Fitting experimental data

1D stretch experiment with rubber

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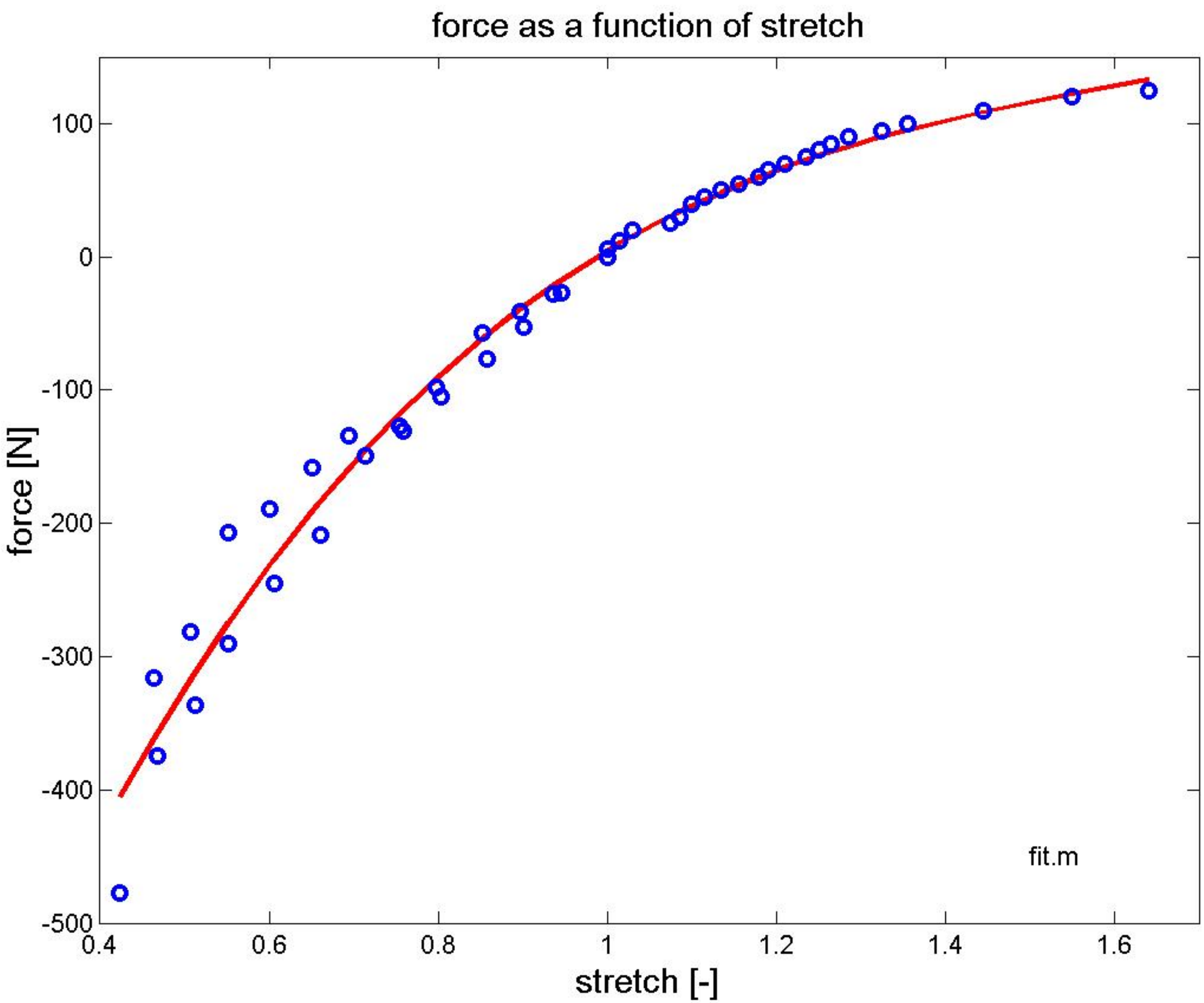
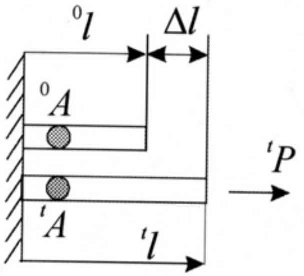




$$\xi = \frac{{}^{t}l}{{}^{0}l} \quad - \quad \text{stretch}$$

Polynomial fit

From a table

${}^{t}P_i$	—	—	—
ξ_i	—	—	—

We could get (assuming for example the 3rd degree polynomial)

$$ {}^{t}P = c_1\xi^3 + c_2\xi^2 + c_3\xi + c_4 \quad \text{(a)} $$

(see function polyfit in Matlab)

The derivative with respect the stretch yields

$$ \frac{d\,{}^{t}P}{d\xi} = 3\xi^2 c_1 + 2\xi c_2 + c_3 $$

$At\ \xi = 1$ (no stretching)

$$ \left.\frac{\partial\,{}^{t}P}{\partial\xi}\right|_{\xi=1} = 3c_1 + 2c_2 + c_3 = k^L = tg\,\alpha \quad (*) $$

Interlude

In the small-strain world we have

$$\sigma = {}_0E\,\varepsilon\,, \qquad \varepsilon = \frac{\Delta l}{{}^0l} = \frac{{}^tl - {}^0l}{{}^0l} = \xi - 1$$

$$\sigma = \frac{{}^tP}{{}^0A}$$

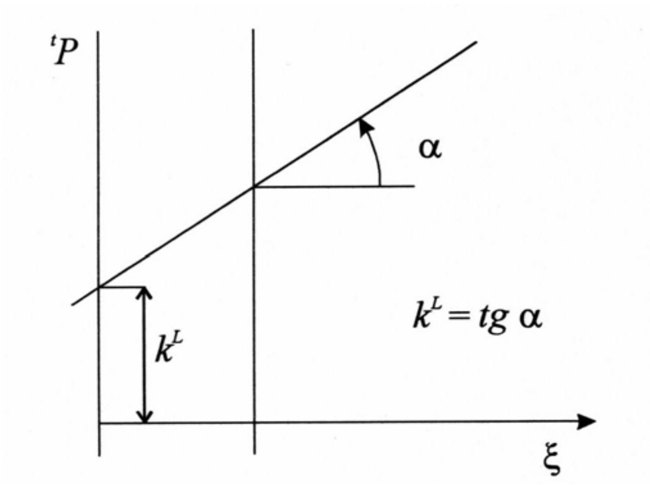
$${}^tP = {}^0A\,{}_0E\,(\xi - 1)$$

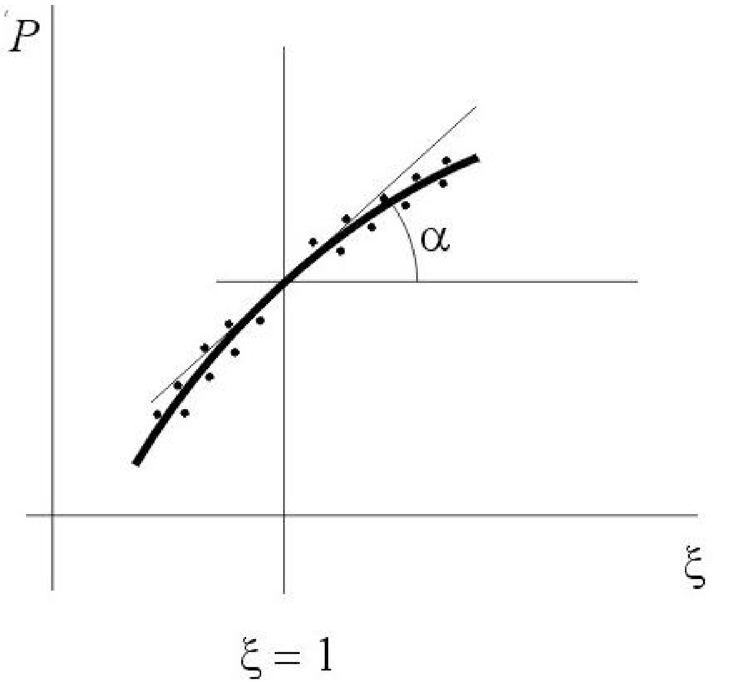
$${}^tP = k^L \xi - k^L \quad \text{— linear relation}$$

$$k^L = {}^0A_0\,E$$

So a sort of an „equivalent“ Young modulus is

$${}_0E = \frac{k^L}{{}^0A} = \frac{1}{{}^0A}\left(3c_1 + 2c_2 + c_3\right)$$





A trivial check

$$\sigma^{eng} = {}^{t}P / {}^{0}A = \frac{{}^{0}A_{0}E(\xi - 1)}{{}^{0}A} = {}_{0}E(\xi - 1)$$

$$= {}_{0}E\left(\frac{{}^{t}l}{{}^{0}l} - 1\right) = {}_{0}E\,\frac{{}^{t}l - {}^{0}l}{{}^{0}l} = {}_{0}E\,\frac{\Delta l}{{}^{0}l} = {}_{0}E\;\varepsilon$$

where

$${}_{0}E = \frac{k^{L}}{{}^{0}A}$$

← from experiment see (*)

For practical FE computations we need $S_{ij} = f\left(E_{ij}\right)$

In 1D case the Green-Lagrange strain tensor is

$$\boxed{E_{11} = E_{GL} = \frac{1}{2}\left(\xi^2 - 1\right) \quad (b)} \quad \Rightarrow \quad \xi = \sqrt{2E+1}$$

$$ {}_t^t\sigma = \frac{{}^t\rho}{{}^0\rho} F \; {}_0^tS \; F^T $$

Assuming uniform deformation $\qquad {}^tx = \frac{{}^tl}{{}^0l} \; {}^0x = \xi \; {}^0x$

we get 1D deformation gradient $\qquad F = \frac{\mathrm{d}\,{}^tx}{\mathrm{d}\,{}^0x} = \xi$

Using the mass conservation law $\qquad {}^0\rho \; {}^0l \; {}^0A \quad = \quad {}^t\rho \; {}^tl \; {}^tA$

$$\frac{{}^t\rho}{{}^0\rho} = \frac{{}^0l \quad {}^0A}{{}^tl \quad {}^tA} = \frac{1}{\xi} \; \frac{{}^0A}{{}^tA}$$

So the relation we are looking for is

$$ {}_t^t\sigma = \frac{1}{\xi}\,\frac{{}^0A}{{}^tA}\,\xi\ {}_0^tS\ \xi = {}_0^tS\ \frac{{}^0A}{{}^tA}\,\xi $$

but

$$ {}_t^t\sigma = \frac{{}^tP}{{}^tA} \Rightarrow {}^tP = {}^tA\ {}_0^tS\ \frac{{}^0A}{{}^tA}\,\xi = {}_0^tS\ {}^0A\,\xi $$

also

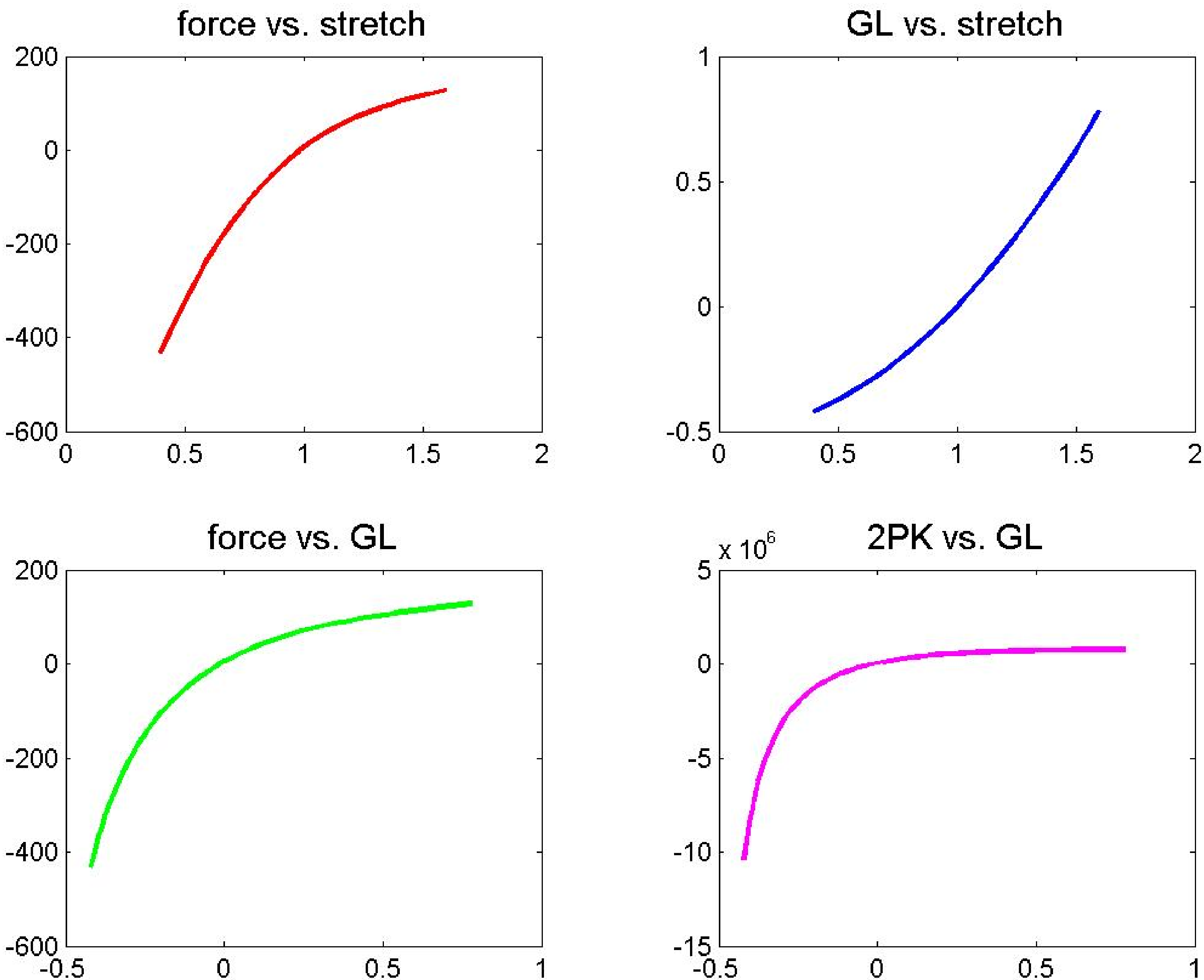
$$ \xi = \sqrt{2E_{GL}+1} $$

Finally

$$ {}^tP = {}_0^tS\ {}^0A\sqrt{2E_{GL}+1} $$

or

$$ {}_0^tS = \frac{{}^tP}{{}^0A\sqrt{2E_{GL}+1}} \qquad \text{(c)} $$


force vs. stretch
200
0
-200
-400
-600
0
0.5
1
1.5
2
GL vs. stretch
1
0.5
0
-0.5
0
0.5
1
1.5
2
force vs. GL
200
0
-200
-400
-600
-0.5
0
0.5
1
2PK vs. GL
x 10^6
5
0
-5
-10
-15
-0.5
0
0.5
1

Check and summary

a) geometry

$$F_{11} = \overline{F} = \xi = \frac{{}^{t}l}{{}^{0}l}$$

b) conservation of mass

$$\frac{{}^{t}\rho}{{}^{0}\rho} = \frac{1}{\xi}\,\frac{{}^{0}A}{{}^{T}A}$$

c) stresses

$${}^{t}_{t}\sigma_{11} = {}^{t}_{t}\overline{\sigma}\,, \qquad {}^{t}_{0}S_{11} = {}^{t}_{0}\overline{S}$$

d) true stress

$${}^{t}_{t}\overline{\sigma} = \frac{{}^{t}P}{{}^{t}A}$$

$$\Rightarrow \quad {}^{t}_{t}\overline{\sigma} = \frac{1}{\xi}\,\frac{{}^{0}A}{{}^{t}A}\;\xi\;{}^{t}_{0}\overline{S}\;\xi = \frac{{}^{0}A}{{}^{t}A}\;{}^{t}_{0}\overline{S}\;\xi$$

Still not enough

What is the relation between ${}^{t}A$ and ${}^{0}A$?

This would depend on other components of deformation gradient, i.e. F_{22}, F_{33}.

And this, in turn would depend on the type of material deformation.

Assuming EQUIVOLUMETRIC deformation $\left({}^{t}V = {}^{0}V, \;\; \mu = 0.5\right)$ - typical for rubber.

We would get

$${}^{t}l \; {}^{t}A = {}^{0}l \; {}^{0}A$$

For $0 < \mu \leq 0.5$

$$\frac{{}^{0}A}{{}^{t}A} = \frac{{}^{t}l}{{}^{0}l} = \xi$$

$$\frac{{}^{0}A}{{}^{t}A} = \frac{{}^{t}l}{{}^{0}l} = \xi$$

And finally

$${}_{t}^{t}\overline{\sigma} = {}_{0}^{t}\overline{S} \; \xi^{2}$$

$${}_{t}^{t}\overline{\sigma} = {}_{0}^{t}\overline{S} \; \xi^{(1+2\mu)}$$

A thought experiment – part 1

Let´s assume that we have a material which behaves linearly in a very wide range of stretch, meaning

$$ {}^{t}P = k\varepsilon = k\frac{\Delta l}{{}^{0}l} = k\frac{{}^{t}l - \Delta l}{{}^{0}l} = k(\xi - 1); $$

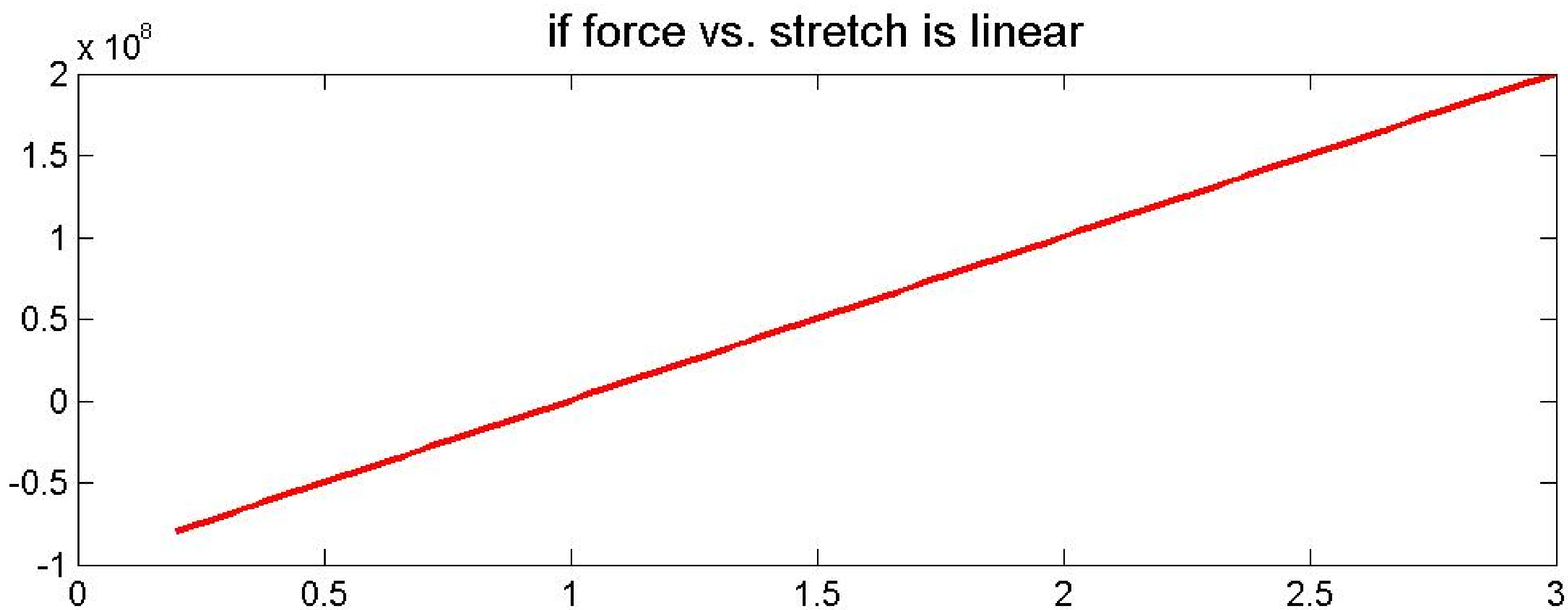
$$ \frac{{}^{t}P}{{}^{0}A} = \frac{k}{{}^{0}A}\varepsilon \quad \sigma^{eng} = {}_{0}E\,\varepsilon \quad \Rightarrow \quad {}_{0}E = k \,/\, {}^{0}A $$

This force expressed by means of the 2nd Piola-Kirchhoff is

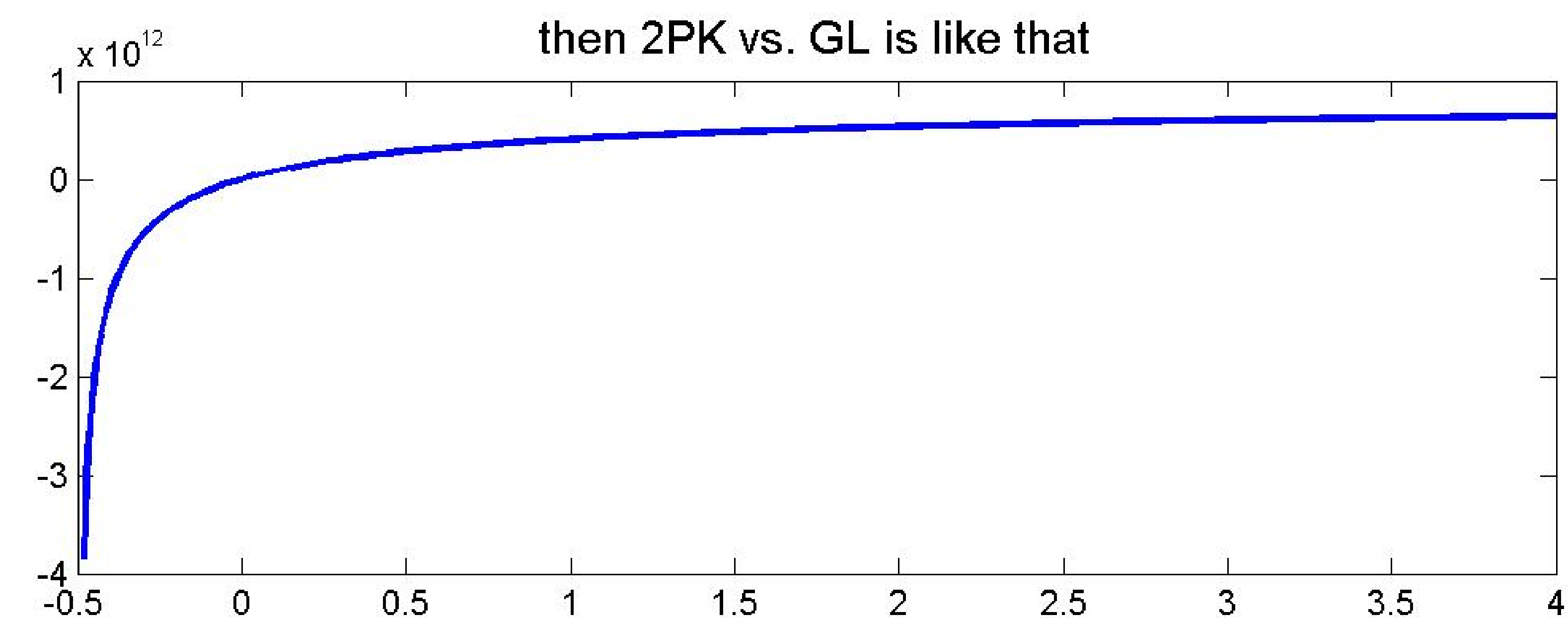
$$ {}^{t}P = {}^{t}_{t}\sigma \; {}^{t}A = {}^{t}_{0}S \; {}^{0}A\,\xi $$

Comparing the last two equations we get

$$ {}^{t}_{0}S = \frac{{}^{t}P}{{}^{0}A\xi} = \frac{k(\xi - 1)}{{}^{0}A\xi} = \frac{k}{{}^{0}A}\left(1 - \frac{1}{\xi}\right) = \frac{k}{{}^{0}A}\left(1 - \frac{1}{\sqrt{2E_{GL} + 1}}\right) $$



if force vs. stretch is linear
x 10^8
2
1.5
1
0.5
0
-0.5
-1
0
0.5
1
1.5
2
2.5
3



then 2PK vs. GL is like that
x 10^12
1
0
-1
-2
-3
-4
-0.5
0
0.5
1
1.5
2
2.5
3
3.5
4

So a linear function ${}^{t}P = f(\xi)$ has a strongly non-linear counterpart in ${}_{0}^{t}S = f(E_{GL})$ for

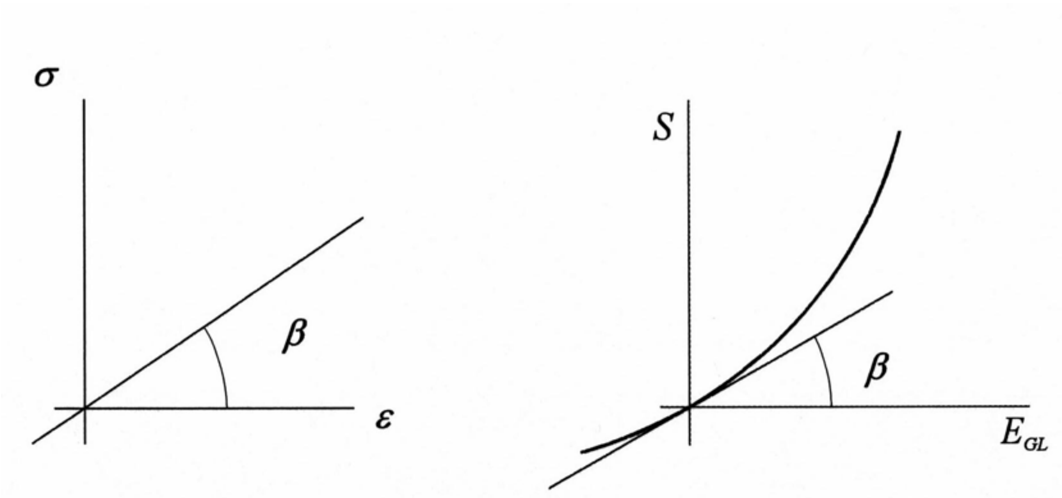
$$ {}_{0}^{t}S = \frac{k}{{}^{0}A}\left(1 - \frac{1}{\sqrt{2E_{GL}+1}}\right), $$

the rate is given by

$$ \frac{d\,{}_{0}^{t}S}{d\,E_{GL}} = -\frac{k}{{}^{0}A}\left(1 - \frac{1}{(2E_{GL}+1)^{3/2}}\right) $$

$$ \left.\frac{d\,{}_{0}^{t}S}{d\,E_{GL}}\right|_{E_{GL}=0} = +\frac{k}{{}^{0}A} = {}_{0}E $$

$$ \Rightarrow \quad tg\beta = {}_{0}E $$



There is the same tangent in origin
– but that’s a trivial conclusion.

A thought experiment – part 2

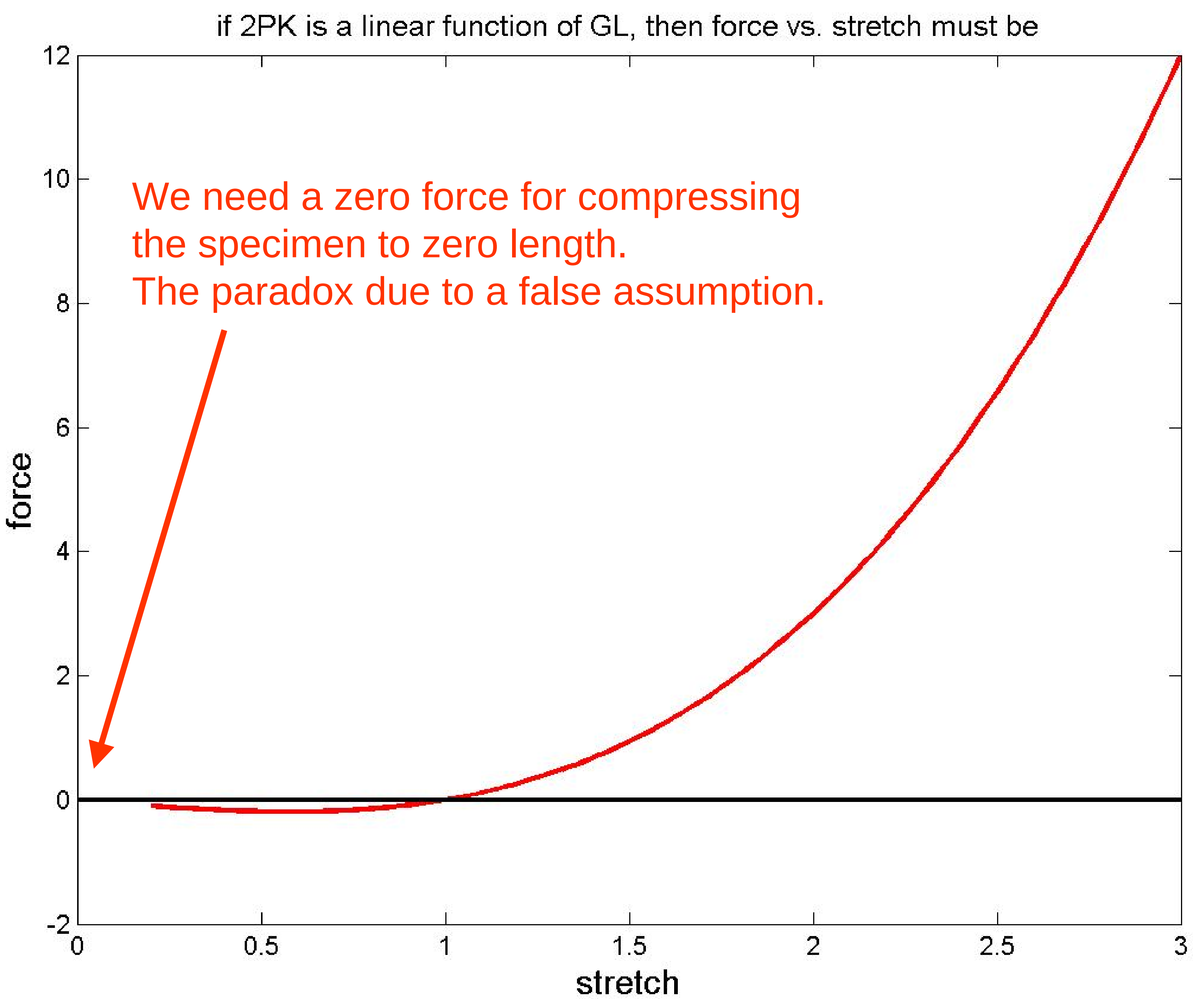
Let's conclude this thought experiment by finding the material properties which would correspond to a linear relation

$$ {}_0^tS = {}_0E \; E_{GL} = \frac{1}{2}\; {}_0E\left(\xi^2 - 1\right) $$

Since $$ {}^tP = {}_0^tS \;\; {}^0A\,\xi $$

by substitution we get

$$ {}^tP = \frac{1}{2}\, E_0 \;\; {}^0A\,\xi\left(\xi^2 - 1\right) $$



if 2PK is a linear function of GL, then force vs. stretch must be
We need a zero force for compressing
the specimen to zero length.
The paradox due to a false assumption.
force
stretch
12
10
8
6
4
2
0
-2
0
0.5
1
1.5
2
2.5
3