Discrete Mathematics 39 (1982) 67–73
North-Holland Publishing Company

PARTITION THEOREMS FOR SYSTEMS OF FINITE SUBSETS OF INTEGERS

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Received 15 January 1980
Revised 12 January 1981

We study generalizations of Ramsey theorem to systems of finite subsets of ω. A system $\mathcal{S}$ of finite subsets of ω is called to be Ramsey if for every partition $\mathcal{S}=\mathcal{S}_1\cup\mathcal{S}_2$ there exists an infinite set $Y\subseteq\omega$ such that $\mathcal{S}_1\cap[Y]^{<\omega}=\emptyset$ or $\mathcal{S}_2\cap[Y]^{<\omega}=\emptyset$. We give some sufficient conditions for a system to be Ramsey. We also prove a theorem which concerns partitions into infinitely many classes. This may be regarded as a common generalization of Erdös–Rado and Nash-Williams theorems.

Introduction

Let $\mathcal{S}$ be a system of finite subsets of ω. Let us say that $\mathcal{S}$ is Ramsey if for every partition $\mathcal{S}=\mathcal{S}_1\cup\mathcal{S}_2$ there exists an infinite set $Y\subseteq\omega$ such that $\mathcal{S}_1\cap[Y]^{<\omega}=\emptyset$ or $\mathcal{S}_2\cap[Y]^{<\omega}=\emptyset$. Now, the Ramsey theorem can be stated as follows: if $\mathcal{S}\subseteq[\omega]^k$, then $\mathcal{S}$ is Ramsey. Not every $\mathcal{S}$ is Ramsey, consider $\mathcal{S}=[\omega]^1\cup[\omega]^2$ and $\mathcal{S}_1=[\omega]^1$, $\mathcal{S}_2=[\omega]^2$. In [3] Nash-Williams gave, inter alia, a sufficient condition for a set system $\mathcal{S}$ to be Ramsey. He proved that if a set system $\mathcal{S}$ does not contain two sets s, t such that s is a proper initial segment of t, then $\mathcal{S}$ is Ramsey. We give a new proof of this fact and show other sufficient conditions. All these conditions are also in some sense necessary.

Another possible generalization of the Ramsey theorem was considered by Erdös and Rado [1]. Let Y be an infinite subset of ω. A partition $[Y]^k=\bigcup_{i=1}^{\infty}\mathcal{S}_i$ is called canonical if there exists $n\in\{0,1,\ldots,k\}$ and $1\leq j_1<j_2<\cdots<j_n\leq k$ such that $\{x_1,x_2,\ldots,x_k\}$, $\{y_1,y_2,\ldots,y_k\}\in[Y]^k$ are elements of the same $\mathcal{S}_i$ iff $x_{j_1}=y_{j_1}$, $x_{j_2}=y_{j_2},\ldots,x_{j_n}=y_{j_n}$. Erdös and Rado proved that for every partition $[\omega]^k=\bigcup_{i=1}^{\infty}\mathcal{S}_i$ there exists an infinite set $Y\subseteq\omega$ such that the partition restricted to the set $[Y]^k$ is canonical. Generalization of the concept of canonical partition to the set systems that contain sets of various finite cardinalities presents some difficulties. We have chosen a definition which is in the case of $[\omega]^k$ a bit weaker but in general case provides an aesthetically pleasing balance between generality and clarity. We prove a "generalization" of Erdös–Rado theorem for Ramsey set systems.

All this is proved via transfinite induction on ω_1. The idea of using transfinite induction occurred to us after reading Ketonen's paper [2].

Notation. $s, t, \ldots$ will always denote finite subsets of ω; $X, Y, Z, \ldots$ infinite subsets of ω; $\mathcal{S}, \mathcal{T}, \ldots$ sets of finite subsets of ω; $\alpha, \beta, \ldots$ countable ordinal numbers; $s \leqslant t$ means s is an initial segment of t ($s < t$: proper initial segment); $\mathcal{S} \upharpoonright X = \mathcal{S} \cap [X]^{<\omega}$ is the restriction of $\mathcal{S}$ to X,

$$\mathcal{S}_{[n]} = \{s \in \mathcal{S} \mid n = \min s\};$$

$$\mathcal{S}_{(n)} = \{s \mid \{n\} \cup s \in \mathcal{S},\ n < \min s\};$$

inclusion is denoted by $\subseteq$; proper inclusion by $\subset$.

Definition. (a) $\mathcal{S}$ is *Ramsey* on X if for every partition $\mathcal{S} = \mathcal{S}_1 \cup \cdots \cup \mathcal{S}_n$ there exists Y such that at most one of the sets $\mathcal{S}_1 \upharpoonright Y \cdots \mathcal{S}_n \upharpoonright Y$ is nonempty.

(b) $\mathcal{S}$ is *Sperner* if there do not exist $s, t \in \mathcal{S}$ such that $s \subset t$.

(c) $\mathcal{S}$ is *thin* if there do not exist $s, t \in \mathcal{S}$ such that $s < t$.

(d) $\mathcal{S}$ is *α-uniform* on X if $\alpha = 0$, $\mathcal{S} = \{\emptyset\}$ or $\alpha > 0$, $\emptyset \notin \mathcal{S}$, and for every $n \in X$, $\mathcal{S}_{(n)}$ is α_n uniform on $X \cap (n, \infty)$, where $\alpha_n + 1 = \alpha$ for every $n \in X$, if α is not limit, or $\{\alpha_n\}_{n \in X}$ is increasing and converging to α if α is limit.

(e) $\mathcal{S}$ is *uniform* on X if $\mathcal{S}$ is α-uniform on X for some α.

For $k \in \omega$, there is exactly one k-uniform system on X, the system of k-element subsets of X. It is easy to show that there are infinitely many α-uniform systems for each $\alpha \geqslant \omega$. A typical example of an ω-uniform system is the set of all $s \subseteq X$ such that the cardinality of s equals to the least element of s. Similarly for $\omega + 1$, take the set of all $s \subseteq X$ such that the cardinality of s is equal to the least but one element of s.

Lemma 1. *If $\mathcal{S}$ is uniform on X, then $\mathcal{S}$ is a maximal thin systems on X.*

Proof. By induction on α such that $\mathcal{S}$ is α-uniform:

(1) $\alpha = 0$, then the assertion is trivial.

(2) $\alpha > 0$ and let the assertion hold for every $\beta < \alpha$. If $s, t \in \mathcal{S}$, $s < t$, then $\min s = \min t = n$; $s - \{n\} < t - \{n\}$ in $\mathcal{S}_{(n)}$, which is a contradiction with the induction hypothesis that $\mathcal{S}_{(n)}$ is thin. Thus $\mathcal{S}$ is thin. If $s \subseteq X$, $n = \min s$, then there is $t \in \mathcal{S}_{(n)}$ such that $t \leqslant s - \{n\}$ or $s - \{n\} \leqslant t$, because $\mathcal{S}_{(n)}$ is minimal. This implies $\{n\} \cup t \leqslant s$ or $s \leqslant \{n\} \cup t$ and $\{n\} \cup t \in \mathcal{S}$, which is the maximality condition for $\mathcal{S}$.

Lemma 2. *If $\mathcal{S}$ is α-uniform on X, $Y \subseteq X$ then $\mathcal{S} \upharpoonright Y$ is α-uniform on Y.*

Proof. By induction on α.

Definition. $\mathcal{R} \subseteq X \times [X]^{<\omega}$ is called *admissible* on X if:

(1) $n \mathcal{R} Y$, $Z \subseteq Y$ implies $n \mathcal{R} Z$ (heredity);

(2) for every $n \in X$, $Y \subseteq X$ there exists $Z \subseteq Y$ such that $n \mathcal{R} Z$ (cofinality).

Lemma 3 (Nash-Williams [3]). *For every admissible $\mathcal{R}$, there exists Y such that for every $n \in Y$, $n\,\mathcal{R}\,(Y \cap (n, \infty))$.*

Proof. Using cofinality of R define a sequence $Y_1 \subseteq Y_2 \subseteq Y_3 \cdots$ such that $n_i\,\mathcal{R}\,Y_{i+1}$ and $n_i < n_{i+1}$ for $n_i = \min Y_i$. Put $Y = \{n_1, n_2, \ldots\}$, and use heredity of $\mathcal{R}$.

Let $\mathcal{S} \neq \emptyset$, put $\mathcal{S}^* = \{t \mid \exists s \in \mathcal{S}, t \preccurlyeq s\}$. Let us call $\mathcal{S}$ *regular* (*singular*) if the relation $\succcurlyeq$ is (is not) well-founded on $\mathcal{S}^*$. ($\succcurlyeq$ is well-founded means that there does not exist an infinite sequence $s_1 \prec s_2 \prec \cdots$).

For $\mathcal{S}$ regular define a countable ordinal $\tau(\mathcal{S})$ – the type of $\mathcal{S}$ – as follows. First define for every $s \in \mathcal{S}^*$ an ordinal $\tau_{\mathcal{S}}(s)$ by well-founded induction:

$$\tau_{\mathcal{S}}(s) = \{\tau_{\mathcal{S}}(t) \mid s \prec t, t \in \mathcal{S}^*\}.$$

Then put $\tau(\mathcal{S}) = \tau_{\mathcal{S}}(\emptyset)$.

The following property enables us to use transfinite induction for regular $\mathcal{S}$'s. For every n, either $\mathcal{S}_{(n)} = \emptyset$ or $\tau(\mathcal{S}_{(n)}) < \tau(\mathcal{S})$. This is because, for $\{n\} \in \mathcal{S}^*$, $\tau(\mathcal{S}_{(n)}) = \tau_{\mathcal{S}}(\{n\}) < \tau_{\mathcal{S}}(\emptyset)$.

Lemma 4. *For every $\mathcal{S}$, X there exists $Y \subseteq X$ such that $\mathcal{S} \restriction Y = \emptyset$ or $\mathcal{S} \restriction Y$ contains a system uniform on Y.*

Proof. Let $\mathcal{S}_1$ consist of the elements of $\mathcal{S}$ minimal with respect to inclusion. $\mathcal{S}_1$ is Sperner and if $\mathcal{S}_1$ satisfies the condition of the lemma, so does $\mathcal{S}$. Hence it is sufficient to prove the lemma for $\mathcal{S}$ Sperner, $\mathcal{S} \neq \emptyset$. Consider these two cases:

(a) $\mathcal{S}$ is singular. Then take a sequence $s_1 \prec s_2 \prec \cdots$, $s_i \in X_1$ of elements of $\mathcal{S}^*$, and put $Y = \bigcup s_i$. Let t be an arbitrary finite subset of Y. Then $t \subset s_i$ for some i. Since $\mathcal{S}$ is Sperner, $t \notin \mathcal{S}$. Thus $\mathcal{S} \restriction Y = \emptyset$.

(b) $\mathcal{S}$ is regular. We shall prove by induction on $\tau(\mathcal{S})$ that for every X, there is $Y \subseteq X$ such that $\mathcal{S} \restriction Y = \emptyset$ or $\mathcal{S} \restriction Y$ is uniform on Y.

Consider the relation $n\,\mathcal{R}\,Z$ defined by the condition: $\mathcal{S}_{(n)} \restriction Z$ is empty or uniform on $Z \cap (n, \infty)$. By Lemma 2, $\mathcal{R}$ is hereditary. If $\mathcal{S}_{(n)} \restriction Z \neq \emptyset$, then $\tau(\mathcal{S}_{(n)}) < \tau(\mathcal{S})$. So we can use the induction hypothesis to show that $\mathcal{R}$ is cofinal. Thus, by Lemma 3, we have some Z such that, for every $n \in Z$, $\mathcal{S}_{(n)} \restriction Z$ is empty or uniform on $Z \cap (n, \infty)$. If $\mathcal{S}_{(n)} \restriction Z = \emptyset$ for infinitely many n's $n \in Z$, then let Y consist of these n's. Otherwise choose n's such that $\mathcal{S}_{(n)} \restriction Z$ is uniform on $Z \cap (n, \infty)$, and corresponding ordinals are either equal or form an increasing sequence.

Theorem 1. *The following statements are equivalent:*

(1) *$\mathcal{S}$ is Ramsey.*

(2) *There exists an X such that $\mathcal{S} \restriction X$ is Sperner.*

(3) *There exists an X such that $\mathcal{S} \restriction X$ is either empty or uniform on X.*

(4) *There exists an X such that $\mathcal{S} \restriction X$ is thin.*

(5) *There exists an X such that for no $Y \subseteq X$ there exist $\mathcal{S}_1$, $\mathcal{S}_2$ uniform on Y, $\mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset$ and $\mathcal{S}_1 \cup \mathcal{S}_2 \subseteq \mathcal{S} \restriction Y$.*

Proof. (1)⇒(2). Let $\mathcal{S}=\mathcal{S}_1\cup\mathcal{S}_2$, where $\mathcal{S}_1$ consists of all the elements of $\mathcal{S}$ minimal w.r.t. inclusion, and $\mathcal{S}_2$ contains the other elements of $\mathcal{S}$. Let X be given by the Ramsey property of $\mathcal{S}$ for the partition $\mathcal{S}=\mathcal{S}_1\cup\mathcal{S}_2$. If $\mathcal{S}_1\restriction X=\emptyset$, then also $\mathcal{S}_2\restriction X=\emptyset$. So we have $\mathcal{S}\restriction X=\mathcal{S}_1\restriction X$ anyway.

(2)⇒(3) By Lemma 4.

(3)⇒(4) By Lemma 1 any uniform $\mathcal{S}$ is thin.

(4)⇒(5) By Lemma 1 union of two uniform systems cannot be thin.

(5)⇒(1) Let $\mathcal{S}=\mathcal{S}_1\cup\mathcal{S}_2\cup\cdots\cup\mathcal{S}_k$. By Lemma 2 and Lemma 4 we can find Y, i such that $\mathcal{S}_i\restriction Y$ is empty or contains a subset uniform on Y. By hypothesis (5), $\mathcal{S}_i\restriction Y$ is nonempty at most for one i.

Lemma 5. *Let $\mathcal{S}$ be uniform on X, and let λ be a function $\lambda:\mathcal{S}\to\omega$ such that $\lambda(s)\notin s$ for every $s\in\mathcal{S}$. Then there is a Y such that $\lambda(s)\notin Y$ for every $s\in\mathcal{S}\restriction Y$.*

Proof. Let $\mathcal{S}$ be α-uniform on X. We shall prove lemma by induction on α. Let $\alpha>0$, ($\alpha=0$ is trivial). Consider the relation $n\,\mathcal{R}\,Y$ defined by the condition

$$\lambda(\mathcal{S}_{[n]}\restriction Y)\cap Y=\emptyset \quad\text{and}\quad |\lambda(\mathcal{S}_{[n]}\restriction Y)\cap[0,n]|\leq 1.$$

Heredity of $\mathcal{R}$ is evident. We shall prove its cofinality. Let $n\in X$; $Y\subseteq X$ be arbitrary. Then, by the induction hypothesis applied to a natural translation of λ to a function defined on $\mathcal{S}_{(n)}$, we have some $Z_1\subseteq Y$ such that

$$\lambda(\mathcal{S}_{[n]}\restriction Z_1)\cap Z_1=\emptyset.$$

Then using the Ramsey property of $\mathcal{S}_{[n]}\restriction Z_1$ we obtain some Z_2, which satisfies also the second part of the condition. Thus $\mathcal{R}$ is adminisible. Let Z be such that $n\,\mathcal{R}\,(Z\cap(n,\infty))$ for every $n\in Z$. Then we have a function $f:Z\to\omega$ such that if $s\in\mathcal{S}\restriction Z$, $\lambda(s)\in Z$, then $\lambda(s)=f(\min s)$. The graph of f has an independent set Y, (since it is 3-colourable). Y is the required set, because if we had $s\in\mathcal{S}\restriction Y$, $\lambda(s)\in Y$, then we would have also $\min s\in Y$ and $f(\min s)=\lambda(s)\in Y$.

Lemma 6. *Let $\mathcal{S}_1,\mathcal{S}_2$ be uniform on X, φ_1,φ_2 one-to-one mappings defined on $\mathcal{S}_1,\mathcal{S}_2$ respectively. Then there exists a Y satisfying one of the following conditions:*

(a) *$\mathcal{S}_1\restriction Y=\mathcal{S}_2\restriction Y$ and $\varphi_1(s)=\varphi_2(s)$ for every $s\in\mathcal{S}_1\restriction Y$, (i.e. $\varphi_1\restriction Y=\varphi_2\restriction Y$),*

(b) *$\varphi_1(\mathcal{S}_1\restriction Y)\cap\varphi_2(\mathcal{S}_2\restriction Y)=\emptyset$.*

Proof. Divide $\mathcal{S}_1$ into two parts by the condition

$$s\in\mathcal{S}_2 \quad\text{and}\quad \varphi_1(s)=\varphi_2(s). \tag{1}$$

Let Z be given by the Ramsey property for this partition. If every $s\in\mathcal{S}_1\restriction Z$ satisfies (1), then we have $\mathcal{S}_1\restriction Z\subseteq\mathcal{S}_2\restriction Z$. By maximality of uniform systems we have $\mathcal{S}_1\restriction Z=\mathcal{S}_2\restriction Z$. Thus (a) is satisfied for $Y=Z$. If every $s\in\mathcal{S}_1\restriction Z$ satisfies the negation of (1), define functions λ_1,λ_2 as follows: for $s\in\mathcal{S}_1$, $\lambda_1(s)\in t-s$ whenever $t-s=\emptyset$ and $\varphi_1(s)=\varphi_2(t)$, otherwise arbitrarily; λ_2 is defined dually. This is always possible, since for given s there is at most one t such that $\varphi_1(s)=\varphi_2(t)$, and vice versa. Let Y be such that

$$\lambda_i(\mathcal{S}_i \restriction Y)\cap Y=\emptyset, \quad i=1,2,$$

according to Lemma 6. We shall prove that Y satisfies (b). Suppose $\varphi_1(s)=\varphi_2(t)$ for some $s\in\mathcal{S}_1\restriction Y$, $t\in\mathcal{S}_2\restriction Y$. As we assume the negation of (1), we must have $s\neq t$. If e.g. $t-s\neq\emptyset$, then $\lambda_1(s)\in t\subseteq Y$, which is a contradiction. Thus $\varphi_1(s)=\varphi_2(t)$ is impossible.

Lemma 7. *Under the assumption of the preceeding lemma, if $\mathcal{S}_1$ is α-uniform, $\mathcal{S}_2$ is β-uniform, and $\alpha\neq\beta$, then* (a) *of the preceding lemma is excluded.*

Proof. It is easy to prove by induction that if $\mathcal{S}$ is α-uniform, then it is not β-uniform for any $\beta\neq\alpha$. Further, α-uniformity is heraditary. Therefore $\mathcal{S}_1$ and $\mathcal{S}_2$ cannot coincide on an infinite set.

Definition. A mapping φ defined on $\mathcal{S}$ is called canonical colouring of $\mathcal{S}$ on X if the following holds:
(1) $\mathcal{S}$ is uniform on X.
(2) There exists a uniform $\mathcal{T}$ and a mapping $f:\mathcal{S}\to\mathcal{T}$ such that
(a) $f(s)\subseteq s$ for every $s\in\mathcal{S}$;
(b) for every $s,t\in\mathcal{S}$, $\varphi(s)=\varphi(t)$ iff $f(s)=f(t)$.
Condition (b) is equivalent with:
(b′) There exists a one-to-one mapping ψ defined on $\mathcal{T}$ such that $\varphi(s)=\psi(f(s))$ for every $s\in\mathcal{S}$.

Roughly speaking, φ is a canonical colouring of $\mathcal{S}$ if the colour of each $s\in\mathcal{S}$ is determined by some subset t of s. The original definition of Erdös and Rado required that also the *position* of t in s is fixed.

Now we shall prove an analogy of Erdös–Rado theorem for Ramsey systems. By Theorem 1, we can restrict ourselves to uniform systems.

Theorem 2. *For every $\mathcal{S}$ uniform on X, and every mapping φ defined on $\mathcal{S}$, there is $Y\subseteq X$ such that $\varphi\restriction(\mathcal{S}\restriction Y)$ is canonical colouring of $\mathcal{S}\restriction Y$ on Y.*

Proof. (I) Let $\mathcal{S}$ be α-uniform. We shall use induction on α. If $\alpha=0$, then the statement is trivial. Suppose now that $\alpha>0$ and the theorem is proved for every $\beta<\alpha$. Define

$$\varphi_{(n)}(s)=\varphi(\{n\}\cup s\} \quad \text{for } s\in\mathcal{S}(n).$$

Let $\mathcal{T}_n, f_n, \psi_n$ be the corresponding uniform system and mappings of (2) and (b′).

The property of being cannonical is obviously hereditary. Therefore, (using the induction hypothesis), $\mathcal{R}$ is admissible, where $n\,\mathcal{R}\,Y$ means $\varphi_{(n)}\restriction(\mathcal{S}_{(n)}\restriction Y)$ is cannonical. Thus we can assume that the $\varphi_{(n)}$'s are cannonical colourings of $\mathcal{S}_{(n)}$ on $X \cap (n, \infty)$.

(II) We can also assume that the ordinals corresponding to $\mathcal{T}_n$, $n \in X$ are either all equal or form an increasing sequence.

(III) We shall show that (if we restrict ourselves on some infinite subset) the following condition holds for every $n < m$.

$$\begin{aligned} &\text{either} \quad \mathcal{T}_n \restriction (m, \infty) = \mathcal{T}_m, \qquad \psi_n \restriction \mathcal{T}_m = \psi_m, \\ &\text{or} \quad \psi_n(\mathcal{T}_n \restriction (m, \infty)) \cap \psi_m(\mathcal{T}_m) = \emptyset. \end{aligned} \tag{1}$$

This follows from Lemma 3, where $m \mathcal{R} Y$ means: "for every n, $n < m$, (1) relativized to Y holds". Again heredity is trivial. Cofinality follows from Lemma 6, where we consider couples $\psi_n \restriction (\mathcal{T}_n \restriction (m, \infty))$, ψ_m by turns for every n, $n < m$.

(IV) Using Ramsey's theorem for graphs we can assume that one part of the alternative (1) holds simultaneously for all n's. According to this we divide the rest of the proof.

(i) $\mathcal{T}_n \restriction (m, \infty) = \mathcal{T}_m$ and $\psi_n \restriction \mathcal{T}_m = \psi_m$ for every $n < m$.

Then we define $Y = X - \{n_0\}$, where $n_0 = \min X$, $\mathcal{T} = \mathcal{T}_{n_0}$, $f(s) = f_n(s - \{n\})$ for $s \in \mathcal{S}$, $n = \min s$, $\psi = \psi_{n_0}$. $\mathcal{T}$ is uniform on Y, $f(s) \subseteq s$ for every $s \in \mathcal{S} \restriction Y$, ψ is one-to-one, since $\mathcal{T}_{n_0}$, f_n, ψ_{n_0} have these properties. It remains to verify the equality from (b′) of the definition of canonical colouring:

$$\varphi(s) = \varphi_{(n)}(s - \{n\}) = \psi_n(f_n(s - \{n\})) = \psi_{n_0}(f_n(s - \{n\})) = \psi(f(s)),$$

where $s \in \mathcal{S} \restriction Y$, $n = \min s$.

(ii) $\psi_n(\mathcal{T}_n \restriction (m, \infty)) \cap \psi_m(\mathcal{T}_m) = \emptyset$ for every $n < m$.

Define sets $\mathcal{T}_{n,(s),m}$ and mappings $\psi_{n,(s),m}$ for every $n < m$, $s \subseteq (n, m]$ as follows: $t \in \mathcal{T}_{n,(s),m} \equiv m < \min t$, $s \cup t \in \mathcal{T}_n$, $\psi_{n,(s),m}(t) = \psi_n(s \cup t)$, for $t \in \mathcal{T}_{n,(s),m}$.

In the same way as done in (3) we can find a suitable infinite subset of X such that for every triple n, m, s, $n < m$, $s \subseteq (n, m]$ either

$$\mathcal{T}_{n,(s),m} = \mathcal{T}_m, \qquad \psi_{n,(s),m} = \psi_m \tag{2}$$

or

$$\psi_{n,(s),m}(\mathcal{T}_{n,(s),m}) \cap \psi_m(\mathcal{T}_m) = \emptyset. \tag{3}$$

For $s = \emptyset$ we have (3) by the assumption of this paragraph, since $\mathcal{T}_{n,(\emptyset),m} = \mathcal{T}_n \restriction (m, \infty)$. For $s \neq \emptyset$, $\mathcal{T}_{n,(s),m}$ is uniform on $X \cap (m, \infty)$ and

$$\tau(\mathcal{T}_{n,(s),m}) < \tau(\mathcal{T}_n) \leq \tau(\mathcal{T}_m).$$

The first inequality here follows e.g. from the formula

$$\mathcal{T}_{n,(s),m} = (\cdots((\mathcal{T}_n)_{(n_1)})_{(n_2)} \cdots)_{(n_k)} \restriction (m, \infty)$$

where $\{n_1, \ldots, n_k\} = s$. The second inequality was secured in (II). This, however, implies that we have (3) also for each $s \neq \emptyset$ (see Lemma 7).

Now define:

$$\mathcal{T}=\{\{n\}\cup t \mid t\in\mathcal{T}_n\},$$
$$f(s)=\{n\}\cup f_n(s-\{n\}) \quad \text{for } s\in\mathcal{S},\ n=\min s,$$
$$\psi(t)=\psi_n(t-\{n\}) \quad \text{for } n=\min t.$$

$\mathcal{T}$ is uniform since $\mathcal{T}_m$'s are uniform and by (II). $f(s)\subseteq s$ is also trivial.

Further we have for all $n<m$,

$$\psi_n(\mathcal{T}_n)=\bigcup_{s\subseteq(n,m]}\psi_{n,(s),m}(\mathcal{T}_{n,(s),m}),$$

therefore $\psi_n(\mathcal{T}_n)\cap\psi_m(\mathcal{T}_m)=\emptyset$ by (3).

If we denote $\psi_{[n]}(t)=\psi_n(t-\{n\})$ for $t\in\mathcal{T}$, $n=\min t$, then $\psi=\bigcup_n\psi_{[n]}$.

It follows that ψ is the union of a family of one-to-one mappings, the ranges of which are disjoint, therefore ψ is one-to-one. It remains only the equality

$$\varphi(s)=\varphi_{(n)}(s-\{n\})=\psi_n(f_n(s-\{n\}))=\psi(\{n\}\cup f_n(s-\{n\}))$$
$$=\psi(f(s)) \qquad \text{for } s\in\mathcal{S},\quad n=\min s.$$

At first glance it may seem that our definition of canonical colouring is rather weak. The next proposition shows that it is not quite so.

Proposition. *Let φ be a canonical colouring of $\mathcal{S}$ on X, let $\mathcal{T}_1, f_1$ and $\mathcal{T}_2, f_2$ be two couples satisfying condition* (2) *of the definition. Then there is Y such that*

$$\mathcal{T}_1\upharpoonright Y=\mathcal{T}_2\upharpoonright Y \quad \textit{and} \quad f_1\upharpoonright Y=f_2\upharpoonright Y.$$

Consequently, the ordinals of $\mathcal{T}_1$ and $\mathcal{T}_2$ are equal.

Proof. Let ψ_1, ψ_2 be the corresponding one-to-one mappings given by (b′). Apply Lemma 6 to these mappings. Then the condition (b) of Lemma 6 is excluded as we have

$$\emptyset\neq\varphi(\mathcal{S}\upharpoonright Y)\subseteq\psi_i(\mathcal{T}_i\upharpoonright Y), \quad i=1,2.$$

Therefore $\mathcal{T}_1$, ψ_1 and $\mathcal{T}_2$, ψ_2 coincide on Y. f_1, f_2 are uniquely determined by ψ_1 and ψ_2. ($f_i(s)$ is the unique $t\in\mathcal{T}_i$, for which $\psi_i(t)=\varphi(s)$.)

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