GENERALIZED SYMMETRIES: QUANTUM GROUPS, NONCOMMUTATIVE AND HIGHER GAUGE THEORIES

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Contents

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1. Introduction

The thesis represents a series of 10 papers:

[P1] B. Jurčo, On coherent states for the simplest quantum groups, Lett.Math. Phys. **21**, 51-58 (1991)

[P2] B. Jurčo, Differential calculus on quantized simple Lie groups, Lett. Math. Phys. **22**, 177-186 (1991)

[P3] B. Jurčo, P. Šťovíček, Quantum dressing orbits on compact groups. Commun. Math. Phys. **152**, 97-126 (1993)

[P4] B. Jurčo, P. Šťovíček, Coherent states for quantum compact groups. Commun. Math. Phys. **182**, 221-251 (1996)

[P5] B. Jurčo, P. Schupp, J. Wess, Noncommutative gauge theory for Poisson manifolds, Nucl. Phys. **B584**, 784-794 (2000)

[P6] B. Jurčo, P. Schupp, J. Wess, Nonabelian noncommutative gauge theory via noncommutative extra dimensions, Nucl.Phys. **B604**, 148-180 (2001)

[P7] B. Jurčo, P. Schupp, J. Wess, Noncommutative line bundle and Morita equivalence, Lett.Math.Phys. **61**, 171 (2002)

[P8] P. Aschieri, B. Jurčo, Gerbes, M5-Brane Anomalies and E8 Gauge Theory, JHEP **0410** 068 (2004)

[P9] P. Aschieri, L. Cantini, B. Jurčo, Nonabelian bundle gerbes, their differential geometry and gauge theory, Commun. Math. Phys. **254**, 367-400 (2005)

[P10] I. Baković, B. Jurčo, The classifying topos of a topological bicategory, Homol. Homotopy Appl. **12(1)**, 279-300 (2010)

Papers [P1-P4] are on quantum groups, papers [P5-P7] are on deformation quantization based noncommutative gauge theory, and papers [P8-P10] are on higher gauge theory. All the papers are in one or another way linked to problems and questions originating from quantum field theory and/or string theory. They all relate to geometry of generalized symmetries and they aim towards a better understanding of the role played by generalized symmetries in quantum field theory and string theory. These generalized symmetries are characterized by their noncommutativity (quantum groups, noncommutative gauge theories) or "nonabelianity" (nonabelian gerbes of higher gauge theory), wherein these to aspects are interrelated. For instance, a nonabelian Yang-Mills theory may be seen, via Seiberg-Witten map, as an example of a noncommutative Yang-Mills theory [P6]. Also, a deformation quantization of an abelian gerbes [1] leads to a honest nonabelian 2-cocycle describing a nonabelian gerbe as known in nonabelian cohomology theory [50]. More generally, the questions and problems addressed in these papers can be seen as being motivated by the attempts of identifying and understanding the fundamental mathematical nature of quantum field theory and string theory.

In the first part, we describe the basic notions and facts relevant to the subject of the thesis and provide some relevant literature. Next, we formulate the goals of the thesis. Afterwards, we discuss the applied methods and give a survey of the results of the above mentioned papers. Some conclusions will be made as well.

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2. Overview about the current stage of the problem and of the literature

2.1. **Quantum Groups; geometry and representation theory.** Origins of the theory of quantum groups can be probably traced back to the year 1981, when the q-deformed algebra $\mathcal{U}_q(su(2))$ appeared in the paper by P.P. Kulish and N.Yu. Reshetikhin [74] in their study of integrable chain models. Leningrad school played a prominent role in the in development of the quantum inverse scattering method, which led to the formal discovery of quantum groups by V.G. Drinfeld [39, 40] and M. Jimbo [59, 60]. Quantum groups of Drinfeld and Jimbo are Hopf algebras, which are one-parameter deformations $\mathcal{U}_h(\mathrm{g})$ of universal enveloping algebras $\mathcal{U}(\mathrm{g})$, where g is a simple complex Lie algebra. Almost simultaneously, S.L. Woronowicz introduced the quantum group $SU_q(2)$, a one-parameter deformation of the algebra of functions on $SU(2)$, which is the dual Hopf algebra to the $U_q(su(2)$ [128]. Subsequently, he developed the general theory of compact quantum matrix groups in [129]. His approach was motivated by the theory of C^*-algebras. Also important for the development of the theory of quantum groups was the work of Yu.I. Manin on quantized coordinate algebras [82]. Since then, the area of quantum groups developed rapidly. It still enjoys attention of both mathematicians and physicist. It is impossible to mention all the developments and results of the theory of quantum groups. Here we will restrict ourselves only to a very brief description of few of them, which are directly related to the subject of the thesis. Similarly, we shall explicitly mention only the literature, which has a direct relationship to the results comprised in papers [P1-P4] and presented in the thesis. Also, we will not go in our discussion of the subject of quantum groups beyond the mid-nineties, when papers [P1-P4] were published. For surveys on quantum groups, we recommend, e.g., the following books [29, 84, 69, 75].

2.1.1. *Hopf algebras of quantized function on Lie groups and quantized universal enveloping algebras.* Quantum group is a Hopf algebra, which can be related via a "semiclassical limit" to a Lie group G or a Lie algebra g. We shall use the standard notation for the product, denote the coproduct by Δ, the counit by ε, and the antipode by S. To each Hopf algebra A (finite dimensional or not) there exists a dual Hopf algebra U := A* (the Hopf dual), the duality given by a properly defined non-degenerate pairing $\langle\cdot,\cdot\rangle$ [29]. In the F-R-T approach [103], which we will use, the Hopf algebra $\mathcal{A} = \mathrm{Fun}_q(G)$ of the so-called quantized functions on G is introduced by the following construction. Let G be a simple complex Lie group belonging to one of the four principal series A_{n-1}, B_n, C_n and D_n. The basic object used is the so-called R-matrix $\mathcal{R} \in \mathrm{Mat}_N(\mathbb{C}) \otimes \mathrm{Mat}_N(\mathbb{C})$. The R-matrix $\mathcal{R}$ depends on a deformation parameter $q = e^{-h} \in \mathbb{C}^\times$ and satisfies the Yang-Baxter equation

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}, \tag{2.1.1}$$

with $N = n$, $N = (2n + 1)$, $N = 2n$ and $N = 2n$ in the respective cases of A_{n-1}, B_n, C_n and D_n. Here and in the sequel, indices like in (2.1.1) refer to factors in the corresponding tensor product. For instance, $\mathcal{R}_{12} = R \otimes I$, etc. The explicit form of the R-matrix cCDSWZan be found, for instance, in [103]. The generators of $\mathcal{A}$ are arranged in a matrix T of dimension $N \times N$. The algebra relations are compactly written as [103]

$$\mathcal{R}T_1T_2 = T_2T_1\mathcal{R}. \tag{2.1.2}$$

Depending on the series, also some additional restrictions (proper versions of unimodularity or orthogonality) are assumed to be valid for the matrix of generators T. The Hopf algebra structure is completed by the respective definitions for the coproduct, the counit and the antipode, which are defined on the generators by

$$\Delta T = T\dot{\otimes}T, \quad \varepsilon(T) = I \quad \text{and} \quad S(T) = T^{-1}, \tag{2.1.3}$$

where $(T\dot{\otimes}T)_{jk} := \sum_\ell T_{j\ell} \otimes T_{\ell k}$.

Let K denote the compact form of G. The Hopf algebra $\mathrm{Fun}_q(G)$ can be endowed with a $*$-involution. Hence, the compact form $\mathrm{Fun}_q(K)$ of $\mathrm{Fun}_q(G)$ can be introduced. The matrix of generators of the compact form $\mathrm{Fun}_q(K)$ will be distinguished by introducing notation U for it. Also, the relations $U^* = U^{-1}$ will be imposed, where $(U^*)_{ij} := (U_{ji})^*$.

In the dual picture, the universal enveloping algebra $\mathcal{U}(\mathrm{g})$ of the Lie algebra g corresponding to G is deformed. The resulting deformation $\mathcal{U}_h(\mathrm{g})$ is defined in terms of the Chevalley generators H_i, X_i^+, X_i^- corresponding to simple roots of g. Again, the compact form $\mathcal{U}_h(\mathrm{k})$ is obtained from $\mathcal{U}_h(\mathrm{g})$, when this is equipped with a proper $*$-involution. We will not describe the defining relations explicitly. These can be found, e.g., in Drinfeld [39, 59] or [103]. In the F-R-T approach [103] the quantum group $\mathcal{U}_h(\mathrm{g})$ can be described by introducing two matrices of generators $L^\pm = (l_{ij}^\pm)_{i,j=1}^N$ and imposing the relations

$$\mathcal{R}_{21}L_1^\pm L_2^\pm = L_2^\pm L_1^\pm \mathcal{R}_{21} \quad \text{and} \quad \mathcal{R}_{21}L_1^+L_2^- = L_2^-L_1^+\mathcal{R}_{21}. \tag{2.1.4}$$

The comultiplication on matrices $L^\pm$ is again the matrix one

$$\Delta L^\pm = L^\pm \dot{\otimes} L^\pm \tag{2.1.5}$$

and the counit is given by

$$e(L^{\pm}) = I. \tag{2.1.6}$$

Formulae for the antipode S will be not given explicitly, these can be found in [103]. The pairing between Hopf algebras $\mathrm{Fun}_q(G)$ and $\mathcal{U}_h(\mathrm{g})$ is given on generators by

$$\langle L^{\pm}, T\rangle = \mathcal{R}^{\pm}, \tag{2.1.7}$$

where $\mathcal{R}^{+} = \mathcal{R}_{21}$ and $\mathcal{R}^{-} = \mathcal{R}^{-1}$. Also here, additional unimodularity and orthogonality relations have to be assumed.

The compact form $\mathcal{U}_h(\mathrm{k})$ can be viewed as the quantization of the generalized Pontryagin dual of the compact group K. The Pontryagin dual is the solvable group AN coming from the Iwasawa decomposition $G = KAN$ [P3]. The $*$-Hopf algebra $\mathrm{Fun}_q(AN)$ is generated by entries of the upper-triangular matrix $\Lambda := S(L^{+})$ and its Hermitian adjoint $\Lambda^{*} = L^{-}$, and is described by relations

$$\mathcal{R}\Lambda_1\Lambda_2 = \Lambda_2\Lambda_1\mathcal{R}, \quad {\Lambda_1}^{*}\mathcal{R}^{-1}\Lambda_2 = \Lambda_2\mathcal{R}^{-1}{\Lambda_1}^{*}, \tag{2.1.8}$$

$$\Delta(\Lambda) = \Lambda\dot{\otimes}\Lambda, \quad \varepsilon(\Lambda) = I \ \text{ and } \ S(\Lambda) = \Lambda^{-1}, \tag{2.1.9}$$

with $\Lambda^{*}_{jj} = \Lambda_{jj}$ and $\prod \Lambda_{jj} = 1$. For the series B_n, C_n and D_n there are additional orthogonality relations to be considered.

2.1.2. *Differential calculi.* We will discuss only first-order differential calculi on the quantum group $\mathcal{A} = \mathrm{Fun}_q(G)$. Any first-order differential calculus can be uniquely extended to an exterior differential calculus on $\mathcal{A}$ [131]. According to [131] a (first-order) differential calculus on the quantum group $\mathcal{A}$ is defined as a couple (Γ, d), where Γ is a bimodule over $\mathcal{A}$ and $d : \mathcal{A} \to \Gamma$ a linear map satisfying:

- for $a, b \in \mathcal{A}$

$$d(ab) = (da)b + a(db) \quad \text{(Leibniz rule)} \ \text{ and} \tag{2.1.10}$$

- any $\rho \in \Gamma$ can be written as

$$\rho = a_k db_k \tag{2.1.11}$$

with some a_k, b_k, $k = 1, \ldots, K$.

A differential calculus is said to be bicovariant if it is:

- left-covariant

$$(a_k db_k = 0) \Rightarrow \Delta(a_k)(\mathrm{id} \otimes d)\Delta(b_k) = 0 \ \text{ and} \tag{2.1.12}$$

- right-covariant

$$(a_k db_k = 0) \Rightarrow \Delta(a_k)(d \otimes \mathrm{id})\Delta(b_k) = 0. \tag{2.1.13}$$

Linear maps $\delta_L : \Gamma \to \mathcal{A} \otimes \Gamma$ and $\delta_R : \Gamma \to \Gamma \otimes \mathcal{A}$ defined by

$$\delta_L(a_k db_k) = \Delta(a_k)(\mathrm{id} \otimes d)\Delta(b_k) \ \text{ and } \ \delta_R(a_k db_k) = \Delta(a_k)(d \otimes \mathrm{id})\Delta(b_k) \tag{2.1.14}$$

give to Γ the structure of a bicovariant bimodule $(\Gamma, \delta_L, \delta_R)$ (see [131] for the precise definition). An element (i.e., a one-form) $\omega \in \Gamma$ is said to be left- or right-invariant if $\delta_L(\omega) = I \otimes \omega$ or $\delta_R(\omega) = \omega \otimes I$, respectively. An element $\omega \in \Gamma$ is said to be bi-invariant if it is simultaneously left- and right-invariant.

Given a differential calculus on $\mathcal{A}$, the linear space ${}_{inv}\Gamma^{*}$ (dual to the linear space ${}_{inv}\Gamma$ of left invariant one-forms) with dual basis elements $\chi_i \in \mathcal{A}^{*}$ can be introduced so that for $a \in \mathcal{A}$

$$da = (\chi_i * a)\omega_i. \tag{2.1.15}$$

In case of the compact form K the differential calculi can be equipped with proper $*$-involutions.

2.1.3. *Representations.* The R-matrix and the related formal algebraic constructions of quantum groups mentioned above have their roots in the 2-dimensional statistical mechanics and the quantum inverse scattering method. As already mentioned, an independent approach was pioneered by Woronowicz [128, 129, 130, 100], who introduced his compact matrix pseudogroups as certain C^*-algebras and discussed also their topological properties. In particular, he described the representations of the algebra $\mathrm{Fun}_q(SU(2))$, for $q \in (0,1)$ in [128]. According to Woronowicz's classification, there exist two inequivalent one-parameter families of $*$-representations, one of them one-dimensional and the other one infinite-dimensional, both parametrized by $\varphi \in S^1$. More explicitly, on the generators

$$U = \begin{pmatrix} a & b \\ -q^{-1}b^* & a^* \end{pmatrix}, \tag{2.1.16}$$

the one-dimensional representation is given by

$$\xi_\varphi(a) = e^{-i\varphi}, \quad \xi_\varphi(b) = 0. \tag{2.1.17}$$

The infinite-dimensional representation is given in an orthonormal basis

$$|n\rangle = A_n^{-1}(a^*)^n|0\rangle, \;\; A_n^2 = (1-q^2)\dots(1-q^{2n}), \;\; n \in \mathbb{N} \tag{2.1.18}$$

by

$$\pi_\varphi(a)|0\rangle = 0, \;\; \pi_\varphi(a^*)|n\rangle = (1-q^{2n})^{1/2}|n-1\rangle, \;\; \pi_\varphi(b)|n\rangle = -q^{n+1}e^{-i\varphi}|n\rangle. \tag{2.1.19}$$

The general case of $\mathrm{Fun}_q(K)$ was treated in [111, 112, 127].

Concerning the theory of finite-dimensional representations of the quantized enveloping algebras $U_h(\mathrm{k})$, the following results are known [77, 101] for generic values of q: The representation theory is not different from the classical case (i.e., $q=1$). Every finite-dimensional representation is completely reducible. Irreducible representations are, up to equivalence, determined by their respective highest (lowest) weights. Induced representations (more precisely, induced corepresentations of the corresponding algebras of quantized function algebras) is described in [96]. Borel-Weil construction for $\mathcal{U}_h(\mathrm{u}(n))$ is given in [13].

For the closely related question of description of q-deformations of homogeneous spaces we refer to, e.g., [79, 99, 108, 126], where various constructions have been proposed. In this thesis, we will describe an independent approach, based on a properly generalized notion of a quantum coherent state in section 5.1.2.

2.1.4. *Quantum double and dressing transformations.* There are two equivalent (modulo a proper completion) descriptions [102] of the quantum double $\mathcal{D}(G)$, which we will denote for simplicity by $\mathcal{D}$. In the first description, $\mathcal{D} \cong \mathrm{Fun}_q(G) \otimes \mathrm{Fun}_q(G)$ as a coalgebra. The algebra structure of $\mathcal{D} \cong \mathrm{Fun}_q(G) \otimes \mathrm{Fun}_q(G)$ can be described as follows. Let $\mathcal{T}$ and $\hat{\mathcal{T}}$ be two copies of matrices of generators corresponding to the two respective factors in $\mathcal{D} \cong \mathrm{Fun}_q(G) \otimes \mathrm{Fun}_q(G)$. We have [27]

$$\mathcal{R}\mathcal{T}_1\mathcal{T}_2 = \mathcal{T}_2\mathcal{T}_1\mathcal{R}, \quad \mathcal{R}\hat{\mathcal{T}}_1\mathcal{T}_2 = \mathcal{T}_2\hat{\mathcal{T}}_1\mathcal{R} \;\text{ and }\; \mathcal{R}\hat{\mathcal{T}}_1\hat{\mathcal{T}}_2 = \hat{\mathcal{T}}_2\hat{\mathcal{T}}_1\mathcal{R}\,. \tag{2.1.20}$$

Of course, also here, there are the antipode, the counit and the additional unimodularity or orthogonality conditions to be considered.

In the second description, $\mathcal{D} \cong \mathcal{U}_h(\mathrm{g})^o \otimes \mathrm{Fun}_q(G)$ as an algebra, with the superscript o denoting, for the future reference, the opposite comultiplication. The relation between the two above descriptions of the quantum double $\mathcal{D} = \mathrm{Fun}_q(G) \otimes \mathrm{Fun}_q(G) = \mathcal{U}_h(\mathrm{g})^o \otimes \mathrm{Fun}_q(G)$ is again given by the Iwasawa decomposition

$$\mathcal{T} = T\Lambda^+ \;\text{ and }\; \hat{\mathcal{T}} = T\Lambda^-\,, \tag{2.1.21}$$

with T being the matrix of generators of $\mathrm{Fun}_q(G)$ (2.1.2) and $\Lambda^\pm$ corresponding to the two matrices $L^\pm$ of generators of $\mathcal{U}_h(\mathrm{g})$ (2.1.4) taken with the opposite comultiplication.

Let us mention, that although not explicit in the above given "coordinate" description of the quantum double $\mathcal{D}$, an important role in the corresponding "universal" constructions is played by a special element $\rho \in \mathcal{U}_h(\mathrm{g})^o \otimes \mathrm{Fun}_q(G)$, which is given in terms of mutually dual bases x_r and a_r

$$\rho = x_r \otimes a_r. \tag{2.1.22}$$

Its basic properties are

$$\rho^{-1} = (\mathrm{id} \otimes S)\rho\,, \tag{2.1.23}$$

$$(\Delta \otimes \mathrm{id})\rho = \rho_{23}\rho_{13}, \;\; (\mathrm{id} \otimes \Delta)\rho = \rho_{12}\rho_{13}\,. \tag{2.1.24}$$

In case of the compact form K, in which $\mathcal{D} = \mathrm{Fun}_q(AN) \otimes \mathrm{Fun}_q(K)$ and $\rho^* = \rho^{-1}$, the canonical element ρ can be given a good meaning. Using ρ, one defines the right and left dressing transformations [102] as coactions

$$\Delta_R : \mathrm{Fun}_q(AN) \to \mathrm{Fun}_q(AN) \otimes \mathrm{Fun}_q(K) : x \mapsto \rho(x \otimes 1)\rho^{-1} \tag{2.1.25}$$

and

$$\Delta_L : \mathrm{Fun}_q(K) \to \mathrm{Fun}_q(AN) \otimes \mathrm{Fun}_q(K) : a \mapsto \rho(1 \otimes a)\rho^{-1}\,. \tag{2.1.26}$$

The dressing transformation can be calculated explicitly on the elements of the matrix $\Lambda^*\Lambda$, the result being

$$\Delta_R(\Lambda^*\Lambda) = U^*\Lambda^*\Lambda U\,, \tag{2.1.27}$$

where on the RHS $\mathrm{Fun}_q(AN)$ has been identified with $\mathrm{Fun}_q(AN) \otimes 1$ and similarly $\mathrm{Fun}_q(K)$ has been identified with $1 \otimes \mathrm{Fun}_q(K)$.

Finally, let us mention that the underlying linear space of the quantum double $\mathcal{D}$ can be equipped with yet another algebra structure [133] (the so-called Heisenberg double, or quantum tangent bundle [106]). We will use the notation $H(\mathcal{D})$ for it. There are again two equivalent descriptions. Let again, with an abuse of notation, $\mathcal{T}$ and $\hat{\mathcal{T}}$ denote the two copies of matrices of generators corresponding to the two respective factors in $H(\mathcal{D}) \cong \mathrm{Fun}_q(G) \otimes \mathrm{Fun}_q(G)$. We have (apart from possible unimodularity or orthogonality conditions)

$$\mathcal{R}\mathcal{T}_1\mathcal{T}_2 = \mathcal{T}_2\mathcal{T}_1\mathcal{R}_{21}\,, \quad \mathcal{R}\hat{\mathcal{T}}_1\mathcal{T}_2 = \mathcal{T}_2\hat{\mathcal{T}}_1\mathcal{R}^{-1} \quad \text{and} \quad \mathcal{R}\hat{\mathcal{T}}_1\hat{\mathcal{T}}_2 = \hat{\mathcal{T}}_2\hat{\mathcal{T}}_1\mathcal{R}_{21}. \tag{2.1.28}$$

An equivalent description, using the decomposition of $H(\mathcal{D}) \cong \mathcal{U}_h(\mathrm{g})^o \otimes \mathrm{Fun}_q(G) \cong \mathrm{Fun}_q(G) \otimes \mathcal{U}_h(\mathrm{g})^o$

$$\mathcal{T} = TL^+ \quad \text{and} \quad \hat{\mathcal{T}} = TL^-, \tag{2.1.29}$$

gives in addition to (2.1.2), (2.1.4) (and possible unimodularity or orthogonality conditions) also [133]

$$L_1^+T_2 = T_2\mathcal{R}_{21}L_1^+ \quad \text{and} \quad L_1^-T_2 = T_2\mathcal{R}_{12}^{-1}L_1^-\,. \tag{2.1.30}$$

In case of the compact form K, also the Heisenberg double $H(\mathcal{D})$ can be equipped with an appropriate $*$-involution.

Concerning classical dressing transformation, in addition to [105] a nice discussion can be found in [81]. The classical dressing orbits coincide with symplectic leaves of a Poisson-Lie group. In case of a simples compact K, the classical dressing orbits are in a 1-1 correspondence with $*$-representations of $Funq(K)$ [111, 112, 127].

2.1.5. *Quantum groups and deformation quantization.* Here we briefly discuss quantum groups in the framework of deformation quantization. We will postpone a more general discussion of the deformation quantization to the sections related to noncommutative gauge theory. The following construction is due to Drinfeld [41]. Let us consider again a complex simple Lie algebra g and the algebra of formal power series $\mathcal{U}(\mathrm{g})[[h]]$ with coefficients in the corresponding enveloping algebra $\mathcal{U}(\mathrm{g})$. There exists a special element

$$\mathcal{F} \in \mathcal{U}(\mathrm{g})[[h]] \otimes \mathcal{U}(\mathrm{g})[[h]] \tag{2.1.31}$$

such that $\mathcal{U}(\mathrm{g})[[h]]$ can be turned into a quantum group (more precisely, into a quasitriangular Hopf algebra [40]), with the standard multiplication and counit induced from $\mathcal{U}(\mathrm{g})[[h]]$, and with the twisted comultiplication Δ_h and antipode S_h given by formulas

$$\Delta_h = \mathcal{F}^{-1}\Delta\mathcal{F}, \quad S_h = u(S)u^{-1}, \tag{2.1.32}$$

with

$$u = \sum \mathcal{F}^{-(1)}S\mathcal{F}^{-(2)}. \tag{2.1.33}$$

Δ and S in the above formulae are the standard comultiplication and antipode induced from $\mathcal{U}(\mathrm{g})$. The formula for the antipode together with the shorthanded notation $\mathcal{F} = \sum \mathcal{F}^{(1)} \otimes \mathcal{F}^{(2)}$ and $\mathcal{F}^{-1} = \sum \mathcal{F}^{-(1)} \otimes \mathcal{F}^{-(2)}$ are taken from [84].

The universal $\mathfrak{R}$-matrix is expressed with help of the symmetric g-invariant element $t \in \mathrm{g} \otimes \mathrm{g}$ (corresponding to the inverse of the Killing matrix) as

$$\mathfrak{R} = \mathcal{F}_{21}^{-1}\exp(ht)\mathcal{F}. \tag{2.1.34}$$

The quantum group structure of $\mathcal{U}(\mathrm{g})[[h]]$ as described above is isomorphic to the quantized enveloping algebra $\mathcal{U}_h(\mathrm{g})$.[1] In case of the compact form k, there exists a compatible $*$-involution on $\mathcal{U}(\mathrm{k})[[h]]$. Unfortunately, no explicit formula for $\mathcal{F}$ is known.

Here we shall be interested in the dual situation, which is nicely described in [124]. In this situation we have the vector space $\mathrm{Fun}(G)[[h]] := C^\infty(G)[[h]]$ with the standard comultiplication and counit, but a deformed multiplication $\star_h$ (star product) and antipode S_h. The corresponding formulas expressing the deformed multiplication and antipode in terms of the undeformed ones, m and S, are:

$$a \star_h b = m(\mathcal{F} * (a \otimes b) * \mathcal{F}^{-1}), \quad \text{and} \quad S_h a = S(u^{-1} * a * u) \tag{2.1.35}$$

[1] For that, of course, one has to interpret the deformation parameter h, in the definition of $\mathcal{U}_h(\mathrm{g})$ as a formal parameter too.

for $a, b \in \mathrm{Fun}(G)[[h]]$. Here, $*$ has been used to denote the actions of $U(g)[[h]]$ on $F(G)[[h]]$ via left and right invariant differential operators. Again, the corresponding quantum group $\mathrm{Fun}(G)[[h]]$ is isomorphic to $\mathrm{Fun}_q(G)$ [41]. As described above, it is the deformation quantization of the corresponding Poisson-Lie groups [2] $(G, \{\cdot, \cdot\}]$ with the Poisson bracket

$$i\{a, b\} = -m(r * (a \otimes b) - (a \otimes b) * r), \quad a, b \in \mathrm{Fun}(G), \tag{2.1.36}$$

where $r \in g \otimes g$ is the classical r-matrix, $\mathfrak{R} = 1 + hr + \ldots$ [105].

2.2. **Noncommutative gauge theory and deformation quantization.** The subject of deformation quantization has a long history. Although its present formulation goes back to the seminal paper by F.Bayen, C.Fronsdal, M. Flato, A. Lichnerowicz and D. Sternheimer [11], the idea of describing quantum mechanics completely in terms of the underlying classical phase space (based on the Weyl transformation and the Wigner map) is due to J.E. Moyal [90] and H.J. Groenewold [53]. Also the idea of noncommutative space-time coordinates was proposed very early. In his letter to Peierls [55], Heisenberg expressed his hope that uncertainty relations for the noncommutative coordinates might provide a natural cut-off for divergences in quantum field theory. First analysis of a quantum theory based on noncommutative coordinates was published by H.S. Snyder [110]. More recently, the noncommutative coordinates appeared in string theory (see, e.g., [122] an the references therein). It was argued that a D-brane world-volume becomes noncommutative in the presence of a non-zero background B-field. In the special case of a constant and non-degenerate 2-form B-field, the noncommutativity of wold-volume coordinates can be explicitly described using the star product of Moyal and Groenewold (also called Moyal-Weyl star product). In this formulation, the gauge theory on the D-branes becomes a noncommutative one, the noncommutativity being caused by the noncommutativity of the coordinates of the D-brane. In mathematics, a considerable progress came through the constructions of star products for any symplectic structure by B.V. Fedosov [47] and for any Poisson structure by M. Kontsevich [72], the latter being a partial result of the celebrated Kontsevich's formality theorem. As in the previous sections, also here we will restrict ourselves only to a very brief description of only few results concerning deformation quantization based gauge theories, which are directly related to the subject of the thesis. Similarly, we shall explicitly mention only the literature, which has a direct relationship to the results comprised in papers [P5-P7] and presented in the thesis. Also, we will not go in our discussion of the subject of noncommutative gauge theories beyond 2002, when last of the papers [P5-P7] was published. For surveys on the topic, we recommend, e.g., the following review articles [45, 123]. For a general discussion of noncommutative geometry (not necessarily based on deformation quantization) see [25].

2.2.1. *Noncommutative gauge theory in string theory.* Let us briefly recall how star products and noncommutative gauge theory arise in string theory [30, 104, 122]: Consider an open string σ-model with a background term

$$S_B = \frac{1}{2i} \int_D \sigma^*(B), \tag{2.2.1}$$

where the integral is over the string world-sheet D (disk), B is a constant, nondegenerate and closed two-form $dB = 0$ and $\sigma : D \to M$ is a smooth map of the world-sheet into the space-time manifold M. In local coordinates $B = 1/2 B_{ij} dx^i \wedge dx^j$. The correlation functions on the boundary of the disc in the so-called "decoupling limit" are

$$\langle f_1(x(\tau_1)) \cdot \ldots \cdot f_n(x(\tau_n)) \rangle_B = \int dx\, f_1 \star \ldots \star f_n, \quad (\tau_1 < \ldots < \tau_n), \tag{2.2.2}$$

with the Weyl-Moyal star product

$$(f \star g)(x) = e^{\frac{i\hbar}{2}\theta^{ij}\partial_i \partial'_j} f(x) g(x')\Big|_{x' \to x}, \tag{2.2.3}$$

which is the deformation quantization of the Poisson structure $\theta = B^{-1}$, $\hbar$ is a formal deformation parameter. More generally, a star product is an associative, $[[\hbar]]$-bilinear product

$$f \star g = fg + \sum_{n=1}^{\infty} (i\hbar)^n \underbrace{B_n(f, g)}_{\text{bilinear}}, \tag{2.2.4}$$

which is the deformation of a Poisson structure θ:

$$[f \stackrel{\star}{,} g] = i\hbar\{f\, ,\, g\} + \mathcal{O}(\hbar^2), \quad \{f\, ,\, g\} = \theta^{ij}(x)\partial_i f\, \partial_j g. \tag{2.2.5}$$

[2]Poisson-Lie-group group is a Lie group equipped with a Poisson structure, wherein the multiplication is a Poisson map

We now perturb the constant B field by adding a gauge potential $a_i(x)$: $B \to B + da$, $S_B \to S_B + S_a$, with

$$S_a = -i \int_{\partial D} d\tau a_i(x(\tau))\partial_\tau x^i(\tau). \tag{2.2.6}$$

Classically we have the naive gauge invariance

$$\delta a_i = \partial_i \lambda, \tag{2.2.7}$$

but in the quantum theory this may depend on the choice of regularization. For the Pauli-Villars regularization (2.2.7) remains a symmetry but if one expands $\exp S_a$ and employs the point-splitting regularization then the functional integral is invariant under noncommutative gauge transformations[3]

$$\hat{\delta}\hat{A}_i = \partial_i \hat{\lambda} + i\hat{\lambda} \star \hat{A}_i - i\hat{A}_i \star \hat{\lambda}, \tag{2.2.8}$$

where $\hat{A}$ denotes the gauge field in the theory defined with the point-splitting regularization. Since a sensible quantum theory should be independent of the choice of regularization, there should be field redefinitions $\hat{A}(a)$, $\hat{\lambda}(a, \lambda)$ (Seiberg-Witten map) that relate (2.2.7) and (2.2.8):

$$\hat{A}(a) + \hat{\delta}_{\hat{\lambda}} \hat{A}(a) = \hat{A}(a + \delta_\lambda a). \tag{2.2.9}$$

It is instructive to study the effect of the extra factor $\exp S_a$ in the correlation function (2.2.2) in more detail: It effectively shifts the coordinates[4]

$$x^i \to x^i + \theta^{ij} \hat{A}_j =: \mathcal{D}x^i. \tag{2.2.10}$$

The shifted coordinates and functions are covariant under noncommutative gauge transformations:

$$\hat{\delta}(\mathcal{D}x^i) = i[\hat{\lambda} \overset{\star}{,} \mathcal{D}x^i]. \tag{2.2.11}$$

The first expression implies (2.2.8) (for θ constant and nondegenerate).

The covariant coordinates (2.2.10) are the background independent operators of [122]; they and, more generally, covariant functions, can also be introduced abstractly in the general case of an arbitrary noncommutative space as we shall discuss in the next section.

2.2.2. *Covariant functions, covariant coordinates.* Take a more or less arbitrary noncommutative space, i.e., an associative unital algebra $\mathcal{A}_x$ of noncommuting variables with multiplication $\star$ and consider (matter) fields ψ on this space. In this section, $\star$ is not necessarily a star product in the sense of the deformation quantization. The fields can be taken to be elements of $\mathcal{A}_x$, or, more generally, a left module of it. The notion of a gauge transformation is introduced as usual [122] [5]

$$\psi \mapsto \Lambda \star \psi, \tag{2.2.12}$$

where Λ is an invertible element of $\mathcal{A}_x$. In analogy to commutative geometry where a manifold can be described by the commutative space of functions on it, we shall refer to the elements of $\mathcal{A}_x$ also as functions. Later we shall focus on the case where the noncommutative multiplication is a star product; the elements of $\mathcal{A}_x$ are then in fact formal power series in the deformation parameter with coefficients being ordinary functions in the usual sense of the word. The left-multiplication of a field with a function $f \in \mathcal{A}_x$ does in general not result in a covariant object because of the noncommutativity of $\mathcal{A}_x$:

$$f \star \psi \mapsto f \star \Lambda \star \psi \neq \Lambda \star (f \star \psi). \tag{2.2.13}$$

(As in ordinary gauge theory the gauge transformation only acts on the fields, i.e., on the elements of the left-module of $\mathcal{A}_x$ and not on the elements of $\mathcal{A}_x$ itself.) To cure (2.2.13), we introduce covariant functions

$$\mathcal{D}f = f + f_A, \tag{2.2.14}$$

that transform under gauge transformations by conjugation as

$$\mathcal{D}f \mapsto \Lambda \star \mathcal{D}f \star \Lambda^{-1}, \tag{2.2.15}$$

by adding 'gauge potentials' f_A with appropriate transformation property[6]

$$f_A \mapsto \Lambda \star [f \overset{\star}{,} \Lambda^{-1}] + \Lambda \star f_A \star \Lambda^{-1}. \tag{2.2.16}$$

[3] In this form this formula is only valid for the Moyal-Weyl star product.

[4] Notation: $\mathcal{D}$ should not be confused with a covariant derivative (but it is related).

[5] We shall often use the infinitesimal version $\delta\psi = i\lambda \star \psi$ of (2.2.12) – this is purely for notational clarity. Other transformations, like, e.g., $\psi \mapsto \psi \star \Lambda$ or $\psi \mapsto \Lambda \star \psi \star \Lambda^{-1}$ can also be considered.

[6] Notation: $[a \overset{\star}{,} b] \equiv a \star b - b \star a \equiv [a, b]_\star$.

Further covariant objects can be constructed from covariant functions; for instance, the '2-tensor'

$$\mathcal{F}(f,g) = [\mathcal{D}f \overset{\star}{,} \mathcal{D}g] - \mathcal{D}([f \overset{\star}{,} g]), \tag{2.2.17}$$

plays the role of covariant noncommutative field strength.

2.2.3. *Canonical structure (constant θ) and noncommutative Yang-Mills theory.* Consider the particular simple case of an algebra $\mathcal{A}_x$ generated by 'coordinates' x^i with canonical commutation relations [7]

$$[x^i \overset{\star}{,} x^j] = i\theta^{ij}, \quad \theta^{ij} \in \mathbb{C}. \tag{2.2.18}$$

This algebra arises in the decoupling limit of open strings in the presence of a *constant* B-field [30, 104, 122]. It can be viewed as the quantization of a Poisson structure with Poisson tensor θ^{ij} and the multiplication $\star$ is then the Weyl-Moyal star product

$$f \star g = f e^{\frac{i}{2}\theta^{ij}\overleftarrow{\partial_i}\otimes\overrightarrow{\partial_j}} g. \tag{2.2.19}$$

(This formula holds only in the present example, where θ^{ij} is constant and we shall also assume that it is non-degenerate). Let us focus on the coordinate functions x^i. The corresponding covariant coordinates are

$$\mathcal{D}x^i = x^i + x^i_A = x^i + \theta^{ij}\hat{A}_j, \tag{2.2.20}$$

where we have used θ to lower the index on $\hat{A}_j$. Using (2.2.18), we see that the transformation (2.2.16) of the noncommutative gauge potential $\hat{A}_j$ is

$$\hat{A}_j \mapsto i\Lambda \star \partial_j(\Lambda^{-1}) + \Lambda \star \hat{A}_j \star \Lambda^{-1}, \tag{2.2.21}$$

or, infinitesimally

$$\delta\hat{A}_j = \partial_j\lambda + i[\lambda, \hat{A}_j]_\star. \tag{2.2.22}$$

The noncommutative field strength

$$\hat{F}_{kl} = \partial_k\hat{A}_l - \partial_l\hat{A}_k - i[\hat{A}_k, \hat{A}_l]_\star \tag{2.2.23}$$

transforms covariantly

$$\hat{F}_{kl} \mapsto \Lambda \star \hat{F}_{kl} \star \Lambda^{-1}. \tag{2.2.24}$$

We have again used θ to lower indices to get (2.2.23) from the definition (2.2.17)

$$i\hat{F}_{kl}\theta^{ik}\theta^{jl} \equiv \mathcal{F}(x^i, x^j) = [x^i_A, x^j]_\star + [x^i, x^j_A]_\star + [x^i_A, x^j_A]_\star. \tag{2.2.25}$$

Note, that we should in general be more careful when using θ to lower indices as in (2.2.20) or (2.2.25) because this may spoil the covariance when θ is not constant as it was in this particular example. Relations (2.2.21), (2.2.23) and (2.2.24) define what is usually called noncommutative Yang-Mills theory (NCYM) in the narrow sense: ordinary Yang-Mills with all matrix products replaced by star products. This simple rule, however, only really works well for the Moyal-Weyl product, i.e. constant θ. In the general case it is wise to stick with the manifestly covariant and coordinate-independent objects defined in (2.2.14) and (2.2.17). The fundamental objects are really the mappings (differential operators) $\mathcal{D}$ and $\mathcal{F}$ in these equations. The transformation of $\mathcal{A} = \mathcal{D} - \mathrm{id} : f \mapsto f_A$ under gauge transformations is exactly so that (2.2.14) transforms by conjugation. The mappings $\mathcal{A} \in \mathrm{Hom}(\mathcal{A}_x, \mathcal{A}_x)$ and $\mathcal{F} \in \mathrm{Hom}(\mathcal{A}_x \wedge \mathcal{A}_x, \mathcal{A}_x)$ play the role of generalized noncommutative gauge potential and noncommutative field strength. There are several reasons, why one needs $\mathcal{A}$ and $\mathcal{D}$ and not just $A^i \equiv \mathcal{A}(x^i)$ (or $\hat{A}_i$, for θ constant): If we perform a general coordinate transformation $x^i \mapsto x^{i'}(x^j)$ and transform A^i (or $\hat{A}_i$) naively as its index structure suggests, then we would obtain objects that are no longer covariant under noncommutative gauge transformations.

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[7] In this section, the deformation parameter $\hbar$ will not be displayed explicitly.

2.3. **Higher gauge theory.** Higher gauge theory is a generalization of gauge theory - such as the theory of principal and vector bundles, connections and the parallel transport - from point particles to the higher-dimensional extended objects. In this context, an abelian gerbe can be viewed as the next level after complex line bundles in realizing integral cohomology classes on a manifold. Complex line bundles are classified (in topology) by their Chern classes, which are integral 2-cohomology classes. An abelian bundle gerbe gives geometric meaning to integral 3-cohomology class, [92, 16]. As in the case of line bundles, abelian bundle gerbes can be described in terms of local "transition functions". However, now the "transition functions" are not functions but local complex line bundles satisfying cocycle conditions for tensor products over triple overlaps of open sets. A more global point of view is to think of an abelian gerbe as a principal $PU(\mathcal{H})$ bundle. Here $PU(\mathcal{H})$ is the projective unitary group in a complex Hilbert space $\mathcal{H}$. In contrast to line bundles, gerbes are generically infinite-dimensional objects; only in the case of a torsion 3-cohomology class one can choose $\mathcal{H}$ to be finite-dimensional. Both of the above realizations of abelian gerbes arise in a natural way in quantum field theory. For instance, in [33, 34] they are related to chiral anomalies and in string theory; and, for instance, in [14, 6, 86] they appear in classification of D-branes in a nontrivial background B-field. For a discussion of relevance of abelian gerbes in in WZW model, TQFT and strings see, e.g., [52, 98, 31], respectively. Abelian (bundle) gerbes are not only a realization of the 3rd cohomology class (the Dixmier-Douady class). One can add geometric structures, a gerbe connection, and (local family of) 2-forms (curving). A gerbe with connection and curving (modulo equivalencies) is a Deligne class on the base manifold (for instance, on a D-brane world-volume); its top form part, the 3-form curvature, gives the Dixmier-Douady class. As before, also here we will restrict ourselves only to a very brief description of only few results concerning gerbes and higher gauge theories, which are directly related to the subject of the thesis. Also, we shall explicitly mention only the literature, which has a direct relationship to the results comprised in papers [P8-P10] and presented in the thesis. For introduction on the higher gauge theories, gerbes, abelian bundle gerbes we recommend, e.g., [12, 16, 87, 93, 57], respectively. Finally, we should also mention that nonabelian gerbes arose in the context of nonabelian cohomology, which goes back to Grothendieck [42, 50, 18] (see [19] or [87] for a concise introduction). The (synthetic) differential geometry of nonabelian gerbes – from the algebraic geometry point of view – is discussed thoroughly in the work of Breen and Messing [17].

2.3.1. *Abelian gerbes.* Line bundles can be described, in a well known manner, using transition functions. Consider a cover $\{O_i\}$ of the manifold M, then a line bundle is given by a set of $U(1)$ valued smooth transition functions $\{\lambda_{ij}\}$ that satisfy $\lambda_{ij} = \lambda_{ji}^{-1}$ and that on triple overlaps $O_{ijk} = O_i \cap O_j \cap O_k$ satisfy the cocycle condition

$$\lambda_{ij}\lambda_{jk} = \lambda_{ik} \ . \tag{2.3.1}$$

In the same spirit, a connection on a line bundle is a set of one-forms $\{\alpha_i\}$ on O_i such that on double overlaps $O_{ij} = O_i \cap O_j$,

$$\alpha_i = \alpha_j + \lambda_{ij} d\lambda_{ij}^{-1} \ . \tag{2.3.2}$$

Actually, we are interested only in isomorphic classes of line bundles with connection, indeed all physical observables are obtained from Wilson loops, and these cannot distinguish between a bundle with connection (λ_{ij}, α_i) and an equivalent one $(\lambda'_{ij}, \alpha'_i)$, that by definition satisfies

$$\lambda'_{ij} = \tilde{\lambda}_i \lambda_{ij} \tilde{\lambda}_j^{-1} \ , \qquad \alpha'_i = \alpha_i + \tilde{\lambda}_i d\tilde{\lambda}_i^{-1} \ , \tag{2.3.3}$$

where $\tilde{\lambda}_i$ are $U(1)$ valued smooth functions on O_i. We are thus led to consider the class $[\lambda_{ij}, \alpha_i]$ of all couples (λ_{ij}, α_i) that satisfy (2.3.2), and where $(\lambda_{ij}, \alpha_i) \sim (\lambda'_{ij}, \alpha'_i)$ iff (2.3.3) holds. The space of all these classes (called Deligne classes) is the Deligne cohomology group $H^1(M, \mathcal{D}^1)$.

Similarly, we can consider the Deligne class $[\lambda_{ijk}, \alpha_{ij}, \beta_i] \in H^2(M, \mathcal{D}^2)$, where now $\lambda_{ijk} : O_{ijk} \to U(1)$ is totally antisymmetric in its indices, $\lambda_{ijk} = \lambda_{jik}^{-1} = \lambda_{kij}$ etc., and satisfies the cocycle condition on quadruple overlaps O_{ijkl}

$$\lambda_{ijk}\lambda_{jkl}^{-1}\lambda_{ikl}\lambda_{ijl}^{-1} = 1 \ . \tag{2.3.4}$$

The connection one-form $\{\alpha_{ij}\}$ satisfies on O_{ijk}

$$\alpha_{ij} + \alpha_{jk} + \alpha_{ki} + \lambda_{ijk} d\lambda_{ijk}^{-1} = 0 \tag{2.3.5}$$

and the curving two-form $\{\beta_i\}$ satisfies on O_{ij}

$$\beta_i - \beta_j + d\alpha_{ij} = 0 \ . \tag{2.3.6}$$

The triple $(\lambda_{ijk}, \alpha_{ij}, \beta_i)$ gives the zero Deligne class if

$$(\lambda_{ijk}, \alpha_{ij}, \beta_i) = D(\tilde{\lambda}_{ij}, \tilde{\alpha}_i) \ , \tag{2.3.7}$$

where D is the Deligne coboundary operator, and $\tilde{\lambda}_{ij} : O_{ij} \to U(1)$ are smooth functions and $\tilde{\alpha}_i$ are smooth one-forms on O_i. Explicitly (2.3.7) reads[8]

$$\lambda_{ijk} = \tilde{\lambda}_{ik}\tilde{\lambda}_{jk}^{-1}\tilde{\lambda}_{ij}^{-1}\,, \tag{2.3.8}$$

$$\alpha_{ij} = -\tilde{\alpha}_i + \tilde{\alpha}_j + \tilde{\lambda}_{ij}d\tilde{\lambda}_{ij}^{-1}\,, \tag{2.3.9}$$

$$\beta_i = d\tilde{\alpha}_i\,. \tag{2.3.10}$$

There is also a geometric structure associated with the triple $(\lambda_{ijk}, \alpha_{ij}, \beta_i)$, it is that of (abelian) gerbe [16] or bundle gerbe [92] (with a connection and curving). Equivalence classes of gerbes with connection and curving are in 1-1 correspondence with Deligne classes, and with abuse of language we occasionally say that $[\mathcal{G}] = [\lambda_{ijk}, \alpha_{ij}, \beta_i]$ is the equivalence class of the gerbe $\mathcal{G} = (\lambda_{ijk}, \alpha_{ij}, \beta_i)$. As before, gauge invariant (physical) quantities can be obtained from the holonomy (Wilson surface), and this depends only on the equivalence class of the gerbe.

Gerbes are also called 1-gerbes in order to distinguish them from 2-gerbes and higher gerbes. In the same way as abelian 1-gerbes were described above, we can define abelian $n-1$-gerbes with curvings using Deligne cohomology classes in $H^n(M, \mathcal{D}^n)$ [16]. Correspondingly, we have characteristic classes in $H^{n+1}(M, \mathbb{Z})$. The case $n = 1$ gives equivalence classes of line bundles with connections, and in this case the characteristic class is the Chern class of the line bundle.

An important example of a 1-gerbe is a torsion gerbe, i.e. a gerbe with a characteristic class being a torsion class (let say an n-torsion) in $H^3(M, \mathbb{Z})$. Such a torsion gerbe can be obtained form a lifting gerbe, i.e. from a gerbe that describes the obstruction of lifting a $PU(n)$ bundle to a $U(n)$ one. We now describe this lifting gerbe and the associated twisted $U(n)$-bundle. Let $P \to M$ be a $PU(n)$ bundle and consider the exact sequence $U(1) \to U(n) \xrightarrow{\pi} PU(n)$. Consider an open cover $\{U_\alpha\}$ of $PU(n)$ with sections $s^\alpha : U_\alpha \subset PU(n) \to U(n)$. We can always choose a good cover $\{O_i\}$ of M such that each transition function g_{ij} of $P \to M$ has image contained in some U_α. Let $G_{ij} = s^\alpha(g_{ij})$, these are $U(n)$ valued functions and satisfy:

$$G_{ik}G_{jk}^{-1}G_{ij}^{-1} = \lambda_{ijk}\,, \tag{2.3.11}$$

where λ_{ijk} is $U(1)$-valued as is easily seen by applying the projection π and using the cocycle relation for the g_{ij} transition functions. We say that G_{ij} are the transition functions for a twisted $U(n)$- bundle and that the lifting gerbe is defined by the twist λ_{ijk}. It is indeed easy to check that the λ_{ijk} satisfy the cocycle condition (2.3.4) on quadruple overlaps O_{ijkl}. A connection for a twisted bundle is a set of Lie$(U(n))$-valued [9] 1-forms A_i such that $\alpha_{ij} \equiv -A_i + G_{ij}A_jG_{ij}^{-1} + G_{ij}dG_{ij}^{-1}$ is a connection for the corresponding gerbe (in particular $\pi_* A$ is a connection on the initial $PU(n)$ bundle P). We restate this construction this way: consider the couple (G_{ij}, A_i), and define

$$\mathbf{D}(G_{ij}, A_i) := (G_{ik}G_{jk}^{-1}G_{ij}^{-1}, -A_i + G_{ij}A_jG_{ij}^{-1} + G_{ij}dG_{ij}^{-1}, \tfrac{1}{n}TrdA_i)\,. \tag{2.3.12}$$

If this triple has abelian entries then it defines a gerbe, and (G_{ij}, A_i) is called a twisted bundle. We also say that the twisted bundle (G_{ij}, A_i) is twisted by the gerbe $\mathbf{D}(G_{ij}, A_i)$. Notice that the nonabelian $\mathbf{D}$ operation becomes the abelian Deligne coboundary operator D if $n = 1$ in $U(n)$ [cf.(2.3.7)].

Following the above discussion of 1-gerbes, for the purposes of this thesis, we understand under an abelian 2-gerbe with curvings on M a quadruple $(\lambda_{ijkl}, \alpha_{ijk}, \beta_{ij}, \gamma_i)$. Here $\lambda_{ijkl} : O_{ijkl} \equiv O_i \cap O_j \cap O_k \cap O_l \to U(1)$ is a Čech 3-cocycle

$$\lambda_{ijkl}\lambda_{ijlm}\lambda_{jklm} = \lambda_{iklm}\lambda_{ijkm} \quad \text{on } O_{ijklm}\,, \tag{2.3.13}$$

and λ_{ijkl} is totally antisymmetric, $\lambda_{ijkl} = \lambda_{jikl}^{-1}$ etc. Next, $\alpha_{ijk} \in \Omega^1(O_{ijk})$, $\beta_{ij} \in \Omega^2(O_{ij})$ and $\gamma_i \in \Omega^3(O_i)$ are a collection of local one, two, and three-forms totally antisymmetric in their respective indices and subject to the following relations:

$$\alpha_{ijk} + \alpha_{ikl} - \alpha_{ijl} - \alpha_{jkl} = \lambda_{ijkl}d\lambda_{ijkl}^{-1} \quad \text{on } O_{ijk}\,, \tag{2.3.14}$$

$$\beta_{ij} + \beta_{jk} - \beta_{ik} = d\alpha_{ijk} \quad \text{on } O_{ijk}\,, \tag{2.3.15}$$

$$\gamma_i - \gamma_j = d\beta_{ij} \quad \text{on } O_{ij}\,. \tag{2.3.16}$$

[8] The Deligne coboundary operator is $D = \pm\delta + d$, the sign factor in front of the Čech coboundary operator depends on the degree of the form D acts on; it insures $D^2 = 0$.

[9] $\mathrm{Lie}(U(n)) :=$ u(n)

The equivalence class of the 2-gerbe with curvings $(\lambda_{ijkl}, \alpha_{ijk}, \beta_{ij}, \gamma_i)$ is given by the Deligne class $[\lambda_{ijkl}, \alpha_{ijk}, \beta_{ij}, \gamma_i]$, where the quadruple $(\lambda_{ijkl}, \alpha_{ijk}, \beta_{ij}, \gamma_i)$ represents the zero Deligne class if it is of the form

(2.3.17) $$\lambda_{ijkl} = \tilde{\lambda}_{ijl}^{-1}\tilde{\lambda}_{jkl}^{-1}\tilde{\lambda}_{ijk}\tilde{\lambda}_{ikl}\,,$$

(2.3.18) $$\alpha_{ijk} = \tilde{\alpha}_{ij} + \tilde{\alpha}_{jk} + \tilde{\alpha}_{ki} + \tilde{\lambda}_{ijk} d\tilde{\lambda}_{ijk}^{-1}\,,$$

(2.3.19) $$\beta_{ij} = \tilde{\beta}_i - \tilde{\beta}_j + d\tilde{\alpha}_{ij}\,,$$

(2.3.20) $$\gamma_i = d\tilde{\beta}_i\,.$$

The above equations are summarized in the expression

(2.3.21) $$(\lambda_{ijkl}, \alpha_{ijk}, \beta_{ij}, \gamma_i) = D(\tilde{\lambda}_{ijk}, \tilde{\alpha}_{ij}, \tilde{\beta}_i)\,,$$

where D is the Deligne coboundary operator, $\tilde{\lambda}_{ijk}$ are $U(1)$-valued functions on O_{ijk} and $\tilde{\alpha}_{ij}, \tilde{\beta}_i$ are respectively 1- and 2-forms on O_{ij} and on O_i.

The Deligne class $[\lambda_{ijkl}, \alpha_{ijk}, \beta_{ij}, \gamma_i] \in H^3(M, \mathcal{D}^3)$ (actually the cocycle $\{\lambda_{ijkl}\}$) defines an integral class $\xi \in H^4(M, \mathbb{Z})$; this is the characteristic class of the 2-gerbe. Moreover, $[\lambda_{ijkl}, \alpha_{ijk}, \beta_{ij}, \gamma_i]$ defines a closed integral 4-form

(2.3.22) $$\frac{1}{2\pi i}G = \frac{1}{2\pi i}d\gamma_i\,.$$

The 4-form G is a representative of $\xi_{\mathbb{R}}$: the image of the integral class ξ in real de Rham cohomology.

2.3.2. *Abelian bundle gerbes.* Here we describe the construction of M. Murray [92], which identifies the geometric objects realizing the classes in $H^3(X, \mathbb{Z})$ in a similar spirit as line bundles realize classes in $H^3(X, \mathbb{Z})$ (see also, e.g., [93, 56, 57]). Let Y be a manifold. Consider a surjective submersion $\wp : Y \to X$, which in particular admits local sections. Let $\{O_i\}$ be the corresponding covering of X with local sections $\sigma_i : O_i \to Y$, i.e., $\wp\sigma_i = id$. We also consider $Y^{[n]} = Y \times_X Y \times_X Y \ldots \times_X Y$, the n-fold fibre product of Y, i.e., $Y^{[n]} := \{(y_1, \ldots y_n) \in Y^n \mid \wp(y_1) = \wp(y_2) = \ldots \wp(y_n)\}$. Given a (complex) line bundle $\mathcal{L}$ over $Y^{[2]}$ we denote by $\mathcal{L}_{12} = p_{12}^*(\mathcal{L})$ the line bundle on $Y^{[3]}$ obtained as a pullback of $\mathcal{L}$ under $p_{12} : Y^{[3]} \to Y^{[2]}$ (p_{12} is the identity on its first two arguments); similarly for $\mathcal{L}_{13}$ and $\mathcal{L}_{23}$. Consider a quadruple $(\mathcal{L}, Y, X, \ell)$, where $\mathcal{L}$ is a line bundle, $Y \to X$ a surjective submersion and ℓ an isomorphism of line bundles $\ell : \mathcal{L}_{12}\mathcal{L}_{23} \to \mathcal{L}_{13}$. We now consider bundles $\mathcal{L}_{12}$, $\mathcal{L}_{23}$, $\mathcal{L}_{13}$, $\mathcal{L}_{24}$, $\mathcal{L}_{34}$, $\mathcal{L}_{14}$ on $Y^{[4]}$ relative to the projections $p_{12} : Y^{[4]} \to Y^{[2]}$ etc. and also the line bundle isomorphisms ℓ_{123}, ℓ_{124}, ℓ_{123}, ℓ_{234} induced by projections $p_{123} : Y^{[4]} \to Y^{[3]}$ etc.

The quadruple $\mathcal{G} = (\mathcal{L}, Y, X, \ell)$, where $Y \to X$ is a surjective submersion, $\mathcal{L}$ is a line bundle over $Y^{[2]}$, and $\ell : \mathcal{L}_{12}L_{23} \to \mathcal{L}_{13}$ an isomorphism of line bundles over $Y^{[3]}$, is called an (abelian) bundle gerbe if ℓ satisfies the cocycle condition (associativity) on $Y^{[4]}$

(2.3.23) $$\begin{array}{ccc} \mathcal{L}_{12}\mathcal{L}_{23}\mathcal{L}_{34} & \xrightarrow{\ell_{234}} & \mathcal{L}_{12}\mathcal{L}_{24} \\ \ell_{123}\big\downarrow & & \big\downarrow \ell_{124} \\ \mathcal{L}_{13}\mathcal{L}_{34} & \xrightarrow{\ell_{134}} & \mathcal{L}_{14}\,. \end{array}$$

Let us also mention that there exists a proper notion of an isomorphism (the so-called stable isomorphism [94]) for abelian bundle gerbes, such that the categories of abelian bundle gerbes and of Čech 2-cocycles (2.3.4) are equivalent.

Further, an abelian bundle 1-gerbe can be equipped with a connection and a curving, so that locally it becomes represented by the full Deligne 2-cocycle (see section 2.3.1). Without going into details, we just notice that the connection on an abelian 1-gerbe $\mathcal{G}$ can be defined as a connection on the line bundle $\mathcal{L}$ fulfilling a more or less obvious compatibility condition on $Y^{[3]}$. Also, the above described twisted principal bundles with connections can be cast in the language of bundle gerbes (cf. bundle gerbe modules of ([6]).

Abelian bundle 2-gerbes have been introduced in [36] and discussed in detail in [116].

2.3.3. *Classifying spaces, classifying topoi.* For a topological group G, principal G-bundles over a (topological) space X are classified by the first Čech cohomology $H^1(X, G)$ of X with coefficients in G. Under some mild conditions, these Čech cohomology classes are in 1-1 correspondence with homotopy classes of maps $[X, BG]$ from X to the classifying space BG (see, e.g., [54]). For example, the elements of

(2.3.24) $$H^1(X, U(1)) \cong H^2(X, \mathbb{Z}) \cong [X, BU(1)]$$

classify line bundles. Thus, characteristic classes for bundles can be obtained as pullbacks of cohomology classes on BG. One way to define the classifying space is to take the geometric realization $|NG|$ of the nerve NG of the group G.

The notion of a principal bundle and of the classifying space can be generalized from the case of a topological group G to the case of a topological category $\mathbb{C}$ [88]. Roughly speaking, a $\mathbb{C}$-principal bundle over X can be defined as a continuous functor from the topological category defined by an ordered open covering $\mathcal{U} = \{U_i\}$ (more generally from a linear order L over X [88]) to the category $\mathbb{C}$, i.e., as a $\mathbb{C}$-valued Čech 1-cocycle. Again, the classifying space $B\mathbb{C}$ is defined as the geometric realization $|N\mathbb{C}|$ of the nerve $N\mathbb{C}$ of the category $\mathbb{C}$. If X is a CW complex and $\mathbb{C}$ has contractible spaces of objects and arrows, concordance classes $Lin_c(X, \mathbb{C})$ of principal $\mathbb{C}$-bundles are in 1-1 correspondence with the homotopy classes of maps $[X, B\mathbb{C}]$:

$$Lin_c(X, \mathbb{C}) \cong [X, B\mathbb{C}]. \tag{2.3.25}$$

The above restrictions on X and $\mathbb{C}$ can be abandoned if one considers, instead of the classifying space $B\mathbb{C}$, the classifying topos $\mathcal{B}\mathbb{C}$ (still, all spaces have to be assumed to be sober, i.e., every closed subset which can not be written as a union of two smaller closed sets is a closure of a unique one point set). The classifying topos $\mathcal{B}\mathbb{C}$ is the so-called Deligne topos $Sh(N\mathbb{C})$ of sheaves on the nerve $N\mathbb{C}$ of the category $\mathbb{C}$. Let us recall that a sheaf S on a simplicial space Y is defined to be a system of sheaves S^n on Y_n, for $n \geq 0$, together with sheaf maps $S(\alpha)\colon Y(\alpha)^* S^n \to S^m$ for each $\alpha\colon [n] \to [m]$. These maps are required to satisfy the proper functoriality conditions [88]. Equipped with properly defined morphisms we have the category of sheaves $Sh(Y)$ on the simplicial space Y. The category $Sh(Y)$ of sheaves on a simplicial space is a topos, which is called the Deligne topos.

There is an equivalence [88] between the category $Lin(X, \mathbb{C})$ of $\mathbb{C}$-principal bundles and category of geometric morphisms $\mathrm{Hom}(Sh(X), Sh(N\mathbb{C}))$ between (Grothendieck) topoi $Sh(X)$ and $Sh(N\mathbb{C})$:

$$Lin(X, \mathbb{C}) \simeq \mathrm{Hom}(Sh(X), Sh(N\mathbb{C})). \tag{2.3.26}$$

Let us motivate the later discussion, for the case of one dimension higher, by considering the example of abelian 1-gerbes. We have

$$H^2(X, U(1)) \cong H^3(X, \mathbb{Z}) \cong [X, B^2U(1)]. \tag{2.3.27}$$

In this situation, $B^2U(1)$ can be given the following interpretation: Starting with $U(1)$ we can consider the strict Lie 2-group (see, e.g., [12]) with only one object, one 1-arrow and 2-arrows being the elements of $U(1)$, or equivalently, the corresponding crossed module [20]. Then the classifying space $B^2U(1)$ is (homotopy) equivalent to the geometric realization of the so-called Duskin nerve [46] of this strict 2-group. The "classifying properties" of geometric realizations of the so-called Duskin's nerves have been investigated in cases of strict Lie 2-groups, topological 2-groups and topological bicategories in [66], [22] and [5], respectively. One of the results of the present thesis is the description of the classifying topos and its properties for any topological bicategory.

3. Goals of the thesis

The thesis has several interrelated goals:

(i) We shall describe bicovariant differential calculi on quantized simple Lie groups. To be more specific, the bicovariant differential calculi are constructed explicitly for the four principal series using the R-matrix approach of the Leningrad school. Such an explicit construction gives a direct relation between the left (or right) invariant vector fields on the quantum group and the corresponding quantized enveloping algebra.

(ii) We shall describe an important class of quantum homogeneous spaces using local coordinate functions generating a noncommutative algebra. This class of quantum homogeneous spaces corresponds to the coadjoint orbits (dressing orbits, including, e.g., flag manifolds) of compact simple Lie groups (for the four principal series) and their generalized Pontryagin duals. In the course of doing that, we shall introduce the proper generalization of Perelomov's coherent states. Also, using differential calculi of the item *(i)*, the representation theory can be related to the noncommutative differential geometry of the quantum dressing orbits in a manner similar to that of the geometric quantization.

(iv) We shall develop an approach to noncommutative gauge theories in the framework of deformation quantization based on an explicit construction of the Seiberg-Witten map, i.e., the map relating the commutative and noncommutative gauge fields. Our construction of the Seiberg-Witten map is based on the Kontsevich's formality theorem and is valid for an arbitrary

(formal) Poisson structure on the underlying manifold (for example, the world-volume of a D-brane).

(v) Based on the results of the item *(iv)*, we give a definition of a noncommutative line bundle in terms of a noncommutative 1-cocycle. The corresponding space of sections is a projective module. The Morita equivalence classes of star products are classified in terms of the action of the Picard group.

(vi) We shall define nonabelian bundle gerbes related to an arbitrary crossed module of Lie groups $G \to D$, hence, generalizing the abelian bundle gerbes of M. Murray. For such nonabelian bundle gerbes, principal G-bundles (equipped with a trivialization under the change of the structure group from G to D) play the role of "transition functions". Based on this, the corresponding theory of connections and curvings (nonabelian B-fields) generalizing some classical constructions know from differential geometry can be developed.

(vii) We shall define "twistings" of nonabelian bundle gerbes by abelian 2-gerbes and describe (locally) their differential geometry. Also, we shall describe a possible application to the cancelation of global anomalies on multiple five-branes.

(viii) Nonabelian bundle gerbes of the item *(vi)* can be viewed as principal bundles with the structure Lie group replaced by a crossed module, i.e., a strict 2-group. Similarly, a principal $\mathbb{C}$-bundle with $\mathbb{C}$ being a topological category, can be generalized to principal $\mathbb{B}$-bundle with $\mathbb{B}$ being a topological bicategory. In this very general setting, we shall construct the corresponding classifying topos for any topological bicategory $\mathbb{B}$.

4. Applied methods

4.1. **Quantum groups.**

4.1.1. *Bicovariant bimodules.* The following general theory of bicovariant bimodules has been given by Woronowicz in [131] and will be used extensively in our construction of differential calculi on quantum groups $\mathcal{A} = \mathrm{Fun}_q(G)$. See also [9] for related discussion. Let us assume that we are given a family of functionals $F = (f_{ij})_{i,j=1}^k \in \mathcal{U} := \mathcal{U}_h(\mathrm{g})$, $k \in \mathbb{N}$, such that

(4.1.1) $$\Delta F = F \dot{\otimes} F$$

and

(4.1.2) $$e(F) = I,$$

and a family of quantum functions $R = (R_{ij})_{i,j=1}^k \in \mathcal{A}$, such that

(4.1.3) $$\Delta R = R \dot{\otimes} R$$

and

(4.1.4) $$e(R) = I.$$

Besides, matrices R and F are supposed to satisfy the following compatibility condition

(4.1.5) $$R_{ij}(a * f_{ih}) = (f_{ji} * a) R_{hi},$$

for all j, h and any $a \in \mathcal{A}$. Here we used the following notation

(4.1.6) $$x * a := a^{(1)}(x, a^{(2)}) \quad \text{and} \quad a * x := (x, a^{(1)}) a^{(2)}$$

for $x \in \mathcal{U}$, $a \in \mathcal{A}$, where $a^{(1)} \otimes a^{(2)} := \Delta a$. Let us now assume a free left module Γ over $\mathcal{A}$ generated by elements ω_i, $i = 1, 2, ..., k$, and let us introduce the right multiplication by elements of A and the left δ_L and right coaction δ_R of A on Γ by the following formulae

(4.1.7) $$(a_i \omega_i) b = a_i (f_{ij} * b) \omega_j,$$

(4.1.8) $$\delta_L(a_i \omega_i) = \Delta(a_i)(1 \otimes \omega_i) \quad \text{and} \quad \delta_R(a_i \omega_i) = \Delta(a_i)(\omega_j \otimes R_{ji}).$$

A theorem of Woronowicz says that the triple $(\Gamma, \delta_L, \delta_R)$ is a bicovariant bimodule and, vice versa, that any bicovariant bimodule is of this form. Elements ω_i, $i = 1, 2..., k$ (left-invariant one-forms) form a basis in the linear subspace ${}_{inv}\Gamma \subset \Gamma$ of all left-invariant elements of Γ. The space Γ of all one-forms on the quantum group $\mathcal{A}$ (and the whole exterior algebra $\Gamma^\wedge$) over A are naturally equipped with a structure of a bicovariant bimodule. Also, given two bicovariant bimodules Γ_1 and Γ_2, we can construct their tensor product $\Gamma_1 \otimes \Gamma_2$, which is again a bicovariant bimodule. The linear basis in ${}_{inv}(\Gamma_1 \otimes \Gamma_2)$ can be chosen as $\omega_{ij} = \omega_i \otimes \omega_j$. In this basis we have $R_{ij,kl} = R^1_{ik} R^2_{jl}$ and $f_{ij,kl} = f^1_{ik} f^2_{jl}$.

Concerning the dual generators χ_i, $i = 1, 2..., k$ of ${}_{inv}\Gamma^*$, i.e., the left-invariant vector fields, we have

$$\Delta\chi_i = \chi_i \otimes f_{ji} + e \otimes \chi_i \quad \text{and} \tag{4.1.9}$$

$$\chi_i \otimes d\omega_i = -\chi_i\chi_j \otimes \omega_i \wedge \omega_j \qquad (\text{Maurer} - \text{Cartan}). \tag{4.1.10}$$

The linear space ${}_{inv}\Gamma^*$ generated by χ_i's is closed under the following "commutator"

$$[\chi_i, \chi_j] = \chi_i\chi_j - \langle f_{kj}, R_{i\ell}\rangle \chi_k\chi_\ell. \tag{4.1.11}$$

This commutator obeys (properly generalized) antisymmetry condition and Jacobi identity [131].

4.1.2. *Coherent states.* The following classical construction is due to A.M. Perelomov [97]. A quantum version of it is given in $[P4]$ and plays an important role in our construction of quantum homogeneous spaces and the related representation theory. Denote by G a simple and simply connected complex Lie group and by $K \subset G$ its compact form. Let $\mathcal{T}^\lambda$ be an irreducible unitary representation of K in $\mathcal{H}_\lambda$ corresponding to a minimal weight λ. $\mathcal{T}^\lambda$ extends unambiguously as a holomorphic representation of G in $\mathcal{H}_\lambda$ (Weyl unitary trick). Let $e_\lambda \in \mathcal{H}_\lambda$ be a normalized weight vector and set

$$\Gamma : G \to \mathcal{H}_\lambda : g \mapsto \mathcal{T}^\lambda(g^{-1})\, e_\lambda\,. \tag{4.1.12}$$

The vector–valued function Γ is a coherent state in the sense of Perelomov. Denote further by $K_0 \subset K$ respectively $P_0 \subset G$ the isotropy subgroups of the point $\mathbb{C}e_\lambda \in \mathbb{P}(\mathcal{H}_\lambda)$. This means that there exists a character χ of P_0, unitary on $K_0 \subset P_0$, such that

$$\mathcal{T}^\lambda(k)\, e_\lambda = \chi(k)\, e_\lambda\,, \text{ for } k \in P_0\,. \tag{4.1.13}$$

The mapping

$$\mathcal{H}_\lambda \ni u \mapsto \langle \Gamma(\cdot), u\rangle \in C^\infty(K) \tag{4.1.14}$$

is injective and so one embeds this way $\mathcal{H}_\lambda$ into the vector space of χ–equivariant functions on K. Sending $(g, k) \in K \times K_0$ to $k^{-1}g \in K$ we get a principal bundle $K \to K_0\backslash K$ and using the 1–dimensional representation χ one associates to it a line bundle over the base space $K_0\backslash K = P_0\backslash G$. Hence χ– equivariant functions on K are identified with sections in this line bundle. Set

$$w_\lambda := \langle e_\lambda, \mathcal{T}^\lambda e_\lambda\rangle \in C^{hol}(G)\,. \tag{4.1.15}$$

The function w_λ is χ–equivariant on K and thus determines a trivialization of the line bundle over the cell given by $w_\lambda(g) \neq 0$. The Gauss decomposition provides a standard way to choose holomorphic coordinates $\{z_j\}$ on this cell. Vectors u from $\mathcal{H}_\lambda$ are then represented by polynomials $\psi_u := {w_\lambda}^{-1}\langle \Gamma, u\rangle$ in the variables $\{{z_j}^*\}$ and so the representation $\mathcal{T}^\lambda$ acts in the space of antiholomorphic functions living on the cell. Finally, we also recall that every operator $B \in \mathrm{Lin}(\mathcal{H}_\lambda)$ is represented by its symbol $\sigma(B) \in C^a(K_0\backslash K)$ or, this is the same, by a real analytic K_0– invariant function on K,

$$\sigma(B) := \{g \mapsto \langle \Gamma(g), B\,\Gamma(g)\rangle\}\,. \tag{4.1.16}$$

The mapping $B \mapsto \sigma(B)$ is injective

4.2. Noncommutative gauge theory based on deformation quantization.

4.2.1. *Deformation quantization via Kontsevich's formality.* Kontsevich's formality map [72], which is the basic tool in our construction of noncommutative gauge theory, is a collection of skew-symmetric multilinear maps U_n for $n = 0, 1, \ldots$, which map tensor products of n polyvector fields to differential operators. More precisely, U_n maps the tensor product of n k_i-vector fields to an m-differential operator, where m is determined by the condition

$$m = 2 - 2n + \sum_{i=1}^{n} k_i. \tag{4.2.1}$$

In particular, U_1 maps a k-vector field to a k-differential operator

$$U_1(\xi_1 \wedge \ldots \wedge \xi_k)(f_1, \ldots, f_k) = \frac{1}{k!} \sum_{\sigma \in \Sigma_k} \mathrm{sgn}(\sigma) \prod_{i=1}^{k} \xi_{\sigma_i}(f_i), \tag{4.2.2}$$

and U_0 is defined to be the ordinary multiplication of functions:

$$U_0(f, g) = fg. \tag{4.2.3}$$

The U_n, $n \geq 1$, satisfy the formality condition [72]

$$d_\mu U_n(\alpha_1, \ldots, \alpha_n) + \frac{1}{2} \sum_{\substack{I \sqcup J = (1,\ldots,n) \\ I,J \neq \emptyset}} \pm [U_{|I|}(\alpha_I)\,,\, U_{|J|}(\alpha_J)]_{\mathrm{G}}$$

$$= \sum_{i<j} \pm U_{n-1}\left([\alpha_i\,,\,\alpha_j]_{\mathrm{S}}, \alpha_1, \ldots, \widehat{\alpha}_i, \ldots, \widehat{\alpha}_j, \ldots, \alpha_n\right), \tag{4.2.4}$$

where

$$d_\mu \mathcal{C} \equiv -[\mathcal{C}\,,\,\mu]_{\mathrm{G}} \tag{4.2.5}$$

is the Gerstenhaber bracket [72], and $\mu(f,g) = f \cdot g$ the commutative multiplication of functions; the hat marks an omitted vector field. See [72, 28] for explicit constructions and more details, and [3, 72] for the definition of the signs in this equation. In the following, we collect the three special cases that we actually use in this thesis.

Consider the formal series (see also [83])

$$\Phi(\alpha) = \sum_{n=0}^{\infty} \frac{(i\hbar)^n}{n!} U_{n+1}(\alpha, \theta, \ldots, \theta). \tag{4.2.6}$$

According to the matching condition (4.2.1), $U_{n+1}(\alpha, \theta, \ldots, \theta)$ is a tridifferential operator for every n if α is a trivector field, it is a bidifferential operator if α is a bivector field, it is a differential operator if α is a vector field and it is a function if α is a function; in all cases θ is assumed to be a bivector field.

A Poisson bivector θ gives rise to a star product via the formality map: According to the matching condition (4.2.1), $U_n(\theta, \ldots, \theta)$ is a bidifferential operator for every n if θ is a bivector field. This can be used to define a product

$$f \star g = \sum_{n=0}^{\infty} \frac{(i\hbar)^n}{n!} U_n(\theta, \ldots, \theta)(f,g) = fg + \frac{i\hbar}{2}\, \theta^{ij} \partial_i f\, \partial_j g + \cdots\,. \tag{4.2.7}$$

The formality condition implies

$$\mathbf{d}_\star \star = i\hbar \Phi(\mathbf{d}_\theta \theta), \tag{4.2.8}$$

or, $[\star\,,\,\star]_{\mathrm{G}} = i\hbar\Phi([\theta\,,\,\theta]_{\mathrm{S}})$, i.e., associativity of $\star$, if θ is Poisson. (If θ is not Poisson, i.e., has non-vanishing Schouten-Nijenhuis bracket $[\theta\,,\,\theta]_{\mathrm{S}}$. then the product $\star$ is not associative, but the non-associativity is nevertheless under control via the formality condition by (4.2.8).)

We can define a linear differential operator[10]

$$\Phi(\xi) = \xi + \frac{(i\hbar)^2}{2} U_3(\xi, \theta, \theta) + \cdots \tag{4.2.9}$$

for every vector field ξ. For θ Poisson the formality condition gives

$$\mathbf{d}_\star \Phi(\xi) = i\hbar \Phi(\mathbf{d}_\theta \xi) = i\hbar\, \mathbf{d}_\theta \xi + \cdots\,. \tag{4.2.10}$$

Vector fields ξ that preserve the Poisson bracket, $\mathbf{d}_\theta \xi = -[\theta\,,\,\xi]_{\mathrm{S}} = 0$, give rise to derivations of the star product (4.2.7): From (4.2.10) and (4.2.5)

$$0 = [\mathbf{d}_\star \Phi(\xi)](f,g) = -[\Phi(\xi)](f \star g) + f \star [\Phi(\xi)](g) + [\Phi(\xi)](f) \star g. \tag{4.2.11}$$

Hamiltonian vector fields $\mathbf{d}_\theta f$ give rise to inner derivations of the star product (4.2.7): We can define a new function[10]

$$\hat{f} \equiv \Phi(f) = f + \frac{(i\hbar)^2}{2} U_3(f, \theta, \theta) + \cdots \tag{4.2.12}$$

for every function f. For θ Poisson the formality condition gives

$$\mathbf{d}_\star \hat{f} = i\hbar \Phi(\mathbf{d}_\theta f). \tag{4.2.13}$$

Evaluated on a function g, this reads

$$[\Phi(\mathbf{d}_\theta f)](g) = \frac{1}{i\hbar}[g \stackrel{\star}{,} \hat{f}]. \tag{4.2.14}$$

The Hamiltonian vector field $\mathbf{d}_\theta f$ is thus mapped to the inner derivation $\frac{i}{\hbar}[\hat{f} \stackrel{\star}{,} \cdot]$.

4.3. **Higher gauge theory.**

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[10] $U_2(\xi, \theta) = 0$ and $U_2(f, \theta) = 0$ by explicit computation of Kontsevich's formulas.

4.3.1. *Crossed modules.* The theory of crossed modules of Lie groups (see, e.g., [20]) will be, in addition to the classical theory of principal bundles and connections [71], one of main tools in our construction of nonabelian bundle gerbes and their differential geometry, including connections and curvings.

Let G and D be two Lie groups. We say that G is a crossed D-module if there is a Lie group morphism $\partial : G \to D$ and a smooth action of D on G $(d, g) \mapsto {}^d g$ such that

$$ {}^{\partial(g)}g' = gg'g^{-1} \text{ (Peiffer condition)} \tag{4.3.1}$$

for $g, g' \in G$, and

$$\partial({}^d l) = d\partial(g)d^{-1} \tag{4.3.2}$$

for $g \in G, d \in D$ hold true. We will use the notation $G \overset{\partial}{\to} D$ or $G \to D$ for the crossed module.

The basic example is the crossed module $G \overset{\partial}{\to} Aut(G)$, where $Aut(G)$ is the automorphism Lie group of the Lie group G and the group homomorphism ∂ is given by the canonical map $Ad : G \to Aut(G)$. Another example is $(1 \to G)$.

There is an obvious notion of a morphism of crossed modules. A morphism between crossed modules $G \overset{\partial}{\to} D$ and $G' \overset{\partial'}{\to} D'$ is a pair of Lie group morphisms $\lambda : G \to G'$ and $\kappa : D \to D'$ such that the diagram

$$\begin{array}{ccc} G & \xrightarrow{\partial} & D \\ \lambda\downarrow & & \downarrow\kappa \\ G' & \xrightarrow{\partial'} & D' \end{array} \tag{4.3.3}$$

commutes, and for any $g \in G$ and $d \in D$ we have the following identity

$$\lambda({}^d g) ={}^{\kappa(d)} \lambda(g). \tag{4.3.4}$$

A crossed module of Lie groups defines naturally a strict Lie 2-group (see, e.g., [4]). The set of objects is $C_0 = \{*\}$, the set of 1-arrows is $C_1 = D$ and the set of 2-arrows is $C_2 = D \times G$. The "vertical" multiplication is given on C_2 by

$$(d, g_1)(\partial(g_1)d, g_2) = (d, g_1 g_2) \tag{4.3.5}$$

and the "horizontal" multiplication is given by

$$(d_1, g_1)(d_2, g_2) = (d_1 d_2, g_1\, {}^{d_1}g_2). \tag{4.3.6}$$

See, e.g., [21] for more details on the relation between crossed modules and strict Lie 2-groups.

4.3.2. *Global worldsheet anomalies of D-branes.* Here we briefly describe the so-called "inflow" mechanism, as it applies to a stack of n D-branes and the corresponding Freed-Witten anomaly [48], [31]. The method described here will be applied later in the thesis (see section 5.3.3) to the case of M5-branes, in which case 1-gerbes will be replaced by 2-gerbes and principal bundles by nonabelian gerbes.

In string theory, the background B-field is naturally interpreted as a 1-gerbe with connection and curving on the spacetime manifold M [31, 70]. Let $[\lambda_{ijk}, \alpha_{ij}, \beta_i]$ be the corresponding Deligne class and H the associated 3-form. Further, let Q be be a cycle embedded in the spacetime manifold M, on which cycle open (super)strings can end (i.e., we have D-branes wrapping Q) and $[\omega_{ijk}, 0, 0]$ be the Deligne class associated with the second Stiefel-Whitney class $\omega_2 \in H^2(Q, \mathbb{Z}_2)$ of the normal bundle of Q (or, which is the same, with its image W_3 in $H^3_{tors}(Q, \mathbb{Z})$). It can be shown [31] that the general condition for a stack of n D-branes to be wrapping the cycle Q in M is the existence of a twisted bundle (G_{ij}, A_i) (2.3.12) on Q such that

$$[\lambda_{ijk}, \alpha_{ij}, \beta_i]|_Q - [\omega_{ijk}, 0, 0] = [\mathbf{D}(G_{ij}, A_i)] + [1, 0, B_Q]\,, \tag{4.3.7}$$

where B_Q is a 2-form on Q. In particular, for the characteristic classes of these gerbes we have (cf. [48]),

$$[H]|_Q - W_3 = \xi_{[\mathbf{D}(G_{ij}, A_i)]}\,, \tag{4.3.8}$$

where $[H]|_Q \equiv \xi_{\mathcal{G}|_Q}$ is the characteristic class of the restriction to Q of the gerbe $\mathcal{G} = (\lambda_{ijk}, \alpha_{ij}, \beta_i)$ associated with the 3-form H, and $W_3 = \beta(\omega_2)$ is the obstruction for having Spin^c structure on the normal bundle of Q Here β is the Bockstein homomorphism associated with the short exact sequence $\mathbb{Z} \overset{\times 2}{\to} \mathbb{Z} \to \mathbb{Z}_2$.

4.3.3. *Nerves, linear orders.* Simplicial spaces [85] coming from (Duskin [46] and possibly other [78, 125, 109]) nerves of topological bicatgeories [11] (and spaces obtained by geometric realizations of these nerves) play a central role in the very last part of this thesis devoted to classifying spaces and topoi. Let us recall that the nerve of a (topological) category is defined as a simplicial space $N\mathbb{C}$ with space of n-simplices $N\mathbb{C}_n$ being the fibred product space $\mathbb{C}_1 \times_{\mathbb{C}_0} \times \ldots \times_{\mathbb{C}_0} \mathbb{C}_1$ of all composable strings of arrows of $\mathbb{C}$. The degeneracy maps $N\mathbb{C}_{n-1} \to N\mathbb{C}_n$ are given by insertions of identity arrows. The face maps $N\mathbb{C}_n \to N\mathbb{C}_{n-1}$ (except the 0th and nth, which are given by dropping the first and the last arrow, respectively) are given by compositions of arrows. In case of a small category, the nerve is just a simplicial set.

Let us recall that the Duskin nerve of a (topological) bicategory $\mathbb{B}$ is a 3-coskeletal simplicial space $N\mathbb{B}$ with 0-simplices the objects x_0 of $2\mathbb{C}$, 1-simplices the 1-arrows $x_0 \stackrel{x_{01}}{\to} x_1$ of $\mathbb{B}$ and 2-simplices are 2-cells which are triangles $x_{02} \stackrel{x_{012}}{\Longrightarrow} x_{01}x_{12}$ filled with a 2-arrow x_{012}.

For the future reference, let us collect few definitions and result, which we will use in section 5.3.4. and which we repeat almost verbatim from [88]. For a simplicial space Y the *geometric realization* $|Y|$ will always mean the thickened (fat) geometric realization. This is defined as a topological space obtained from the disjoint sum $\sum_{n\geq 0} X_n \times \Delta^n$ by the the equivalence relations

$$(\alpha^*(x), t) \sim (x, \alpha(t)) \tag{4.3.9}$$

for all injective (order-preserving) arrows $\alpha\colon [n] \to [m] \in \Delta$, any $x \in X_m$ and any $t \in \Delta^n$, where Δ^n is the standard topological n-simplex. In the case of a so-called good simplicial space [5] (e.g., all Y_n re CW-complexes), this geometric realization is homotopy equivalent to the geometric realization of the underlying simplicial set of Y, which is defined as above but allowing for all arrows in Δ.

Linear order over a topological space X is a sheaf L on X together with a subsheaf $O \subseteq L \times_X L$ such that for each point $x \in X$ the stalk L_x is nonempty and linearly ordered by the relation $y \leq z$ iff $(y, z) \in O_x$, for $y, z \in L_x$. A mapping $L \to L'$ between two linear orders over X is a mapping of sheaves restricting for each $x \in X$ to an order preserving map of stalks $L_x \to L'_x$. This defines a category of linear orders on X.

A linear order L on X defines an obvious topological category with L as space of objects and the order subsheaf $O \subseteq L \times_X L$ as space of arrows. Hence, we can speak of a nerve NL of the linear order L. This nerve is obviously a simplicial sheaf on X (a simplicial space with étale maps into X).

For any space X and any simplicial space Y write $Lin(X, Y)$ for the category of linear orders (L, aug) on X equipped with a simplicial map (augmentation) $\mathrm{aug} : NL \to Y$ from the nerve of L to Y. A morphism $(L, \mathrm{aug}) \to (L', \mathrm{aug}')$ in $Lin(X, Y)$ is a map of linear orders $L \to L'$ such that the induced map $NL \to NL'$ on the nerves respects the augmentations.

We call two objects $E_0, E_1 \in Lin(X, Y)$ concordant if there exists an $E \in Lin(X \times [0, 1], Y)$ such that we have $E_0 \cong i_0^*(E)$ and $E_0 \cong i_1^*(E)$ under the obvious inclusions $i_0, i_1 : X \hookrightarrow X \times [0, 1]$. $Lin_c(X, Y)$ will denote the collection of concordance classes of objects from $Lin(X, Y)$.

Let Y be a simplicial space. For any space X there is a natural equivalence of categories (cf. section 2.3.3)

$$\mathrm{Hom}(Sh(X), Sh(Y)) \simeq Lin(X, Y). \tag{4.3.10}$$

Here $\mathrm{Hom}(Sh(X), Sh(Y))$ is the category of geometric morphisms of topoi $Sh(X)$ and $Sh(Y)$, with morphisms being natural transformations. On homotopy classes of topos morphisms we have the natural bijection

$$[Sh(X), Sh(Y)] \cong Lin_c(X, Y). \tag{4.3.11}$$

Let X be a CW-complex and Y be a locally contractible simplicial space. There is a natural bijection between homotopy classes of maps $[X, |Y|]$ and concordance classes $Lin_c(X, Y)$.

5. Results of the thesis

5.1. **Quantum groups.**

5.1.1. *Differential calculi.* In paper [P2], bicovariant differential calculi on quantum groups $\mathrm{Fun}_q(G)$ (2.1.2) are constructed using the F-R-T approach and Woronowicz's theory.

Let us consider the vector corepresentation of $\mathcal{A} = \mathrm{Fun}_q(G)$ given by the matrix of the generators $T = (t_{ij})_{i,j=1}^N$ (2.1.2). We have a natural bicovariant bimodule such that $R = T$ (cf. 4.1.1-4.1.2). It is easily seen that we can choose both $F = S(L^{\pm})^t$ (cf. 4.1.3-4.1.4) in this case. Another natural bicovariant bimodule is given by the choice $R = S(T)^t$. In this case we can take $F = L^{\pm}$. We denote the bicovariant bimodules thus obtained as Γ_1, Γ_2, Γ_1^c and Γ_2^c according to the choices

$$\Gamma_1 : \; R = T, \qquad F = S(L^+)^t, \tag{5.1.1}$$

[11] see, e.g., [8] for the definition of a bicategory

$$\Gamma_2 : R = T, \qquad F = S(L^-)^t, \tag{5.1.2}$$

$$\Gamma_1^c : R = S(T)^t, \qquad F = L^-, \tag{5.1.3}$$

$$\Gamma_2^c : R = S(T)^t, \qquad F = L^+. \tag{5.1.4}$$

Taking the tensor product $\Gamma = \Gamma_1 \otimes \Gamma_1^c$ we get a new bicovariant bimodule. For other choices of tensor products ($\Gamma_1^c \otimes \Gamma_1, \Gamma_2 \otimes \Gamma_2^c$ and $\Gamma_2^c \otimes \Gamma_2$) all that follows is analogous. The choices $\Gamma_1 \otimes \Gamma_2^c, \Gamma_1^c \otimes \Gamma_2, \Gamma_2 \otimes \Gamma_1^c$ and $\Gamma_2^c \otimes \Gamma_1$ are not interesting, since they lead to the trivial differential calculi ($da = 0$, for all $a \in \mathcal{A}$). According to the general theory of Woronowicz, the bicovariant bimodule Γ can be described as follows.

Proposition 5.1. *Let $(\omega_{ij})_{i,j=1}^N$ be the basis for ${}_{inv}\Gamma$. Right multiplication is given in terms of the left one by*

$$\omega_{ij} a = ((id \otimes S(l_{ki}^+) l_{jl}^-)\Delta a)\omega_{kl} \tag{5.1.5}$$

and the right coaction by

$$\delta_R(\omega_{ij}) = \omega_{kl} \otimes t_{ki} S(t_{jl}). \tag{5.1.6}$$

Our choice of the bicovariant bimodule Γ is motivated by the particular form of the coaction (5.1.6). It follows that the linear space ${}_{inv}\Gamma$ contains a bi-invariant element $\tau = \sum \omega_{ii}$, which can be used to define a derivative on $\mathcal{A}$.

Theorem 5.2. *For $a \in \mathcal{A}$, the derivative*

$$da = \tau a - a\tau \tag{5.1.7}$$

defines a first-order bicovariant differential calculus. The first-order calculus (5.1.7) extends to a unique bicovariant exterior differential calculus by setting

$$d\theta = \tau \wedge \theta - (-1)^k \theta \wedge \tau, \tag{5.1.8}$$

where k is the degree of a homogeneous element $\theta \in \Gamma^\wedge$.

Let us mention that the bi-invariance of the one-form τ is essential for the differential calculus based on (5.1.7) to be a bicovariant one. In case of a compact form K, the derivative d will define a "$*$-calculus". In the case of $SU_q(2)$, we get in this way the $4D_+$ calculus of Woronowicz [100]. A more explicit description of the above described differential calculi is given by the following corollary.

Corollary 5.3. *For $a \in \mathcal{A}$, we have*

$$da = ((id \otimes (S(l_{ki}^+) l_{il}^- - \delta_{kl} e)\Delta a)\omega_{kl}. \tag{5.1.9}$$

Let us denote by

$$\chi_{ij} = S(l_{ik}^+) l_{ki}^- - \delta_{ij} e, \tag{5.1.10}$$

or more compactly

$$\chi = S(L^+)L^- - Ie, \tag{5.1.11}$$

the matrix of left-invariant vector fields χ_{ij} on $\mathcal{A}$. The "commutators" (4.1.11) ($[\chi', \chi] = \sum S(\chi_{(1)})\chi' \chi_{(2)}$) among the elements χ_{ij} of the basis dual to the ω_{ij} can now be obtained directly from relations (2.1.4) between the functionals $l_{ij}^\pm$ or from the fact that $d^2 a = 0$ for any $a \in \mathcal{A}$. With the notation $\lambda_{ijkl,mnop} = (S(l_{oi}^+) l_{jp}^-)(t_{mk} S(t_{ln}))$, we have

$$[\chi_{ij}, \chi_{kl}] = \chi_{ij}\chi_{kl} - \lambda_{mnop,ijkl}\chi_{mn}\chi_{op} = -\delta_{kl}\chi_{ij} + \lambda_{ssmn,ijkl}\chi_{mn}. \tag{5.1.12}$$

In a more compact notation using the R-matrix $\mathcal{R}$

$$\mathcal{R}_{21}^{-1}\chi_1\mathcal{R}_{12}^{-1}\chi_2 - \chi_2\mathcal{R}_{21}^{-1}\chi_1\mathcal{R}_{12}^{-1} = \chi_2\mathcal{R}_{21}^{-1}\mathcal{R}_{12}^{-1} - \mathcal{R}_{21}^{-1}\mathcal{R}_{12}^{-1}\chi_2. \tag{5.1.13}$$

Now we can describe a direct relation between the quantum group generated by $(\chi_{ij})_{i,j=1}^N$ and the quantized enveloping algebra $\mathcal{U}_h(\mathrm{g})$ of [103] generated by functionals $(l_{ij}^\pm)_{i,j=1}^N$. Let us introduce matrix $L = (l_{ij})_{i,j=1}^N$, $L = S(L^+)L^-$. The upper and lower triangular matrices $S(L^+)$ and L^- can be constructed from L by its decomposition into triangular parts. In this sense, the quantum group generated by χ's and the quantized enveloping $\mathcal{U}_h(\mathrm{g})$ of FRT are equivalent. Let us now discuss very briefly the classical limit. We have $\mathcal{R} = 1 + \hbar r + \ldots$, where $q = e^\hbar$ and r is the corresponding classical r-matrix, $L^\pm = 1 + \hbar\eta^\pm + \ldots$, with $\eta^\pm$ matrices of generators of the corresponding Lie algebra g. The matrix elements of $\chi = (\eta^- - \eta^+)$ are no longer linearly independent in this limit, and the linear space spanned by these is just the Lie algebra g. As a result, the classical differential calculus on the group G is obtained as a quotient. Here we should mention that closely related results were reported independently in [37, 10]. Concerning related work coauthored by the author of the thesis: In [44] vector fields on quantum doubles (called in this paper "complex quantum groups" [27, 26]), was constructed. A short survey on differential calculi on quantum groups up to 1994 can be found in [65].

5.1.2. *Dressing orbits, coherent states and representations.* In paper [P3], the orbits of the left dressing transformation Δ_L (2.1.26) on the compact simple Lie group K are described.

Let W be the corresponding Weyl group and $\mathbb{T}^r \subset K$ is the maximal torus (i.e., $r = \operatorname{rank} K$). The classical dressing orbits as well as the irreducible $*$-representations of $\mathrm{Fun}_q(K)$ are in a one to one correspondence with elements in $W \times \mathbb{T}^r$ [111, 112, 127], from where it is also known that the description of representations of $\mathrm{Fun}_q(K)$ can be reduced to the description of those of $\mathrm{Fun}_q(SU(2))$. In this case, we can arrange the the generators of $\mathrm{Fun}_q(SU(2))$ and its dual $\mathrm{Fun}_q(AN)$ in the respective 2×2 matrices

$$U = \begin{pmatrix} a & b \\ -q^{-1}b^* & a^* \end{pmatrix}, \quad \text{and} \quad \Lambda = \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix}. \tag{5.1.14}$$

Dressing orbit corresponding to an element of the form $(1, t) \in W \times \mathbb{T}$ is zero-dimensional. The quantization of such an orbit $\mathcal{O}_\tau$ gives trivially $\mathcal{O}_t = \mathbb{C}$ and $\Delta_L(1) = 1 \otimes 1$. The corresponding one-dimensional representation of $\mathrm{Fun}_q(SU(2))$ is explicitly given by an algebra homomorphism $\psi : \mathrm{Fun}_q(SU(2)) \to \mathcal{O}_t$, given on generators by $a \mapsto t$, $b \mapsto 0$. The result of the quantization is, of course, the one-dimensional representation (2.1.17).

The quantization of the dressing orbit (with the complex dimension 1) corresponding to the nontrivial Weyl element $w \in W$ is less trivial and is one of the results of [P3]. We will summarize it in the following theorem.

Theorem 5.4. *The (quantized) orbit $\mathcal{O}_w$ corresponding to the nontrivial Weyl element w is generated by elements z and z^* fulfilling relations*

$$1 + zz^* = q^{-2}(1 + z^* z). \tag{5.1.15}$$

The dressing transformation on the orbit $\mathcal{O}_w$ is given explicitly as

$$\Delta_L(z) = \alpha^2 \otimes z - \beta\alpha \otimes 1. \tag{5.1.16}$$

The homomorphism $\psi : \mathrm{Fun}_q(SU(2)) \to \mathcal{O}_w$ defining the corresponding representation (2.1.19) is given by

$$a \mapsto (1 + zz^*)^{-1/2} z, \quad \text{and} \quad b \mapsto (1 + zz^*)^{-1/2}. \tag{5.1.17}$$

Moreover, the representation Hilbert space $\mathcal{H}$ can be described as a completion with respect to an appropriate scalar product of the space of complex polynomials $\mathcal{O}_a = \mathbb{C}[z^*]$ in the variable z^*. Hence, the result of quantization is equivalent to the infinite-dimensional representation (2.1.19). The paper [P3] also contains, among other results, a construction, in terms of the universal element ρ, of the quantum Weyl element introduced in [73, 80] and an interpretation of the above described representations as Berezin's quantizations of the corresponding classical dressing orbits. Further results based on [P3] are given in [114].

The paper [P4] is concerned with the dual situation. Thus, it deals with the right dressing transformations Δ_R (2.1.25) and describes a quantization of a the big cell for a coadjoint orbit (which is identified as a dressing orbit) of a compact form K of a simple Lie group G belonging to the one of the classical series A_r, B_r, C_r and D_r. The idea was to introduce a proper notion of a coherent state in the sense of Perelomov [97], since the classical construction of coherent states can be viewed as an inverse procedure to the method of orbits. The definition of the coherent state in paper [P4] is a nontrivial generalization of the definition given in [P1] in the case of A_1.

Let us discuss some of the results of [P4] in more detail. A subset Π_0 of the set of simple roots $\Pi = \{\alpha_1, \ldots, \alpha_r\}$ of g defines a Lie subalgebra $\mathfrak{k}_0$ of $\mathfrak{k}$ and a Lie subalgebra $\mathfrak{p}_0$ of g. For instance, $\Pi_0 = \emptyset$ in the generic case of a flag manifold. Let us denote the corresponding subgroups as $K_0 \subset K$ and $P_0 \subset G$. The coadjoint orbit is determined by a subset Π_0 and holomorphic coordinate functions on the big cell are introduced via a quantum analogue of the Gauss decomposition. Hence, the generators of $\mathcal{C}_a$ form a block upper triangular matrix Z with unit blocks on the diagonal, the structure of the blocks being unambiguously determined by the set Π_0. In addition to the R-matrix $\mathcal{R}$, a further matrix $\mathcal{Q}$ can be defined by setting some of the entries of $\mathcal{R}$ to zero. Again, the structure of $\mathcal{Q}$ is completely determined by Π_0. We have the following proposition.

Proposition 5.5. *The elements of Z fulfill the relations*

$$\mathcal{R}\, \mathcal{Q}_{12}^{-1} Z_1 \mathcal{Q}_{12} Z_2 = \mathcal{Q}_{21}^{-1} Z_2 \mathcal{Q}_{21} Z_1 \mathcal{R}\,. \tag{5.1.18}$$

Also, there are additional relation originating from the respective unimodularity and orthogonality relations on the respective matrices T.

In the generic case of the flag manifold, Z is upper triangular with units on the diagonal and $\mathcal{Q}$ is the diagonal $\mathcal{Q} = \operatorname{diag}(\mathcal{R})$.

The basic role in [P4] is played by the vacuum functional $\langle\cdot\rangle$ and the quantum coherent state Γ. The vacuum functional $\langle\cdot\rangle$, is a linear functional on $\mathcal{U}_h(\mathrm{k})$ defined by

$$\langle X\rangle := \langle e_\lambda, \tau_\lambda(X)e_\lambda\rangle\,, \tag{5.1.19}$$

where $e_\lambda \in \mathcal{H}_\lambda$ is a normalized lowest weight vector and $\mathcal{H}_\lambda$ is the representation space space of τ_λ. By the duality between $\mathcal{U}_h(\mathrm{k})$ and $\mathrm{Fun}_q(K)$, the vacuum functional can be interpreted as an element in $\mathrm{Fun}_q(K)$ denoted by w_λ and having the following properties:

$$S\,w_\lambda = {w_\lambda}^*,\ \varepsilon(w_\lambda) = 1, \text{ and } \quad w_{\lambda_1+\lambda_2} = w_{\lambda_1}w_{\lambda_2} = w_{\lambda_2}w_{\lambda_1}. \tag{5.1.20}$$

Its image $\chi := p_0(w_\lambda)$ under the restriction morphism $p_0 : \mathcal{A} = \mathrm{Fun}_q(K) \to \mathrm{Fun}_q(K_0)$ gives a unitary character

$$\Delta\chi = \chi\otimes\chi, \quad S\chi = \chi^* = \chi^{-1}. \tag{5.1.21}$$

Now we can define the quantum coherent state Γ, generalizing to the present setting the classical construction of A.M. Perelomov.

Definition 5.6. *Let ρ be the universal element (2.1.22) of the quantum double $\mathcal{D}$ corresponding to K. The quantum coherent state is defined by*

$$\Gamma := (\tau_\lambda\otimes S)\rho(e_\lambda\otimes 1) \in \mathcal{H}_\lambda\otimes Fun_q(K), \tag{5.1.22}$$

Now we can represent $\mathcal{H}_\lambda$ as a subspace $\mathcal{M}_\lambda$ of $\mathcal{C}_a$ by the injective mapping

$$\mathcal{H}_\lambda \ni u \mapsto \psi_u := w_\lambda^{-1}\langle\Gamma, u\rangle \in \mathcal{C}_a. \tag{5.1.23}$$

This observation can be used to introduce on $\mathcal{C}_a$ the structure of a $\mathcal{U}_h(\mathrm{k})$-module. For $f\in\mathcal{C}_a$ and $X\in\mathcal{U}_h(\mathrm{g})$ we do have

$$X\cdot f := {w_\lambda}^{-1}\,\xi_X\cdot(w_\lambda f). \tag{5.1.24}$$

In this equation, ξ_X denotes the dualized right dressing action. We have the following theorem.

Theorem 5.7. *$\mathcal{M}_\lambda$ is a cyclic $\mathcal{U}_h(\mathrm{k})$-submodule in $\mathcal{C}_a$ with the cyclic vector 1.*

Also, this module structure of $\mathcal{M}_\lambda$ can be described in terms of noncommutative Cartan calculus on the "quantum" coadjoint orbit in a form reminiscent of geometric quantization of classical coadjoint orbits [P4]. Finally, let us mention that the paper [P1], dealing with the special case of $\mathcal{U}_h(\mathrm{su}(2))$, was a predecessor of paper [P4]. In [P1], also coherent states for the deformation of the Heisenberg-Weyl algebra (independently on [15], and [51]) and for the discrete series of representation of the noncompact form $\mathcal{U}_h(\mathrm{su}(1,1))$ have been introduced and studied. In view of the above definition 5.6 and theorem 5.7 the results of [P1] concerning $\mathcal{U}_h(\mathrm{su}(2))$ can be seen as a simple corollary. In this case the matrix Z is a 2×2 upper triangular matrix with units on its diagonal, the nonzero matrix element being a complex number $z\in\mathbb{C}$.

Some further development of the results of [P4] by its authors are presented in [118, 119, 120, 121], [62]. Concerning coherent states, the case of quantum $SU(3)$ was discussed independently in [115]. Finally, let us mention that in a related work [68] of the author of the thesis, the quantum double and the Heisenberg double were interpreted within the deformation quantization perspective.

5.2. Noncommutative gauge theories and deformation quantization.

5.2.1. *Seiberg-Witten map from Kontsevisch's formality.* In papers [P5] and [P6], a framework for constructing noncommutative gauge theories based on the Seiberg-Witten map (2.2.9) was developed. Using the formality map of Kontsevich, the Seiberg-Witten map was constructed, in the abelian case, explicitly for an arbitrary Poisson structure. The main idea of these papers is based on the observation that the "semiclassical" version of the Seiberg-Witten map can be understood as a formal version of the well-known Moser's lemma [89] from symplectic geometry and that the Konsevich's formality can be used to "quantize" this.

Let us consider an associative algebra $\mathfrak{A} = (C^\infty(M)[[\hbar]],\star)$ that is a deformation quantization of a Poisson structure θ over some manifold M. For an arbitrary Poisson structure θ a star product $\star$ exists and can be expressed in terms of the Kontsevich formality map [72], so we shall restrict ourselves to this case. Consider an abelian gauge theory on the Poisson manifold M. In the local description, the gauge potential, field strength and infinitesimal gauge transformations are

$$a = a_i dx^i, \qquad f = \frac{1}{2}f_{ij}\,dx^i\wedge dx^j = da, \qquad f_{ij} = \partial_i a_j - \partial_j a_i, \qquad \delta_\lambda a = d\lambda. \tag{5.2.1}$$

We will first construct a semiclassical version of the Seiberg-Witten map, where all star commutators are replaced by Poisson brackets. The construction is essentially a formal generalization of Moser's lemma to Poisson manifolds. Let us consider the nilpotent coboundary operator of the Poisson cohomology (see [38])

$$\mathbf{d}_\theta = -[\,\cdot\,,\,\theta]_{\mathrm{S}}, \tag{5.2.2}$$

where $[\,,\,]_{\mathrm{S}}$ is the Schouten-Nijenhuis bracket and $\theta = \frac{1}{2}\theta^{ij}\partial_i \wedge \partial_j$ is the Poisson bivector. Acting with $\mathbf{d}_\theta$ on a function f gives the Hamiltonian vector field corresponding to f

$$\mathbf{d}_\theta f = \{\,\cdot\,, f\} = \theta^{ij}(\partial_j f)\partial_i. \tag{5.2.3}$$

It is natural to introduce a "vector field"

$$\mathbf{a}_\theta = a_i \mathbf{d}_\theta x^i = \theta^{ji} a_i \partial_j \tag{5.2.4}$$

corresponding to the abelian gauge potential a and a bivector field

$$\mathbf{f}_\theta = \mathbf{d}_\theta \mathbf{a}_\theta = -\frac{1}{2}\theta^{ik} f_{kl}\theta^{lj}\,\partial_i \wedge \partial_j \tag{5.2.5}$$

corresponding to the abelian field strength $f = da$. We have $\mathbf{d}_\theta \mathbf{f}_\theta = 0$, due to $\mathbf{d}_\theta^2 \propto [\theta\,,\,\theta]_{\mathrm{S}} = 0$ (Jacobi identity).

We are now ready to perturb the Poisson structure θ by introducing a one-parameter deformation θ_t with $t \in [0,1]$:[12]

$$\partial_t \theta_t = \mathbf{f}_{\theta_t} \tag{5.2.6}$$

with initial condition $\theta_0 = \theta$. In local coordinates:

$$\partial_t \theta_t^{ij} = -(\theta_t f \theta_t)^{ij}, \qquad \theta_0^{ij} = \theta^{ij}, \tag{5.2.7}$$

with formal solution given by the geometric series

$$\theta_t = \theta - t\theta f\theta + t^2\theta f\theta f\theta - t^3\theta f\theta f\theta f\theta \pm \cdots = \theta\frac{1}{1+tf\theta}, \tag{5.2.8}$$

if f is not explicitly θ-dependent. (The differential equations (5.2.6), (5.2.7) and the rest of the construction do make sense even if f or a are θ-dependent). θ_t is a Poisson tensor for all t because $[\theta_t\,,\,\theta_t]_{\mathrm{S}} = 0$ at $t = 0$ and

$$\partial_t[\theta_t\,,\,\theta_t]_{\mathrm{S}} = -2\mathbf{d}_{\theta_t}\mathbf{f}_{\theta_t} \propto [\theta_t\,,\,\theta_t]_{\mathrm{S}}. \tag{5.2.9}$$

The evolution (5.2.6) of θ_t is generated by $\mathbf{a}_\theta$:

$$\partial_t\theta_t = \mathbf{d}_{\theta_t}\mathbf{a}_{\theta_t} = -[\mathbf{a}_{\theta_t}\,,\,\theta_t]_{\mathrm{S}}. \tag{5.2.10}$$

This Lie derivative can be integrated to a flow

$$\rho_a^* = \exp(\mathbf{a}_{\theta_t} + \partial_t)\exp(-\partial_t)|_{t=0} \tag{5.2.11}$$

that relates the Poisson structures $\theta' = \theta_1$ and $\theta = \theta_0$. We define a semi-classical (semi-noncommutative) generalized gauge potential

$$A_a = \rho_a^* - \mathrm{id}. \tag{5.2.12}$$

Under an infinitesimal gauge transformation $a \mapsto a + d\lambda$, the $\mathbf{a}_\theta$ (5.2.4) changes by a Hamiltonian vector field $\mathbf{d}_\theta\lambda = \theta^{ij}(\partial_j\lambda)\partial_i$:

$$\mathbf{a}_\theta \mapsto \mathbf{a}_\theta + \mathbf{d}_\theta\lambda. \tag{5.2.13}$$

Let us compute the effect of this gauge transformation on the flow (5.2.11).

Proposition 5.8. *Under the infinitesimal gauge transformation $a \mapsto a + d\lambda$, we have to the first order in λ*

$$\rho_{a+d\lambda}^* = (\mathrm{id} + \mathbf{d}_\theta\tilde{\lambda}) \circ \rho_a^*, \quad \textit{i.e.,} \quad \rho_{a+d\lambda}^*(f) = \rho_a^*(f) + \{\rho_a^*(f), \tilde{\lambda}\} \tag{5.2.14}$$

and

$$A_{a+d\lambda} = A_a + \mathbf{d}_\theta\tilde{\lambda} + \{A_a, \tilde{\lambda}\}, \tag{5.2.15}$$

with

$$\tilde{\lambda}(\lambda, a) = \sum_{n=0}^{\infty} \frac{(\mathbf{a}_{\theta_t} + \partial_t)^n(\lambda)}{(n+1)!}\bigg|_{t=0}. \tag{5.2.16}$$

[12] In this notation the equations resemble those of Moser's original lemma, which deals with the symplectic 2-form ω, the inverse of θ (provided it exists). There, e.g., $\partial_t\omega_t = f$ for $\omega_t = \omega + tf$.

Hence, the equations (5.2.16) and (5.2.12) with (5.2.11) are explicit semi-classical versions of the Seiberg-Witten map. The semi-classical (semi-noncommutative) generalized field strength evaluated on two functions (e.g. coordinates) f, g is

$$F_a(f,g) = \{\rho^* f\,,\, \rho^* g\} - \rho^*\{f\,,\, g\} = \rho^*\left(\{f\,,\, g\}' - \{f\,,\, g\}\right). \tag{5.2.17}$$

Abstractly as a 2-cochain:

$$F_a = \rho^* \circ \frac{1}{2}(\theta' - \theta)^{ik}\partial_i \wedge \partial_k = \rho^* \circ \frac{1}{2}(f')_{jl}\theta^{ij}\theta^{kl}\partial_i \wedge \partial_k \tag{5.2.18}$$

with $\theta' f = \theta f'$, or

$$f' = \frac{1}{1 + f\theta} f, \tag{5.2.19}$$

which we recognize as the noncommutative field strength (with lower indices) for constant f, θ [122]. The general result for non-constant f, θ is thus simply obtained by the application of the covariantizing map ρ^* (after raising indices with θ's).

The Seiberg-Witten map in the semiclassical regime for constant θ has previously been discussed in [35, 58], where it was understood as a coordinate redefinition that eliminates fluctuations around a constant background.

We will now use Kontsevich's formality theorem to quantize everything.

The construction mirrors the semiclassical one, the exact correspondence is given by the formality maps U_n (4.2.4) that are skew-symmetric multilinear maps that take n polyvector fields into a polydifferential operator. We start with the differential operator

$$\mathbf{a}_\star = \sum_{n=0}^{\infty} \frac{(i\hbar)^n}{n!} U_{n+1}(\mathbf{a}_\theta, \theta, \ldots, \theta), \tag{5.2.20}$$

which is the image of $\mathbf{a}_\theta$ under the formality map (4.2.9); then we use the coboundary operator $\mathbf{d}_\star$ (of the Hochschild cohomology) to define a bidifferential operator

$$\mathbf{f}_\star = \mathbf{d}_\star \mathbf{a}_\star. \tag{5.2.21}$$

This is the image of $\mathbf{f}_\theta = \mathbf{d}_\theta \mathbf{a}_\theta$ under the formality map:

$$\mathbf{f}_\star = \sum_{n=0}^{\infty} \frac{(i\hbar)^{n+1}}{n!} U_{n+1}(\mathbf{f}_\theta, \theta, \ldots, \theta). \tag{5.2.22}$$

The t-dependent Poisson structure (5.2.6) induces a t-dependent star product via (4.2.7)

$$g \star_t h = \sum_{n=0}^{\infty} \frac{(i\hbar)^n}{n!} U_n(\theta_t, \ldots, \theta_t)(g,h). \tag{5.2.23}$$

The t-derivative of this equation is

$$\partial_t(g \star_t h) = \sum_{n=0}^{\infty} \frac{(i\hbar)^{n+1}}{n!} U_{n+1}(\mathbf{f}_{\theta_t}, \theta_t, \ldots, \theta_t)(g,h), \tag{5.2.24}$$

where we have used (5.2.6) and the skew-symmetry and multi-linearity of U_n. Comparing with (5.2.22) we find

$$\partial_t(g \star_t h) = \mathbf{f}_{\star_t}(g,h), \tag{5.2.25}$$

or, shorter, as an operator equation: $\partial_t(\star_t) = \mathbf{f}_{\star_t}$. But $\mathbf{f}_{\star_t} = \mathbf{d}_{\star_t}\mathbf{a}_{\star_t} = -[\mathbf{a}_{\star_t}\,,\, \star_t]_{\mathrm{G}}$, so the t-evolution is generated by the differential operator $\mathbf{a}_{\star_t}$ and can be integrated to a flow

$$\mathcal{D}_a = \exp(\mathbf{a}_{\star_t} + \partial_t)\exp(-\partial_t)\big|_{t=0}\,, \tag{5.2.26}$$

that relates the star products $\star' = \star_1$ and $\star = \star_0$, and that defines the generalized noncommutative gauge potential

$$\mathcal{A}_a = \mathcal{D}_a - \mathrm{id}. \tag{5.2.27}$$

The transformation of $\mathbf{a}_\star$ under an infinitesimal gauge transformation $a \mapsto a + d\lambda$ can be computed from (5.2.13) with the help of (4.2.13):

$$\mathbf{a}_\star \mapsto \mathbf{a}_\star + \frac{1}{i\hbar}\mathbf{d}_\star\hat{\lambda}. \tag{5.2.28}$$

We have the following theorem, the deformation quantization version of the above proposition 5.8.

Theorem 5.9. *Under the infinitesimal gauge transformation $a \mapsto a + d\lambda$, we have the following transformation of the quantum flow and the noncommutative gauge potential*

$$\mathcal{D}_{a+d\lambda} = (\mathrm{id} + \frac{1}{i\hbar}\mathbf{d}_\star\hat{\Lambda}) \circ \mathcal{D}_a, \quad i.e., \quad \mathcal{D}_{a+d\lambda}(f) = \mathcal{D}_a f + \frac{i}{\hbar}[\hat{\Lambda} \stackrel{\star}{,} \mathcal{D}_a f] \tag{5.2.29}$$

$$\mathcal{A}_{a+d\lambda} = \mathcal{A}_a + \frac{1}{i\hbar}\left(\mathbf{d}_\star\hat{\Lambda} - \hat{\Lambda} \star \mathcal{A} + \mathcal{A} \star \hat{\Lambda}\right), \tag{5.2.30}$$

with

$$\hat{\Lambda}(\lambda, a) = \sum_{n=0}^{\infty} \frac{(\mathbf{a}_{\star_t} + \partial_t)^n(\hat{\lambda})}{(n+1)!}\bigg|_{t=0}. \tag{5.2.31}$$

Hence, the equations (5.2.27) with (5.2.26) and (5.2.31) are explicit versions of the abelian Seiberg-Witten map to all orders in $\hbar$. They are unique up to (noncommutative) gauge transformations. Perhaps more importantly, this construction provides us with an explicit version of the "covariantizer" $\mathcal{D}_a$ (the equivalence map that sends coordinates and functions to their covariant analogs) in terms of a finite number of (classical) fields a_i. The noncommutative gauge parameter (5.2.31) also satisfies the consistency condition

$$\delta_\alpha\hat{\Lambda}(\beta, a) - \delta_\beta\hat{\Lambda}(\alpha, a) = \frac{i}{\hbar}[\hat{\Lambda}(\alpha, a) \stackrel{\star}{,} \hat{\Lambda}(\beta, a)], \tag{5.2.32}$$

with $\delta_\alpha(a_i) = \partial_i\alpha$, $\delta_\alpha(\beta) = 0$, that follows from computing the commutator of abelian gauge transformations on a covariant field [64].

The generalized noncommutative field strength evaluated on two functions (or coordinates) f, g is

$$\mathcal{F}_a(f, g) = \mathcal{D}_a\left([f \stackrel{\star'}{,} g] - [f \stackrel{\star}{,} g]\right). \tag{5.2.33}$$

Up to order θ^2 the series for $\mathcal{A}_a$ and Λ agree with the semiclassical results. In components:

$$\mathcal{A}_a(x^i) = \theta^{ij}a_j + \frac{1}{2}\theta^{kl}a_l(\partial_k(\theta^{ij}a_j) - \theta^{ij}f_{jk}) + \ldots, \tag{5.2.34}$$

$$\hat{\Lambda} = \lambda + \frac{1}{2}\theta^{ij}a_j\partial_i\lambda + \frac{1}{6}\theta^{kl}a_l(\partial_k(\theta^{ij}a_j\partial_i\lambda) - \theta^{ij}f_{jk}\partial_i\lambda) + \ldots. \tag{5.2.35}$$

In paper [P6], also a possible extension to the case of nonabelian noncommutative gauge theories was attempted by applying the above described construction to the abelian gauge field on the product Poisson manifold ($M \times \mathrm{g}^*, \theta \times \theta_{\mathrm{g}^*}$, with the dual g^* to the Lie algebra g equipped with its standard Poisson structure. As an application, the properties the Dirac-Born-Infeld action were discussed. More physics-oriented papers closely related to the subject of the thesis - and coauthored by the author of the thesis - are [2, 32, 63, 61, 64].

5.2.2. *Noncommutative line bundles and Morita equivalence of star products.* In paper [P7], some global properties of abelian noncommutative gauge theories constructed above have been described. The idea was to identify the consistency condition (5.2.32) as an infinitesimal noncommutative 1-cocycle. This leads to the following construction.

Let (M, θ) be a general Poisson manifold, and let $\star$ be the Kontsevich's deformation quantization of the Poisson tensor θ. Further, let us consider a good covering $\{U^i\}$ of M. In this situation a noncommutative line bundle $\mathcal{L}$ can be defined by a collection of $\mathbb{C}[[\hbar]]$-valued local transition functions $G^{ij} \in C^\infty(U^i \cap U^j)[[\hbar]]$ (that can be thought valued in the enveloping algebra of $U(1)$, see [64]), and a collection of maps $\mathcal{D}^i : C^\infty(U^i)[[\hbar]] \to C^\infty(U^i)[[\hbar]]$, formal power series in $\hbar$, starting with the identity, and with coefficients being differential operators, such that

$$G^{ij} \star G^{jk} = G^{ik} \tag{5.2.36}$$

on $U^i \cap U^j \cap U^k$, $G^{ii} = 1$ on U^i, and

$$\mathrm{Ad}_\star G^{ij} = \mathcal{D}^i \circ (\mathcal{D}^j)^{-1} \tag{5.2.37}$$

on $U^i \cap U^j$ or, equivalently, $\mathcal{D}^i(f) \star G^{ij} = G^{ij} \star \mathcal{D}^j(f)$ for all $f \in C^\infty(U^i \cap U^j)[[\hbar]]$. The transitions functions G_{ij} give a finite version of the infinitesimal noncommutative 1-cycle (5.2.32). With this definition, the local maps $\mathcal{D}^i$ can be used to define *globally* a new star product $\star'$ (because the inner automorphisms $\mathrm{Ad}_\star G^{ij}$ do not affect $\star'$)

$$\mathcal{D}^i(f \star' g) = \mathcal{D}^i f \star \mathcal{D}^i g\,. \tag{5.2.38}$$

We say that two line bundles $\mathcal{L}_1 = \{G_1^{ij}, \mathcal{D}_1^i, \star\}$ and $\mathcal{L}_2 = \{G_2^{ij}, \mathcal{D}_2^i, \star\}$ are equivalent if there exists a collection of invertible local functions $H^i \in C^\infty(U^i)[[\hbar]]$ such that

$$G_1^{ij} = H^i \star G_2^{ij} \star (H^j)^{-1} \tag{5.2.39}$$

and

$$\mathcal{D}_1^i = \mathrm{Ad}_\star H^i \circ \mathcal{D}_2^i \; . \tag{5.2.40}$$

The tensor product of two line bundles $\mathcal{L}_1 = \{G_1^{ij}, \mathcal{D}_1^i, \star_1\}$ and $\mathcal{L}_2 = \{G_2^{ij}, \mathcal{D}_2^i, \star_2\}$ is well defined if $\star_2 = \star_1'$ (or $\star_1 = \star_2'$.) Then the corresponding tensor product is a line bundle $\mathcal{L}_2 \otimes \mathcal{L}_1 = \mathcal{L}_{21} = \{G_{12}^{ij}, \mathcal{D}_{12}^{ij}, \star_1\}$ defined as

$$G_{12}^{ij} = \mathcal{D}_1^i(G_2^{ij}) \star_1 G_1^{ij} = G_1^{ij} \star_1 \mathcal{D}_1^j(G_2^{ij}) \tag{5.2.41}$$

and

$$\mathcal{D}_{12}^i = \mathcal{D}_1^i \circ \mathcal{D}_2^i \; . \tag{5.2.42}$$

The order of indices of $\mathcal{L}_{21}$ indicates the bimodule structure of the corresponding space of sections to be defined later, whereas the first index on the G_{12}'s and $\mathcal{D}_{12}$'s indicates the star product (here: $\star_1$) by which the objects multiply.

A section $\Psi = (\Psi^i)$ is a collection of functions $\Psi^i \in C^\infty_\mathbb{C}(U^i)[[\hbar]]$ satisfying consistency relations

$$\Psi^i = G^{ij} \star \Psi^i \tag{5.2.43}$$

on all intersections $U^i \cap U^j$. With this definition, the space of sections $\mathcal{E}$ is a right $\mathcal{A}_x = (C^\infty(M)[[\hbar]], \star)$ module. We shall use the notation $\mathcal{E}_{\mathcal{A}_x}$ for it. The right action of the function $f \in \mathcal{A}_x$ is the regular one

$$\Psi . f = (\Psi^k \star f) \; . \tag{5.2.44}$$

Using the maps $\mathcal{D}^i$, it is easy to turn $\mathcal{E}$ also into a left $\mathcal{A}_x{}' = (C^\infty(M)[[\hbar]], \star')$ module ${}_{\mathcal{A}_x{}'}\mathcal{E}$. The left action of $\mathcal{A}_x{}'$ is given by

$$f . \Psi = (\mathcal{D}^i(f) \star \Psi^i) \; . \tag{5.2.45}$$

It is easy to check, using (5.2.37), that the left action (5.2.45) is compatible with (5.2.43). From the property (5.2.38) of the maps $\mathcal{D}^i$, we find

$$f.(g.\Psi) = (f \star' g).\Psi \; . \tag{5.2.46}$$

Together we have a bimodule structure ${}_{\mathcal{A}_x{}'}\mathcal{E}_{\mathcal{A}_x}$ on the space of sections.

There is an obvious way of tensoring sections. The section

$$\Psi_{12}^i = \mathcal{D}_1^i(\Psi_2^i) \star_1 \Psi_1^i \tag{5.2.47}$$

is a section of the tensor product line bundle (5.2.41), (5.2.42). Tensoring of line bundles naturally corresponds to tensoring of bimodules.

We have the following proposition.

Proposition 5.10. *Let M be a compact manifold. Then the space of sections $\mathcal{E}$ as a right $\mathcal{A}_x$-module is projective of finite type. The same holds if $\mathcal{E}$ is considered as a left $\mathcal{A}_x'$ module. The two algebras $\mathcal{A}_x$ and $\mathcal{A}_x'$ are Morita equivalent.*

Concerning the converse, let $L \in \mathrm{Pic}(M) \cong H^2(M, \mathbb{Z})$ be a (complex) line bundle on M (compact) and c its Chern class. Let F be a curvature two form on M whose cohomology class $[F]$ is (the image in $\mathbb{R}$ of) the Chern class c. Consider the formal Poisson structure θ' given by the geometric series

$$\theta' = \theta(1 + \hbar F \theta)^{-1}. \tag{5.2.48}$$

In this formula θ and F are understood as maps $\theta : T^*M \to TM$, $F : TM \to T^*M$ and θ' is the result of the indicated map compositions. With these assumptions we have the following theorem.

Theorem 5.11. *Any star product $\star'$ on M, which is Morita equivalent to the star product $\star$ quantizing the Poisson structure θ, must be (up to a global isomorphism) the deformation quantization of θ' corresponding to a $c \in H^2(M, \mathbb{Z})$.*

This construction depends only on the integer cohomology class c, indeed if c is the trivial class then $F = da$ and the corresponding quantum line bundle is trivial, i.e.,

$$G^{ij} = (H^i)^{-1} \star H^j \; . \tag{5.2.49}$$

In this case the linear map

$$\mathcal{D} = \mathrm{Ad}_\star H^i \circ \mathcal{D}^i \tag{5.2.50}$$

defines a global equivalence (a stronger notion than Morita equivalence) of $\star$ and $\star'$.

We should mention that our results concerning deformation quantization of line bundles and Morita equivalence were particularly inspired by [23] and [24]. The first paper describes Morita equivalence in the semiclassical limit and in the second paper an independent approach to deformation quantization of vector bundles is developed. In the paper [1] coauthored by the author of the thesis, abelian gerbes were quantized using the above described noncommutative line bundles and applied to deformation quantization of twisted Poisson structures of [76, 95, 107]. Deformation quantization of abelian bundle leads to honest nonabelian 2-cocycles of the kind that we will describe in the next sections devoted to higher gauge theories.

5.3. **Higher gauge theory.**

5.3.1. *Nonabelian bundle gerbes.* In paper [P9], the main goal was to generalize the theory of abelian bundle gerbes and their differential geometry, due to Murray [92], to the nonabelian case. Hence, in contrary to the previous approaches to nonabelian gerbes (e.g., [50, 18, 19, 17]), our study was from the differential geometry viewpoint. We believe that it is primarily in this context that nonabelian gerbes structures can appear and can be recognized in physics. It is for example in this context that one would like to have a formulation of Yang-Mills theory with higher forms. The idea followed in [P9] was to replace the "transition" line bundles $\mathcal{L}$ by G-principal bundles with additional structure, which would allow to multiply them.

Let $(G \xrightarrow{\partial} D)$ be a crossed module of Lie groups and X a manifold. Let $P \to X$ be a left principal G-bundle, such that the principal D-bundle $D \times_\partial P$ is trivial with a trivialization defined by a section (i.e., a left G-equivariant smooth map) $\mathfrak{d} : P \to D$. We call the couple $(P, \mathfrak{d})$ a $(G \to D)$-bundle. Two $(G \to D)$-bundles $(P, \mathfrak{d})$ and $(P', \mathfrak{d}')$ over X are isomorphic if they are isomorphic as left G-bundles by an isomorphism $\ell : P \to P'$ and $\mathfrak{d}'\ell = \mathfrak{d}$. Obviously, a pullback of a $(G \to D)$-bundle is again a $(G \to D)$-bundle.

The $(G \to D)$-bundle $(P, \mathfrak{d})$ is also a right principal G-bundle with the right action of G given by $p.l = {}^{\mathfrak{d}(p)}(g).p$ for $p \in P, l \in G$. The left and right actions commute, hence, P has naturally the structure if a principal G-bibundle [49, 50, 18]. The section $\mathfrak{d}$ is G-biequivariant. Let $(P, \mathfrak{d})$ and $(\tilde{P}, \tilde{\mathfrak{d}})$ are two $(G \to D)$-bundles over X. Let us define an equivalence relation on the Whitney sum $P \oplus \tilde{P} = P \times_X \tilde{P}$ by $(pl, \tilde{p}) \sim (p, g\tilde{p})$, for $(p, \tilde{p}) \in P \oplus \tilde{P}$ and $g \in G$. Then $(P\tilde{P} := (P \oplus \tilde{P})/\sim, \mathfrak{d}\tilde{\mathfrak{d}})$ with $\mathfrak{d}\tilde{\mathfrak{d}}([p, \tilde{p}]) := \mathfrak{d}(p)\tilde{\mathfrak{d}}(\tilde{p})$ is a $(G \to D)$-bundle.

Let Y be a manifold. Consider a surjective submersion $\wp : Y \to X$, which in particular admits local sections. Let $\{O_i\}$ be the corresponding covering of X with local sections $\sigma_i : O_i \to Y$, i.e., $\wp\sigma_i = id$. We also consider $Y^{[n]} = Y \times_X Y \times_X Y \ldots \times_X Y$, the n-fold fibre product of Y, i.e., $Y^{[n]} := \{(y_1, \ldots y_n) \in Y^n \mid \wp(y_1) = \wp(y_2) = \ldots \wp(y_n)\}$. Given a $(G \to D)$-bundle $\mathcal{P} = (P, \mathfrak{d})$ over $Y^{[2]}$ we denote by $\mathcal{P}_{12} = p_{12}^*(\mathcal{P})$ the crossed module bundle on $Y^{[3]}$ obtained as a pullback of $\mathcal{P}$ under $p_{12} : Y^{[3]} \to Y^{[2]}$ (p_{12} is the identity on its first two arguments); similarly for $\mathcal{P}_{13}$ and $\mathcal{P}_{23}$. Consider a quadruple $(\mathcal{P}, Y, X, \ell)$, where $\mathcal{P} = (P, \mathfrak{d})$ is a crossed module bundle, $Y \to X$ a surjective submersion and ℓ an isomorphism of crossed module bundles $\ell : \mathcal{P}_{12}\mathcal{P}_{23} \to \mathcal{P}_{13}$. We now consider bundles $\mathcal{P}_{12}$, $\mathcal{P}_{23}$, $\mathcal{P}_{13}$, $\mathcal{P}_{24}$, $\mathcal{P}_{34}$, $\mathcal{P}_{14}$ on $Y^{[4]}$ relative to the projections $p_{12} : Y^{[4]} \to Y^{[2]}$ etc. and also the crossed module isomorphisms ℓ_{123}, ℓ_{124}, ℓ_{123}, ℓ_{234} induced by projections $p_{123} : Y^{[4]} \to Y^{[3]}$ etc. Now we can define a $(G \to D)$-bundle gerbe for a general crossed module of Lie groups.

Definition 5.12. *The quadruple $(\mathcal{P}, Y, X, \ell)$, where $Y \to X$ is a surjective submersion, $\mathcal{P}$ is a crossed module bundle over $Y^{[2]}$, and $\ell : \mathcal{P}_{12}\mathcal{P}_{23} \to \mathcal{P}_{13}$ an isomorphism of crossed module bundles over $Y^{[3]}$, is called a crossed module bundle gerbe if ℓ satisfies the cocycle condition (associativity) on $Y^{[4]}$*

$$\begin{array}{ccc} \mathcal{P}_{12}\mathcal{P}_{23}\mathcal{P}_{34} & \xrightarrow{\ell_{234}} & \mathcal{P}_{12}\mathcal{P}_{24} \\ {\scriptstyle \ell_{123}}\big\downarrow & & \big\downarrow{\scriptstyle \ell_{124}} \\ \mathcal{P}_{13}\mathcal{P}_{34} & \xrightarrow{\ell_{134}} & \mathcal{P}_{14}\,. \end{array} \tag{5.3.1}$$

Abelian bundle gerbes as introduced in [92], [93] are $(U(1) \to 1)$-bundle gerbes. More generally, if $A \to 1$ is a crossed module then A is necessarily an abelian group and an abelian bundle gerbe can be identified as an $(A \to 1)$-bundle gerbe.

A $(1 \to G)$-bundle gerbe is the same thing as a G-valued function g on $Y^{[2]}$ satisfying on $Y^{[3]}$ the cocycle relation $g_{12}g_{23} = g_{23}$ and hence, a principal G-bundle on X (more precisely, a descent datum of a principal G-bundle).

The stable isomorphism of two $(G \to D)$-bundle gerbes is defined as follows.

Definition 5.13. *Two crossed module bundle gerbes $(\mathcal{P}, Y, X, \ell)$ and $(\mathcal{P}', Y', X, \ell')$ are stably isomorphic if there exists a crossed module bundle $\mathcal{Q} \to \bar{Y} = Y \times_X Y'$ such that over $\bar{Y}^{[2]}$ the crossed module bundles $q^*\mathcal{P}$ and $\mathcal{Q}_1 q'^*\mathcal{P}'\mathcal{Q}_2^{-1}$ are isomorphic. The corresponding isomorphism $\tilde{\ell} : q^*\mathcal{P} \to \mathcal{Q}_1 q'^*\mathcal{P}'\mathcal{Q}_2^{-1}$ should satisfy on $\bar{Y}^{[3]}$ (with an obvious abuse of notation) the condition*

$$\tilde{\ell}_{13}\ell = \ell'\tilde{\ell}_{23}\tilde{\ell}_{12}\,. \tag{5.3.2}$$

In the above definition, q and q' are projections onto first and second factor of $\bar{Y} = Y \times_X Y'$ and $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are the pullbacks of $\mathcal{Q} \to \bar{Y}$ to $\bar{Y}^{[2]}$ under respective projections form $\bar{Y}^{[2]}$ to $\bar{Y}$ etc.

Locally, bundle gerbes can be described in terms of 2-cocycles as follows. First, let us notice that the trivializing cover $\{O_i\}$ of the map $\wp : Y \to X$ defines a new surjective submersion $\wp' : Y' = \coprod O_i \to X$. The local sections of $Y \to X$ define a map $f : Y' \to Y$, which is compatible with the maps $\wp$ and $\wp'$, i.e., such that $\wp f = \wp'$. We know that crossed module bundle gerbes $\mathcal{G}_f$ and $\mathcal{G}$ are stably isomorphic. Hence, we can again assume $Y = \coprod O_i$. For simplicity, we assume that the covering $\{O_i\}$ is a good one. We have the following proposition.

Proposition 5.14. *A crossed module bundle gerbe can be locally described by a 2-cocycle* (d_{ij}, g_{ijk})*, where the smooth maps* $d_{ij} : O_{ij} \to D$ *and* $g_{ijk} : O_{ijk} \to G$ *fulfill the following conditions:*

$$d_{ij}d_{jk} = \partial(g_{ijk})d_{ik} \quad \text{on} \quad O_{ijk} \tag{5.3.3}$$

and

$$g_{ijk}g_{ikl} = {}^{d_{ij}}g_{jkl}g_{ijl} \quad \text{on} \quad O_{ijkl}\,. \tag{5.3.4}$$

2-cocycles (d_{ij}, g_{ijk}) *and* (d'_{ij}, g'_{ijk}) *corresponding to stably isomorphic* $(G \to D)$*-bundle gerbes are related by*

$$d'_{ij} = d_i\partial(g_{ij})d_{ij}d_j^{-1} \tag{5.3.5}$$

$${}^{d_i^{-1}}g'_{ijk} = g_{ij}\,{}^{d_{ij}}g_{jk}g_{ijk}g_{ik}^{-1}, \tag{5.3.6}$$

with smooth functions $g_{ij} : O_{ij} \to G$ *and* $d_i : O_i \to D$*.*

These are, of course, the well-known formulas from non-abelian cohomology theory, for nonabelian 2-cocycles (see, e.g. [18, 19, 87]). Also, as shown in [P9], the following theorem holds true.

Theorem 5.15. *Stable isomorphism classes of crossed module bundle gerbes are one to one with stable isomorphism classes of 2-cocycles.*

Let us mention, that it is possible to develop the theory of connections and curvings on nonabelian bundle gerbes in the framework described in this section. This is also one of the results of paper [P9]. The construction is rather involved, since in contrary to the abelian case it is not enough to consider ordinary connections of the special G-principal bundles P over $Y^{[2]}$, which enter the definition of the nonabelian bundle gerbe. However, as introduced and studied in [P9], nonabelian bundle gerbes, connections and curvings are very natural concepts also in classical differential geometry. We will give the corresponding local description in the next section. We just mention the following result of [P9], which can be proved using the partition of unity.

Theorem 5.16. *On each crossed module bundle gerbe there exist a connection and a curving.*

5.3.2. *Twisting nonabelian gerbes.* In paper [P8], we developed, based on [P9] the theory of twisted nonabelian gerbes on the level of cocycles. Our aim was to generalize the twisted principal bundles and also to generalize the "inflow" mechanism, as described above for D-branes, to the case of M5-branes. The notion of a twisted 1-gerbe (2-gerbe module) can be introduced performing a similar construction as in the case of a twisted principal bundle (2.3.11). For concreteness, we assume the crossed module to be the one of the form $G \to Aut(G)$. While twisted nonabelian bundles are described by nonabelian transition functions $\{G_{ij}\}$, twisted nonabelian gerbes are described by transition functions $\{f_{ijk}, \varphi_{ij}\}$ that are respectively valued in G and in $Aut(G)$, $f_{ijk} : O_{ijk} \to G$, $\varphi_{ij} : O_{ij} \to Aut(G)$, and where the action of φ_{ij} on $U(1)$ is trivial: $\varphi_{ij}|_{U(1)} = id$. The twisted cocycle relations now read

$$\lambda_{ijkl} = f_{ikl}^{-1}f_{ijk}^{-1}\varphi_{ij}(f_{jkl})f_{ijl}\,, \tag{5.3.7}$$

$$\varphi_{ij}\varphi_{jk} = Ad_{f_{ijk}}\varphi_{ik}\ , \tag{5.3.8}$$

where $\{\lambda_{ijkl}\}$ is $U(1)$-valued. It is not difficult to check that $\{\lambda_{ijkl}\}$ is a Čech 3-cocycle and thus defines a 2-gerbe (without curvings). In the particular case $\lambda_{ijkl} = 1$ equations (5.3.7), (5.3.8) define a nonabelian 1-gerbe (without curvings).

One can also consider twisted gerbes with connection 1-forms: $(f_{ijk}, \varphi_{ij}, a_{ij}, \mathcal{A}_i)$ where $a_{ij} \in \mathrm{Lie}(G)\otimes\Omega^1(O_{ij})$, $\mathcal{A}_i \in \mathrm{Lie}(Aut(G)) \otimes \Omega^1(O_i)$, and twisted gerbes with curvings:

$$(f_{ijk}, \varphi_{ij}, a_{ij}, \mathcal{A}_i, B_i, d_{ij}, H_i) \tag{5.3.9}$$

where B_i, d_{ij} are 2-forms and H_i 3-forms, all of them $\mathrm{Lie}(G)$-valued; $B_i \in \mathrm{Lie}(G)\otimes\Omega^2(O_i)$, $d_{ij} \in \mathrm{Lie}(G)\otimes\Omega^2(O_{ij})$, $H_i \in \mathrm{Lie}(G) \otimes \Omega^3(O_i)$. Before defining a twisted 1-gerbe we need to introduce some more notation. Given an element $X \in \mathrm{Lie}(Aut(G))$, we can construct a map (a 1-cocycle) $T_X : G \to \mathrm{Lie}(G)$ in the following way,

$$T_X(h) \equiv [he^{tX}(h^{-1})]\ , \tag{5.3.10}$$

where $[he^{tX}(h^{-1})]$ is the tangent vector to the curve $he^{tX}(h^{-1})$ at the point 1_G. Given a Lie$(Aut(G))$-valued form $\mathcal{A}$, we write $\mathcal{A} = \mathcal{A}^\rho X^\rho$ where $\{X^\rho\}$ is a basis of Lie$(Aut(G))$. We then define $T_\mathcal{A}$ as

$$T_\mathcal{A} \equiv \mathcal{A}^\rho T_{X^\rho} \; . \tag{5.3.11}$$

We use the same notation $T_\mathcal{A}$ for the induced map on Lie(G). Now we extend this map to allow $T_\mathcal{A}$ to act on a Lie(G)-valued form $\eta = \eta^\alpha Y^\alpha$, where $\{Y^\alpha\}$ is a basis of Lie(G), by $T_\mathcal{A}(\eta) = \eta^\alpha \wedge T_\mathcal{A}(Y^\alpha)$. Also, we define

$$\mathcal{K}_i \equiv d\mathcal{A}_i + \mathcal{A}_i \wedge \mathcal{A}_i \, , \tag{5.3.12}$$

$$k_{ij} \equiv da_{ij} + a_{ij} \wedge a_{ij} + T_{\mathcal{A}_i}(a_{ij}) \, . \tag{5.3.13}$$

Definition 5.17. *A twisted 1-gerbe is a set* $(f_{ijk}, \varphi_{ij}, a_{ij}, \mathcal{A}_i, B_i, d_{ij}, H_i)$ *such that,* $\varphi_{ij}|_{U(1)} = id$, $T_{\mathcal{A}_i}|_{U(1)} = 0$,

$$\varphi_{ij}\varphi_{jk} = Ad_{f_{ijk}} \varphi_{ik} \, , \tag{5.3.14}$$

$$\mathcal{A}_i + ad_{a_{ij}} = \varphi_{ij} \mathcal{A}_j \varphi_{ij}^{-1} + \varphi_{ij} d\varphi_{ij}^{-1} \, , \tag{5.3.15}$$

$$d_{ij} + \varphi_{ij}(d_{jk}) = f_{ijk} d_{ik} f_{ijk}^{-1} + T_{\mathcal{K}_i + ad_{B_i}}(f_{ijk}) \, , \tag{5.3.16}$$

$$\varphi_{ij}(H_j) = H_i + d\, d_{ij} + [a_{ij}, d_{ij}] + T_{\mathcal{K}_i + ad_{B_i}}(a_{ij}) - T_{\mathcal{A}_i}(d_{ij}) \, , \tag{5.3.17}$$

and such that $\mathbf{D}_H(f_{ijk}, \varphi_{ij}, a_{ij}, \mathcal{A}_i, B_i, d_{ij}) \equiv (\lambda_{ijkl}, \alpha_{ijk}, \beta_{ij}, \gamma_i)$ *has* $U(1)$*- and* Lie$(U(1))$*-valued elements, where*

$$\lambda_{ijkl} \equiv f_{ikl}^{-1} f_{ijk}^{-1} \varphi_{ij}(f_{jkl}) f_{ijl} \, , \tag{5.3.18}$$

$$\alpha_{ijk} \equiv a_{ij} + \varphi_{ij}(a_{jk}) - f_{ijk} a_{ik} f_{ijk}^{-1} - f_{ijk} d f_{ijk}^{-1} - T_{\mathcal{A}_i}(f_{ijk}) \, , \tag{5.3.19}$$

$$\beta_{ij} \equiv \varphi_{ij}(B_j) - B_i - d_{ij} + k_{ij} \, , \tag{5.3.20}$$

$$\gamma_i \equiv H_i - dB_i + T_{\mathcal{A}_i}(B_i) \, , \tag{5.3.21}$$

and where the the same notation φ_{ij} *has been used for the induced map* $\varphi_{ij} : O_{ij} \to Aut(Lie(G))$.

If there is zero on the LHS of equations (5.3.19), (5.3.20), (5.3.21) and 1 on the LHS of eq. (5.3.18), equations (5.3.14)-(5.3.21) define a nonabelian gerbe with connection and curving. A little algebra shows that in the less trivial situation we have the following proposition.

Proposition 5.18. *Assume that* λ_{ijkl} *is* $U(1)$*-valued and* α_{ijk}*,* β_{ij} *and* γ_i *are* Lie$(U(1))$*-valued, then the equations of the above definition guarantee that* $(\lambda_{ijkl}, \alpha_{ijk}, \beta_{ij}, \gamma_i)$ *is a 2-gerbe with connections and curvings; hence the name twisted 1-gerbe for the set* $(f_{ijk}, \varphi_{ij}, a_{ij}, \mathcal{A}_i, B_i, d_{ij}, H_i)$.

We say that the nonabelian gerbe $(f_{ijk}, \varphi_{ij}, a_{ij}, d_{ij}, A_i, B_i, H_i)$ is twisted by the 2-gerbe $(\lambda_{ijkl}, \alpha_{ijk}, \beta_{ij}, \gamma_i)$.

In a recent preprint [67] by the author of the thesis, twisted bundle gerbes are identified as bundle 2-gerbes with particular structure 2-crossed modules.

5.3.3. *Global worldsheet anomalies of M5-branes.* In order to describe the "inflow" mechanism for M5-branes, a slight generalization of twisted nonabelian gerbes to the case of the crossed module $\tilde{\Omega}G \to PG$ is needed. Here, G is a (simply-connected, compact, simple) Lie group, $\tilde{\Omega}G$ is the centrally extended group of based smooth loops and PG is the based path group of paths starting at the identity. The case relevant for a (six-dimensional, compact, oriented) M5-brane V embedded in an 11-dimensional spacetime spin manifold Y is that of $G = E_8$ [43]. The case relevant to string group (see, e.g., [117, 7] for models of string group) and string structures is $G = Spin(n)$ (see, e.g., [57], for a short discussion).

Comparing with the case of D-branes living the 10-dimensional spacetime, there is now a "3-form" G replacing the "2-form" field B. Based on the discussion in [43], this field G together with the metric give rise to an abelian 2-gerbe with curvings, its restriction to V being referred as the Chern-Simons 2-gerbe CS. Let $[CS]$ be the corresponding Deligne class. Also, there exists an torsion element $\theta \in H^4(V, \mathbb{Z})$ [132], replacing it this situation the integral Stiefel-Whitney class W_3 from the D-brane case, with the corresponding Deligne class $[\vartheta_{ijkl}, 0, 0, 0]$. The following condition generalizing that of (4.3.7) has been proposed in paper [P8].

Conjecture 5.19. *In order for a "stack" of M5-branes to be wrapping the cycle* V *in* Y*, there should exist a twisted nonabelian gerbe* $(f_{ijk}, \varphi_{ij}, a_{ij}, \mathcal{A}_i, B_i, d_{ij})$ *satisfying [cf. (4.3.7)]*

$$[CS] - [\vartheta_{ijkl}, 0, 0, 0] = [\mathbf{D}_H(f_{ijk}, \varphi_{ij}, a_{ij}, \mathcal{A}_i, B_i, d_{ij})] + [1, 0, 0, C_V] \, , \tag{5.3.22}$$

where $[1, 0, 0, C_V]$ *is the trivial Deligne class associated with a global 3-form* C_V*.*

Some arguments, based on homotopy properties of E_8, supporting this condition are given in [P8].

5.3.4. *Classifying topos of a topological bicategory.* In paper [P10], some established results [88] on classifying spaces and topoi were put together in a new way, with consequences for bicategories.

Let $\mathbb{B}$ be a topological bicategory. In analogy with the case of a topological category we have the following definition.

Definition 5.20. *The classifying topos $\mathcal{B}\mathbb{B}$ of the topological bicategory $\mathbb{B}$ is defined as $Sh(N\mathbb{B})$, the topos of sheaves on the Duskin nerve $N\mathbb{B}$. Similarly, the classifying space $B\mathbb{B}$ of a topological bicategory $\mathbb{B}$ is the geometric realization $|N\mathbb{B}|$ of its nerve $N\mathbb{B}$.*

Also, for a topological bicategory $\mathbb{B}$ write $Lin(X,\mathbb{B})$ for the category of linear orders over X equipped with an augmentation aug: $NL \to N\mathbb{B}$.

Definition 5.21. *An object E of $Lin(X,\mathbb{B})$ is called a Duskin principal $\mathbb{B}$- bundle. We call two Duskin principal $\mathbb{B}$-bundles E_0 and E_1 on X concordant if there exists a Duskin principal $\mathbb{B}$-bundle on $X \times [0,1]$ such that we have the equivalences $E_0 \simeq i_0^*(E)$ and $E_0 \simeq i_1^*(E)$ under the obvious inclusions $i_0, i_1 \colon X \hookrightarrow X \times [0,1]$.*

We can consider a linear order L as a locally trivial bicategory (with only trivial 2-morphisms). In this case the Duskin nerve of L coincides with the ordinary nerve of L which justifies the same notation NL for both nerves. Therefore, an augmentation $NL \to N\mathbb{B}$ is the same, by the nerve construction, as a continuous normal lax functor $L \to \mathbb{B}$. Similarly to the case of topological category (2.3.26) we have the following "classifying" property of the classifying topos $\mathcal{B}\mathbb{B}$.

Theorem 5.22. *For a topological bicategory $\mathbb{B}$ and a topological space X there is a natural equivalence of categories*

$$\mathrm{Hom}(Sh(X), \mathcal{B}\mathbb{B}) \simeq Lin(X,\mathbb{B}). \tag{5.3.23}$$

On homotopy classes of topos morphisms we have the natural bijection

$$[Sh(X), \mathcal{B}\mathbb{B}] \cong Lin_c(X,\mathbb{B}). \tag{5.3.24}$$

Let us recall that the topological bicategory $\mathbb{B}$ is locally contractible if its spaces of objects, 1-arrows and 2-arrows are locally contractible. The "classifying" property of the classifying space $B\mathbb{B}$ now follows as a corollary.

Corollary 5.23. *For a locally contractible bicategory $\mathbb{B}$ and a CW-complex X there is a natural bijection*

$$[X, B\mathbb{B}] \cong Lin_c(X,\mathbb{B}). \tag{5.3.25}$$

If, in addition, the topological bicategory $\mathbb{B}$ is a so-called "good" one [5] then the above is true also if we use, instead of the thickened geometric realization of the nerve, the geometric realization of the underlying simplicial set. The case of a good topological bicategory, as well as the sufficient conditions for a bicategory being a good one, are discussed in [5]. Thus, as a corollary we have a slight generalization of the result of Baas, Bökstedt and Kro [5]. As shown in [P10], similar results apply also to other types of nerves of bicategories, such as the Lack-Paoli [78], Tansamani [125] and Simpson [109] nerve.

6. Conclusions

The author of the thesis believes that the results described in the thesis fit well into the recent trends of exploring possible generalizations of the symmetry principles underlying our present understanding of quantum field theory. Since generalized symmetries discussed in this thesis arose naturally in both mathematics (operator algebras, category theory) and physics (integrable systems, string theory), there is a good chance that their study will help us in our attempts of identifying and understanding the fundamental mathematical structure of quantum field and string theory.

Concerning more specific questions, the author believes that the results on quantum groups and noncommutative gauge theories comply with the recent interest in noncommutative geometries. In this respect, the results on quantum groups may be useful for the further development of the theory of homogeneous quantum spaces, their differential geometry and a deeper understanding of the related representation theory. The results on noncommutative gauge theories may be useful for construction of realistic models of quantum field theories and their more rigorous investigations. An interesting problem would be, for instance, to understand homogenous quantum spaces from the deformation quantization perspective and, based on this, to develop the related noncommutative gauge theories on these. Related to the results concerning higher gauge theories, these comply with the current trends in categorification of mathematics and physics. Abelian gerbes already proved to be relevant to interrelated problems in mathematics (string structures, twisted K-theory, elliptic cohomology) and physics (global worldsheet anomalies and holonomy of D-branes). One can speculate, that further extended objects in string theory and/or their charges can be described using non-abelian higher gauge theories, in analogy with D-branes.

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