Continuum Mechanics, part 3

Kinematics_3

- Small strains
- Principal stretch
- Summary for kinematics
- Rigid body motion
- Motion and flow

In case of small strains and rotations

deformation gradient

We know that $[Z] = [F] - [I]$

symmetric and antisymmetric parts

material displacement gradient $\longrightarrow$ $[Z] = [\varepsilon] + [\Omega]$

infinitessimal Cauchy strain

infinitessimal rotation

From it follows

$$[F] = [I] + [\varepsilon] + [\Omega]$$

right Cauchy-Green def. tensor

then $$[C] = [F]^T[F] = \left([I] + [\varepsilon]^T + [\Omega]^T\right)\left([I] + [\varepsilon] + [\Omega]\right)$$

$$= \left([I] + [\varepsilon] - [\Omega]\right)\left([I] + [\varepsilon] + [\Omega]\right)$$

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this is exact, so far. Neglecting squares and products of $[\varepsilon]$ and $[\Omega]$ we get

$$[C] = [U]^2 \cong [I] + 2[\varepsilon]$$

And to the same order of approximation we ca[n] express the <u>left stretch tensor</u>

$$[U] \cong [I] + [\varepsilon]$$

and

$$[U]^{-1} \cong [I] - [\varepsilon]$$

similarly for <u>rotation tensor</u>

$$[R] = [F][U]^{-1} \cong \left([I] + [\varepsilon] + [\Omega]\right)\left([I] - [\varepsilon]\right) \cong [I] + [\Omega]$$

PRINCIPAL STRETCHES AND PRINCIPAL AXES OF DEFORMATION

We have already shown $\tilde{\lambda}\{m\} = [F]\{m_o\}$ ...see 22

Let's assume, that a particular line element stretches but not rotate during the motion.

Then $\{m\}$ becomes $\{m_o\}$ and $[F]$ becomes $[U]$ and

$$[U]\{m_o\} = \tilde{\lambda}\{m_o\}$$

which is a standard eigenvalue problem.

$[U]$ is symmetric, has real eigenvalues $\Rightarrow$ $\tilde{\lambda}_1 \geq \tilde{\lambda}_2 \geq \tilde{\lambda}_3$

these values are so called principal stretches

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the corresponding eigenvectors determine the principal axes of $[U]$.

It means that the deformation $[U]$ consists of pure extensions along the three coordinate directions the magnitude of principal extensions is $\tilde{\lambda}_i$.

Since $[C] = [U]^2$, the eigenvalues of $[C]$ are squares of eigenvalues of $[U]$. So

$$\lambda_i = \tilde{\lambda}_i^2 \quad .... \quad \text{see } 25 \qquad \lambda_i > 0$$

Similarly the Green-Lagrange strain tensor $[E] = \frac{1}{2}\left([C]-[I]\right)$ has eigenvalues $\frac{1}{2}(\lambda_i - 1)$

shift theorem

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Polar decomposition   an example

Let's assume we have a set of material points whose coordinates are given by

$$a_x = r\cos\varphi$$

$$a_y = r\sin\varphi \quad , \quad r = 1$$

Let the deformation is described by $x_i = x_i(a_j, t)$ that is particularly by

$$x_1 = \sqrt{3}\, a_1 + a_2$$

$$x_2 = 2a_2$$

$$x_3 = a_3 \quad \ldots \quad \text{2D deformation}$$

Deformation gradient is

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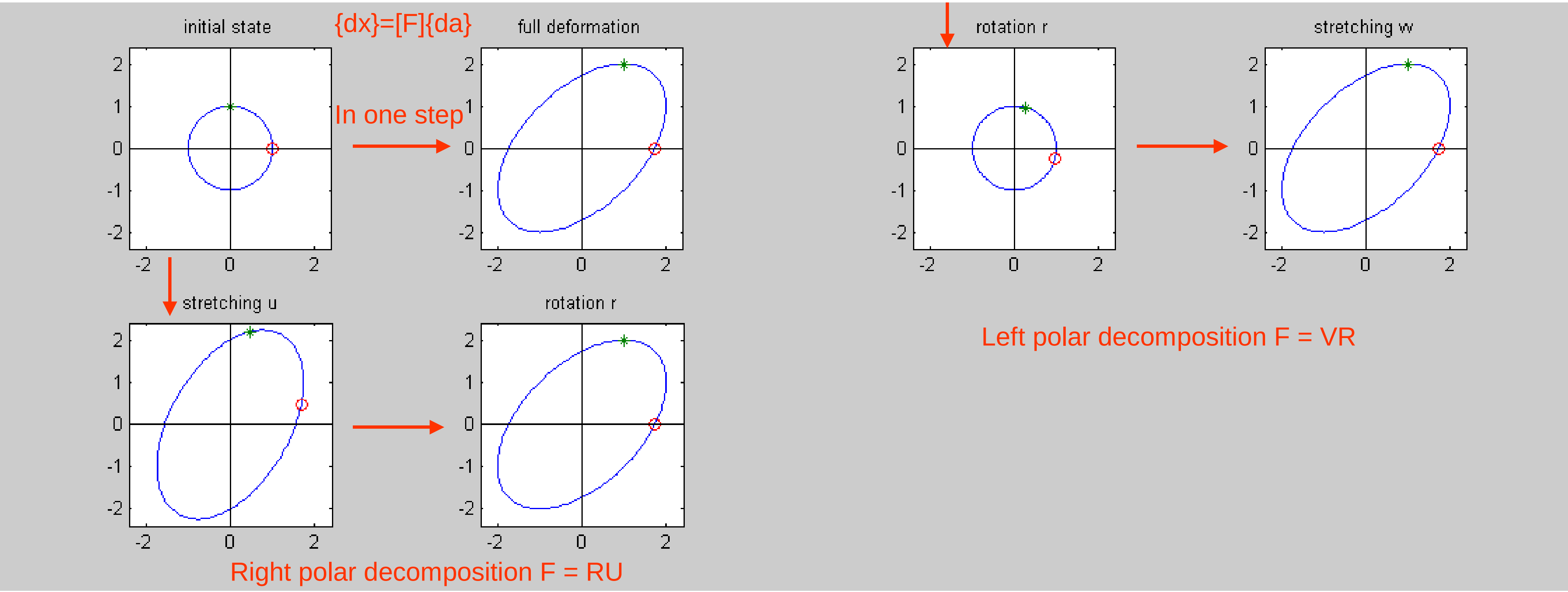
$$[F] = \left[\frac{\partial x_i}{\partial a_j}\right] = \begin{bmatrix} \sqrt{3} & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow$$ constant homogeneous deformation

Decomposition is given by $[C] = [F]^T[F]$

$$([\psi],[\lambda]) = \mathrm{eig}([C]); \quad [U] = [\psi][\Lambda]^{1/2}[\psi]^T;$$

$$[R] = [F][U]^{-1}; \quad [V] = [R][U][R]^T$$

The case of homogeneous deformation applied to a circle



For more details see the program polardec.m

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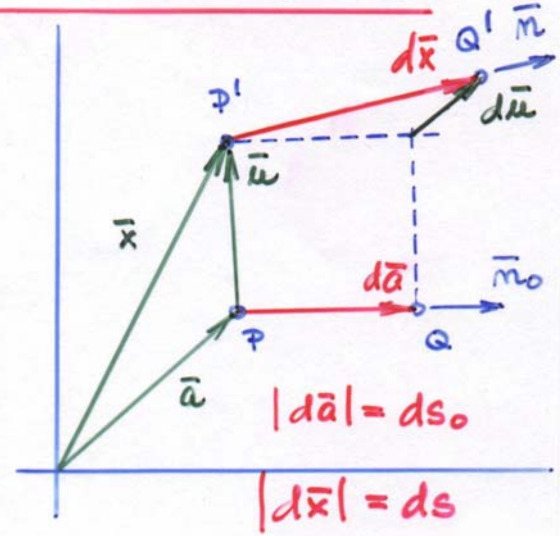
GRAND SUMMARY FOR KINEMATICS



$x_i = x_i(a_j, t) \qquad a_j = a_j(x_k, t)$

$\{dx\} = [F]\{da\} \qquad [F] = \dfrac{\partial x_i}{\partial a_j}$

$\{du\} = [Z]\{da\} \qquad [Z] = \dfrac{\partial u_i}{\partial a_j}$

$\{du\} = [\bar{Z}]\{dx\} \qquad [\bar{Z}] = \dfrac{\partial u_i}{\partial x_j}$

$[Z] = [F] - [I], \quad [\bar{Z}] = [I] - [F]^{-1}$

$\tilde{\lambda}^2 = \{n_o\}^T [C] \{n_o\} \qquad [C] = [F]^T[F] \qquad \tilde{\lambda} = ds/ds_o$

$1/\tilde{\lambda}^2 = \{n\}^T [\bar{C}]^{-1} \{n\} \qquad [\bar{C}] = [F][F]^T$

$[E] = \frac{1}{2}\left([C] - [I]\right) \qquad [A] = \frac{1}{2}\left([I] - [\bar{C}]^{-1}\right)$

$(ds)^2 - (ds_o)^2 = 2\{dx\}^T [A] \{dx\} = 2\{da\}^T [E] \{da\}$

$[F] = [R][U] \qquad \left([\phi], [\Lambda]\right) = eig\left([C]\right)$

$[U] = [\phi][\Lambda]^{-1/2}[\phi]^T, \; [R] = [F][U]^{-1}$

SMALL STRAINS AND ROTATIONS

$$[\varepsilon] = \frac{1}{2}\left([Z] + [Z]^T\right) = \frac{1}{2}\left([F]^T + [F] + [I]\right)$$

$$[\Omega] = \frac{1}{2}\left([Z] - [Z]^T\right) \; ; \; [Z] = [\bar{Z}]$$

$$[F] = [I] + [\varepsilon] + [\Omega]$$

$$\left.\begin{aligned} [C] &= [I] + 2[\varepsilon] \\ [U] &= [I] + [\varepsilon] \\ [U]^{-1} &= [I] - [\varepsilon] \end{aligned}\right\} \text{approximation}$$

RIGID-BODY MOTION

$$[F] = [Q]$$

$$[C] = [Q]^T[Q] = [I]$$

$$[\bar{C}] = [Q][Q]^T = [I]$$

$$[E] = \frac{1}{2}\left([C] - [I]\right) = [\emptyset]$$

$$[A] = \frac{1}{2}\left([I] - [\bar{C}]^{-1}\right) = [\emptyset]$$

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NOTATION and TERMINOLOGY

F deformation gradient tensor

deformační gradient

Z material displacement gradient tensor

materiálový gradient posunů

$\bar{Z}$ spatial displacement gradient tensor

prostorový gradient posunů

$\tilde{\lambda}$ stretch ratio in direction of $\bar{n}$ and n_0 resp.

poměrné protažení ve směru

C right Cauchy - Green deformation tensor (Green)

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$\bar{C}$ left Cauchy-Green deformation tensor

E Green-Lagrange strain tensor

A Almansi (Euler) strain tensor

Polar decomposition $F \to RU = VR$

R rotation tensor

U right stretching tensor (t. rotace)

V left stretching tensor (t. protažení

ε Cauchy infinitesimal strain tensor

Ω infinitesimal rotation tensor

Q rigid-body rotation tensor

AN INTERLUDE

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THE PHYSICAL MEANING OF THE LOGARITHMIC STRAIN

$$\varepsilon = \lg\left({}^t\ell / {}^0\ell\right)$$

$$d\varepsilon = \frac{{}^0\ell}{{}^t\ell}\,\frac{1}{{}^0\ell}\,d\,{}^t\ell = \frac{d\,{}^t\ell}{{}^t\ell}$$

AN INCREMENT

$$\frac{d\varepsilon}{dt} = \frac{d\,{}^t\ell / dt}{{}^t\ell}$$

WITH RESPECT TIME

$$\dot{\varepsilon} = \frac{\dot{u}}{{}^t\ell} = \frac{\Delta\ell / \Delta t}{{}^t\ell}$$

- INSTANTANEOUS VARIABLE
- RATE OF STRAIN DEPENDS ON THE RATE OF DISPLACEMENT
- SOMETIMES IT IS WRITTEN IN THE INCREMENTAL FORM

$$\Delta\varepsilon = \frac{\Delta u}{{}^t\ell} = \frac{\Delta\ell}{{}^t\ell}$$

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LOGARITHMIC STRAIN IS OFTEN CALLED THE TRUE STRAIN

$\varepsilon = \int \dot{\varepsilon}\, dt$ _ SO IT IS A CUMMULATIVE QUANTITY

Rigid-body motion

The equation $x_i = x_i\,(a_j, t)$ describes the motion of a body. Using this equation the position x_i of each particle a_j at time t can be calculated.

In a rigid-body motion the body moves without changing its shape. The distances between any two particles do not change. Neither does the angle between two lines joining a particle to two other particles.

Each rigid body motion can be decomposed into translation and rotation.

Translation is a rigid-body motion of a body in which every particle undergoes the same trajectory – has the same displacements. This motion can be described by the equation

$$x_i = a_i + c_i\,(t), \tag{1}$$

where vector c is independent of position and depends on time only.

Rotation is a rigid-body motion in which every particle follows the circular trajectory about the axis of rotation with the same angular velocity.

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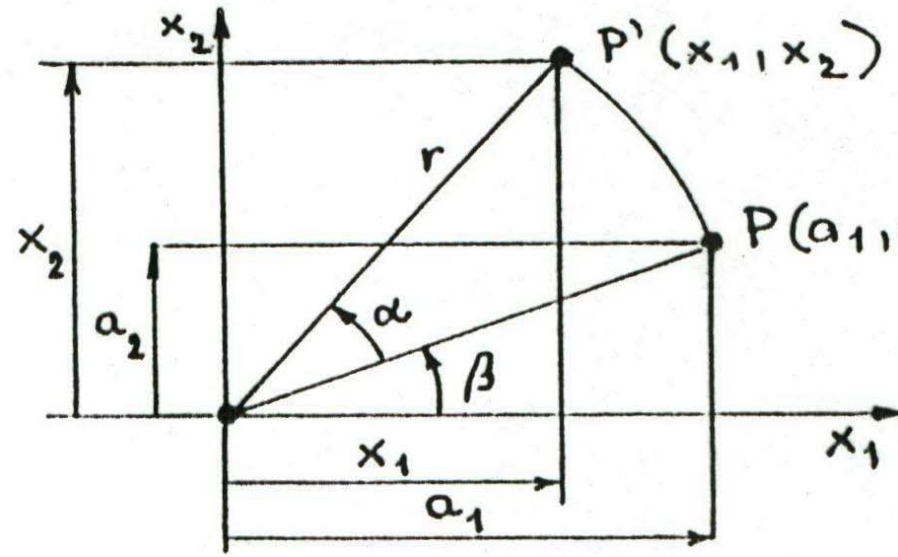
Rotation about a coordinate axis, say x_3

the coordinates of the current position of the rotated particle



$\{x\} \leftarrow \{a\}$

can be found easily

$$x_1 = r\cos(\alpha+\beta) = r(\cos\alpha\cos\beta - \sin\alpha\sin\beta) = a_1\cos\alpha - a_2\sin\alpha$$
$$x_2 = r\sin(\alpha+\beta) = r(\sin\alpha\cos\beta - \cos\alpha\sin\beta) = a_1\sin\alpha + a_2\cos\alpha$$

So that

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$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} \text{ or } \{x\} = [Q]\{a\}$$

It is obvious that rotation is an orthogonal transformation (length is preserved) and so the inverse motion is described by

$$\{a\} = [Q]^T \{x\}$$

It should be reminded that for a rigid-body motion the deformation gradient [F] becomes rotation tensor [R]

$$[F]=[Q]$$

and then the right Cauchy-Green deformation tensor is

$$[C]=[F]^T[F]=[Q]^T[Q]=[I]$$

and has the constant value throughout a rigid-body motion. the same conclusion is valid for the left Cauchy-Green deformation tensor

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$$\overline{[C]}=[F][F]^T=[I].$$

It is obvious that Green-Lagrange and Almansi strain tensors are equal to zero in a rigid-body motion

$$[E]=\frac{1}{2}([C]-[I])=[0],$$

$$[A]=\frac{1}{2}([I]-\overline{[C]}^{-1})=[0].$$

MOTION AND FLOW

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Terms used to describe the continuous change of configuration of a continuum. Sometimes a motion leading to a permanent deformation as, for example, in plasticity.

Velocity

MATERIAL DESCRIPTION

$$v_i = v_i(a_j, t) = \frac{dx_i}{dt} = \dot{x}_i$$

SPATIAL DESCRIPTION

$$v_i = v_i(x_j, t) = \frac{D}{Dt} u_i(x_j, t) = \frac{\partial u_i(x_j, t)}{\partial t} + v_k(x_j, t) \frac{\partial u_i(x_j, t)}{\partial x_k}$$

Acceleration

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MATERIAL DESCRIPTION

$$z_i = z_i(a_j, t) = \frac{d v_i(a_j, t)}{dt} = \dot{v}_i = \ddot{u}_i = \ddot{x}_i$$

SPATIAL DESCRIPTION

$$z_i = z(x_j, t) = \frac{D v_i(x_j, t)}{Dt} = \frac{\partial v_i(x_j, t)}{\partial t} + v_k(x_j, t)\frac{\partial v_i(x_j, t)}{\partial x_k}$$

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VELOCITY GRADIENT

$$L_{ij} = \frac{\partial v_i}{\partial x_j}$$

and its symmetric and unsymmetric parts are $L_{ij} = D_{ij} + W_{ij}$

Rate of deformation tensor is (tensor rychlosti deformace)

$$D_{ij} = \frac{1}{2}\left(L_{ij} + L_{ji}\right) = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$$

$D_{ij} = D_{ji}$ symmetric part $[D] = [D]^T$

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Other terms used for this tensor

rate of strain
strain rate
velocity strain

Antisymmetric part of velocity gradient is

$$W_{ij} = \frac{1}{2}\left(L_{ij} - L_{ji}\right)$$

$$W_{ij} = - W_{ji}$$

and is called <u>spin tensor</u> (vorticity tensor)

For small deformation gradients $[Z] \doteq [\bar{Z}]$

and $\frac{\partial u_i}{\partial a_j} \doteq \frac{\partial u_i}{\partial x_j}$. From it follows that Cauchy

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial a_j} + \frac{\partial u_j}{\partial a_i}\right) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

Differentiating we get

$$\Delta\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial \Delta u_i}{\partial x_j} + \frac{\partial \Delta u_j}{\partial x_i}\right)$$

The first glimpse of incremental approach

On multiplying $D_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$ by Δt

$$D_{ij}\,\Delta t = \frac{1}{2}\left(\frac{\partial v_i \Delta t}{\partial x_j} + \frac{\partial v_j \Delta t}{\partial x_i}\right) = \frac{1}{2}\left(\frac{\partial \Delta u_i}{\partial x_j} + \frac{\partial \Delta u_j}{\partial x_i}\right)$$

Comparing the marked equations we can write

$$D_{ij} = \frac{\Delta \varepsilon_{ij}}{\Delta t} \; ; \quad \Delta t \to 0 \quad D_{ij} = \dot{\varepsilon}_{ij}$$

which explains the physical meaning of D_{ij}.

The rate of deformation tensor is the time rate of infinitesimal Cauchy strain tensor

And by similar reasoning we obtain

$$W_{ij} = \frac{\Delta \Omega_{ij}}{\Delta t}$$

The spin tensor is the time rate of infinitesimal rotation tensor

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It should be noticed that the rate-of-deformation tensor $[D]$ has properties which are analogous to those of infinitesimal strain $[\varepsilon]$ – symmetry, principal axes, components of D_{ij} obey compatibility relations, etc.

Components of W_{ij} can be found in so called spin or vorticity vector which is defined

$$\bar{w} = \text{curl}\ \bar{v} = \text{rot}\ \bar{v} = \begin{vmatrix} \bar{e}_1 & \bar{e}_2 & \bar{e}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ v_1 & v_2 & v_3 \end{vmatrix} = \left(\bar{\nabla}_t \times \bar{v} \right) =$$

$$= \begin{Bmatrix} \frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3} \\ \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1} \\ \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \end{Bmatrix} = \begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix} .$$

Bruk $\quad W_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} \right) =$

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$$\frac{1}{2}\begin{bmatrix} 0 & \frac{\partial v_1}{\partial x_2} - \frac{\partial v_2}{\partial x_1} & \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1} \\ \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} & 0 & \frac{\partial v_2}{\partial x_3} - \frac{\partial v_3}{\partial x_2} \\ \frac{\partial v_3}{\partial x_1} - \frac{\partial v_1}{\partial x_3} & \frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3} & 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \cancel{0} & -w_3 & w_2 \\ w_3 & \cancel{0} & -w_1 \\ -w_2 & w_1 & \cancel{0} \end{bmatrix}$$

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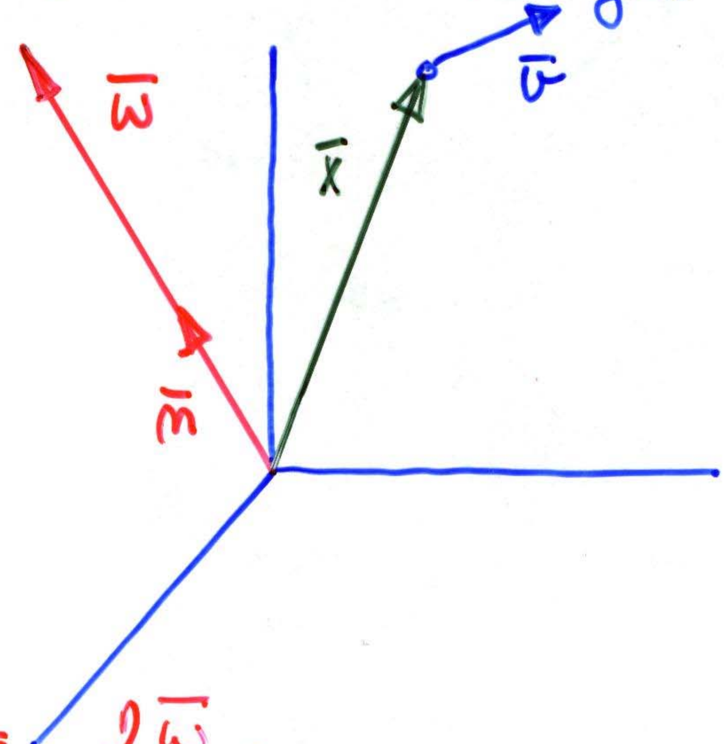
We can recall that in a rigid-body rotation with angular speed $\bar{\omega}$ about an axis through 0 with a unit vector $\bar{n}$ the velocity of a particle can be expressed by



$$\bar{v} = \bar{\omega} \times \bar{x}$$

substituting this into

$$\bar{w} = \mathrm{rot}\, \bar{v} = \bar{\nabla}_t \times \bar{v} = \bar{\nabla}_t \times \bar{\omega} \times \bar{x} = 2\bar{\omega}.$$

In a rigid body motion the velocity gradient becomes

$$L_{ij} = \frac{\partial v_i}{\partial x_j} \qquad \bar{v} = \bar{\omega} \times \bar{x} = \begin{vmatrix} \bar{e}_1 & \bar{e}_2 & \bar{e}_3 \\ \omega_1 & \omega_2 & \omega_3 \\ x_1 & x_2 & x_3 \end{vmatrix} = \left\{ \begin{matrix} \omega_2 x_3 - \omega_3 x_2 \\ \omega_3 x_1 - \omega_1 x_3 \\ \omega_1 x_2 - \omega_2 x_1 \end{matrix} \right\}$$

$$\Rightarrow [L] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} = \frac{1}{2} \underbrace{\begin{bmatrix} 0 & -W_3 & W_2 \\ W_3 & 0 & -W_1 \\ -W_2 & W_1 & 0 \end{bmatrix}}_{W_{ij}}$$

In a rigid-body motion

$$\Rightarrow [L] = [W] \; ; \; [D] = [\cancel{0}]$$

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It should be reminded that the material time derivative of deformation gradient [F] can be expressed by means of velocity gradient [L].

$$\frac{D}{Dt}[F] = \frac{D}{Dt}\left[\frac{\partial x_i}{\partial a_j}\right] = \frac{\partial v_i}{\partial a_j}$$

Using the chain rule

$$\frac{\partial v_i}{\partial a_j} = \frac{\partial v_i}{\partial x_k}\frac{\partial x_k}{\partial a_j} = L_{ik}\, F_{kj}$$

So that $[\dot{F}] = [L][F]$ or $[L] = [\dot{F}][F]^{-1}$.

In case of small displacement gradient we have $[Z] \doteq [\bar{Z}]$ and $[F] \doteq [F]^{-1} \doteq [I]$ and from it follows that

$$[L] = [\dot{F}]$$

and also

$$[D] = [\dot{\varepsilon}] \; ; \quad [W] = [\dot{\Omega}]$$

rate of deformation

time rate of Cauchy strain tensor

spin

time rate of infinitesimal rotation