608.
[4] *C. Chicone*: Lyapunov-Schmidt reduction and Melnikov integrals for bifurcation of periodic solutions in coupled oscillators. J. Differ. Equations *112* (1994), 407–447.
[5] *C. Chicone*: Ordinary Differential Equations with Applications. Springer, New York, 1999.
[6] *Ph. Hartman*: Ordinary Differential Equations. John Wiley, New York, 1964.
[7] *M. Y. Li, J. S. Muldowney*: Dynamics of differential equations on invariant manifolds. J. Differ. Equations *168* (2000), 295–320.

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