BOUNDARY VALUE PROBLEM WITH AN INNER POINT FOR THE SINGULARLY PERTURBED SEMILINEAR DIFFERENTIAL EQUATIONS

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Abstract. In this paper we investigate the problem of existence and asymptotic behavior of solutions for the nonlinear boundary value problem

$$\varepsilon y'' + ky = f(t, y), \quad t \in \langle a, b\rangle,\ k < 0,\ 0 < \varepsilon \ll 1$$

satisfying three point boundary conditions. Our analysis relies on the method of lower and upper solutions and delicate estimations.

Keywords: singular perturbation, boundary value problem, upper solution, lower solution

MSC 2010: 34B16

1. Introduction

We will consider the three point boundary value problem

$$\varepsilon y'' + ky = f(t, y), \quad t \in \langle a, b\rangle,\ k < 0,\ 0 < \varepsilon \ll 1, \tag{1.1}$$

$$y(a) = y(c) = y(b), \quad a < c < b. \tag{1.2}$$

We can view this equation as the mathematical model of the nonlinear dynamical system with a high-speed feedback. Moreover, this class of equations has special significance in connection with applications involving nonlinear vibrations. We focus on the existence and asymptotic behavior of solutions $y_\varepsilon(t)$ for ε belonging to a non-resonant set and on an estimate of the difference between the solution $y_\varepsilon(t)$ of (1.1), (1.2) and a singular solution $u(t)$ of the equation $ku = f(t, u)$.

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This is a singularly perturbed problem because the order of the differential equation drops when ε becomes zero. The situation in the present case is complicated by the fact that there is an inner point in the boundary conditions, in contrast to the "standard" boundary conditions as the Dirichlet problem, Neumann problem, Robin problem, periodic boundary value problem ([1], [4], [5]), for example. In the problem considered there does not exist a positive solution $\tilde{v}_\varepsilon$ of DE $\varepsilon y'' - my = 0$, $m > 0$, $0 < \varepsilon$ (i.e. $\tilde{v}_\varepsilon$ is convex) such that $\tilde{v}_\varepsilon(c) - \tilde{v}_\varepsilon(a) = u(c) - u(a) > 0$ and $\tilde{v}_\varepsilon(t) \to 0^+$ for $t \in \langle a, b\rangle$ and $\varepsilon \to 0^+$, which could be used to solve this problem by the method of upper and lower solutions. We will define the correction function $v_\varepsilon^{(\mathrm{corr})}(t)$ which will allow us to apply the method.

As was said before, we apply the method of upper and lower solutions and some delicate estimates to prove the existence of a solution for problem (1.1), (1.2) which converges uniformly to the solution u of the reduced problem (i.e. if we let $\varepsilon \to 0^+$ in (1.1)) on every compact subset of the interval (a, b) for $\varepsilon \to 0^+$.

As usual (cf. [3]), we say that $\alpha_\varepsilon \in C^2(\langle a, b\rangle)$ is a lower solution for problem (1.1), (1.2) if $\varepsilon\alpha_\varepsilon''(t) + k\alpha_\varepsilon(t) \geqslant f(t, \alpha_\varepsilon(t))$ and $\alpha_\varepsilon(c) - \alpha_\varepsilon(a) = 0$, $\alpha_\varepsilon(b) - \alpha_\varepsilon(c) \leqslant 0$ for every $t \in \langle a, b\rangle$. An upper solution $\beta_\varepsilon \in C^2(\langle a, b\rangle)$ satisfies $\varepsilon\beta_\varepsilon''(t) + k\beta_\varepsilon(t) \leqslant f(t, \beta_\varepsilon(t))$ and $\beta_\varepsilon(c) - \beta_\varepsilon(a) = 0$, $\beta_\varepsilon(b) - \beta_\varepsilon(c) \geqslant 0$ for every $t \in \langle a, b\rangle$.

Theorem 1.1 [2], [3]. *If α_ε, β_ε are respectively lower and upper solutions for* (1.1), (1.2) *such that $\alpha_\varepsilon \leqslant \beta_\varepsilon$, then there exists a solution y_ε of* (1.1), (1.2) *with $\alpha_\varepsilon \leqslant y_\varepsilon \leqslant \beta_\varepsilon$.*

Denote $H(u) = \{(t, y); \quad a \leqslant t \leqslant b, |y - u(t)| < d(t)\}$, where $d(t)$ is the positive continuous function on $\langle a, b\rangle$ such that

$$d(t) = \begin{cases} |u(c) - u(a)| + \delta & \text{for } a \leqslant t \leqslant a + \frac{1}{2}\delta, \\ \delta & \text{for } a + \delta \leqslant t \leqslant b - \delta, \\ |u(b) - u(c)| + \delta & \text{for } b - \frac{1}{2}\delta \leqslant t \leqslant b, \end{cases}$$

δ is a small positive constant and $u \in C^2$ is a solution of the reduced equation $ku = f(t, u)$ on $\langle a, b\rangle$. We will assume that such a solution u exists. Further, we will write $s(\varepsilon) = O(r(\varepsilon))$ when $0 < \lim\limits_{\varepsilon \to 0^+} |s(\varepsilon)/r(\varepsilon)| < \infty$.

2. Main result

Theorem 2.1. *Let $f \in C^1(H(u))$ satisfy the condition*

$$\left|\frac{\partial f(t,y)}{\partial y}\right| \leqslant w < -k \quad \textit{for every } (t,y) \in H(u) \textit{ (hyperbolicity condition).}$$

Then there exists ε_0 such that for every $\varepsilon \in (0,\varepsilon_0\rangle$ the problem (1.1), (1.2) has a unique solution satisfying the inequality

$$-v_\varepsilon^{(\mathrm{corr})}(t) - \hat v_\varepsilon(t) - C\varepsilon \leqslant y_\varepsilon(t) - (u(t) + v_\varepsilon(t)) \leqslant \hat v_\varepsilon(t) + C\varepsilon \quad \textit{for } u(c) - u(a) \geqslant 0$$

and

$$-\hat v_\varepsilon(t) - C\varepsilon \leqslant y_\varepsilon(t) - (u(t) + v_\varepsilon(t)) \leqslant v_\varepsilon^{(\mathrm{corr})}(t) + \hat v_\varepsilon(t) + C\varepsilon \quad \textit{for } u(c) - u(a) \leqslant 0$$

on $\langle a,b\rangle$ where

$$\begin{aligned} v_\varepsilon(t) &= \frac{u(c)-u(a)}{D}\cdot\bigl(\mathrm{e}^{\sqrt{m/\varepsilon}(b-t)} - \mathrm{e}^{\sqrt{m/\varepsilon}(t-b)} + \mathrm{e}^{\sqrt{m/\varepsilon}(t-c)} - \mathrm{e}^{\sqrt{m/\varepsilon}(c-t)}\bigr),\\ \hat v_\varepsilon(t) &= \frac{|u(b)-u(c)|}{D}\cdot\bigl(\mathrm{e}^{\sqrt{m/\varepsilon}(t-a)} - \mathrm{e}^{\sqrt{m/\varepsilon}(a-t)} + \mathrm{e}^{\sqrt{m/\varepsilon}(c-t)} - \mathrm{e}^{\sqrt{m/\varepsilon}(t-c)}\bigr),\\ D &= \bigl(\mathrm{e}^{\sqrt{m/\varepsilon}(b-a)} + \mathrm{e}^{\sqrt{m/\varepsilon}(c-b)} + \mathrm{e}^{\sqrt{m/\varepsilon}(a-c)}\bigr)\\ &\quad - \bigl(\mathrm{e}^{\sqrt{m/\varepsilon}(a-b)} + \mathrm{e}^{\sqrt{m/\varepsilon}(b-c)} + \mathrm{e}^{\sqrt{m/\varepsilon}(c-a)}\bigr), \end{aligned}$$

$m = -k - w$, $C = m^{-1}\max\{|u''(t)|; t \in \langle a,b\rangle\}$ and the function

$$\begin{aligned} v_\varepsilon^{(\mathrm{corr})}(t) = \frac{w|u(c)-u(a)|}{\sqrt{m\varepsilon}}\cdot\Bigl[&-O(1)\frac{v_\varepsilon(t)}{(u(c)-u(a))}\\ &+ O(\mathrm{e}^{\sqrt{m/\varepsilon}(a-c)})\frac{\hat v_\varepsilon(t)}{|u(b)-u(c)|} + tO(\mathrm{e}^{\sqrt{m/\varepsilon}\chi(t)})\Bigr] \end{aligned}$$

is positive for $t \in (a,b\rangle$.

R e m a r k 1. The function $v_\varepsilon(t)$ satisfies

(1) $\varepsilon v_\varepsilon'' - m v_\varepsilon = 0$,

(2) $v_\varepsilon(c) - v_\varepsilon(a) = -(u(c) - u(a))$, $v_\varepsilon(b) - v_\varepsilon(c) = 0$,

(3) $v_\varepsilon(t) \geqslant 0$ ($\leqslant 0$) is decreasing (increasing) for $u(c) - u(a) \geqslant 0$ ($\leqslant 0$),

(4) $v_\varepsilon(t)$ converges uniformly to 0 for $\varepsilon \to 0^+$ on every compact subset of $(a,b\rangle$,

(5) $v_\varepsilon(t) = (u(c) - u(a))O\bigl(\mathrm{e}^{\sqrt{m/\varepsilon}\chi(t)}\bigr)$ where $\chi(t) = a - t$ for $a \leqslant t \leqslant \frac12(b+c)$ and $\chi(t) = t - b + a - c$ for $\frac12(b+c) < t \leqslant b$.

The function $\hat{v}_\varepsilon(t)$ satisfies

(1) $\varepsilon\hat{v}_\varepsilon'' - m\hat{v}_\varepsilon = 0$,
(2) $\hat{v}_\varepsilon(c) - \hat{v}_\varepsilon(a) = 0$, $\hat{v}_\varepsilon(b) - \hat{v}_\varepsilon(c) = |u(b) - u(c)|$,
(3) $\hat{v}_\varepsilon(t) \geqslant 0$ is increasing,
(4) $\hat{v}_\varepsilon(t)$ converges uniformly to 0 for $\varepsilon \to 0^+$ on every compact subset of $\langle a, b)$,
(5) $\hat{v}_\varepsilon(t) = |u(b) - u(c)| O\big(\mathrm{e}^{\sqrt{m/\varepsilon}\hat{\chi}(t)}\big)$ where $\hat{\chi}(t) = t - b$ for $\frac{1}{2}(a+c) \leqslant t \leqslant b$ and $\hat{\chi}(t) = c - b + a - t$ for $a \leqslant t < \frac{1}{2}(a+c)$.

The correction function $v_\varepsilon^{(\mathrm{corr})}(t)$ will be determined precisely in the next section.

3. The correction function $v_\varepsilon^{(\mathrm{corr})}(t)$

Consider the linear problem

$$\varepsilon y'' - my = -2w\,|v_\varepsilon(t)|\,, \quad t \in \langle a, b\rangle,\ \varepsilon > 0 \tag{3.1}$$

with the boundary condition (1.2).

We apply the method of upper and lower solutions. We define

$$\alpha_\varepsilon(t) = 0$$

and

$$\beta_\varepsilon(t) = \frac{2w}{m}\max\{|v_\varepsilon(t)|\,, t \in \langle a, b\rangle\} = \frac{2w}{m}\,|v_\varepsilon(a)|\,.$$

Obviously, $|v_\varepsilon(a)| = 2wm^{-1}|u(c) - u(a)|\big(1 + O\big(\mathrm{e}^{\sqrt{m/\varepsilon}(a-c)}\big)\big)$ and the constant functions α_ε and β_ε are lower and upper solutions for problem (3.1), (1.2). Thus on the basis of Lemma 1.1 there exists a unique solution $y_\varepsilon^{\mathrm{Lin}}$ of the linear problem (3.1), (1.2) for every ε such that

$$0 \leqslant y_\varepsilon^{\mathrm{Lin}}(t) \leqslant \frac{2w}{m}\,|u(c) - u(a)|\,\big(1 + O\big(\mathrm{e}^{\sqrt{m/\varepsilon}(a-c)}\big)\big)$$

on $\langle a, b\rangle$. The solution we denote by $v_\varepsilon^{(\mathrm{corr})}(t)$ i.e. is

$$v_\varepsilon^{(\mathrm{corr})}(t) \stackrel{\mathrm{def}}{=} y_\varepsilon^{\mathrm{Lin}}(t)$$

and we compute $v_\varepsilon^{(\mathrm{corr})}(t)$ exactly:

$$v_\varepsilon^{(\mathrm{corr})}(t) = -\frac{(\psi_\varepsilon(a) - \psi_\varepsilon(c))}{(u(c) - u(a))}v_\varepsilon(t) + \frac{(\psi_\varepsilon(c) - \psi_\varepsilon(b))}{|u(b) - u(c)|}\hat{v}_\varepsilon(t) + \psi_\varepsilon(t)$$

where

$$\psi_\varepsilon(t) = \frac{w|u(c)-u(a)|}{D\sqrt{m\varepsilon}} t\big(\mathrm{e}^{\sqrt{m/\varepsilon}(b-t)} + \mathrm{e}^{\sqrt{m/\varepsilon}(t-b)} - \mathrm{e}^{\sqrt{m/\varepsilon}(c-t)} - \mathrm{e}^{\sqrt{m/\varepsilon}(t-c)}\big).$$

Hence

$$\begin{aligned}
\psi_\varepsilon(a) - \psi_\varepsilon(c) &= \frac{w|u(c)-u(a)|}{D\sqrt{m\varepsilon}} a\big(\mathrm{e}^{\sqrt{m/\varepsilon}(b-a)} + \mathrm{e}^{\sqrt{m/\varepsilon}(a-b)} \\
&\quad - \mathrm{e}^{\sqrt{m/\varepsilon}(c-a)} - \mathrm{e}^{\sqrt{m/\varepsilon}(a-c)}\big) \\
&\quad - \frac{w|u(c)-u(a)|}{D\sqrt{m\varepsilon}} c\big(\mathrm{e}^{\sqrt{\frac{m}{\varepsilon}}(b-c)} + \mathrm{e}^{\sqrt{m/\varepsilon}(c-b)} - 2\big) \\
&= \frac{w|u(c)-u(a)|}{\sqrt{m\varepsilon}} O(1), \\
\psi_\varepsilon(c) - \psi_\varepsilon(b) &= \frac{w|u(c)-u(a)|}{D\sqrt{m\varepsilon}} c\big(\mathrm{e}^{\sqrt{m/\varepsilon}(b-c)} + \mathrm{e}^{\sqrt{m/\varepsilon}(c-b)} - 2\big) \\
&\quad - \frac{w|u(c)-u(a)|}{D\sqrt{m\varepsilon}} b\big(2 - \mathrm{e}^{\sqrt{m/\varepsilon}(c-b)} - \mathrm{e}^{\sqrt{m/\varepsilon}(b-c)}\big) \\
&= \frac{w|u(c)-u(a)|}{\sqrt{m\varepsilon}} O\big(\mathrm{e}^{\sqrt{m/\varepsilon}(a-c)}\big), \\
\psi_\varepsilon(t) &= \frac{w|u(c)-u(a)|}{\sqrt{m\varepsilon}} O\big(\mathrm{e}^{\sqrt{m/\varepsilon}\chi(t)}\big).
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
v_\varepsilon^{(\mathrm{corr})}(t) = \frac{w|u(c)-u(a)|}{\sqrt{m\varepsilon}} \cdot \Big[& -O(1)\frac{v_\varepsilon(t)}{(u(c)-u(a))} \\
&+ O\big(\mathrm{e}^{\sqrt{m/\varepsilon}(a-c)}\big)\frac{\hat{v}_\varepsilon(t)}{|u(b)-u(c)|} + tO\big(\mathrm{e}^{\sqrt{m/\varepsilon}\chi(t)}\big)\Big].
\end{aligned}$$

4. Proof of theorem

Proof. First we will consider the case $u(c) - u(a) \geqslant 0$. We define the lower solutions by

$$\alpha_\varepsilon(t) = u(t) + v_\varepsilon(t) - v_\varepsilon^{(\mathrm{corr})}(t) - \hat{v}_\varepsilon(t) - \Gamma_\varepsilon$$

and the upper solutions by

$$\beta_\varepsilon(t) = u(t) + v_\varepsilon(t) + \hat{v}_\varepsilon(t) + \Gamma_\varepsilon.$$

Here $\Gamma_\varepsilon = \varepsilon\Delta/m$ where Δ is the constant which shall be defined below, $\alpha \leqslant \beta$ on $\langle a, b\rangle$ and satisfy the boundary conditions prescribed for the lower and upper solutions of (1.1), (1.2).

Now we show that $\varepsilon\alpha_\varepsilon''(t) + k\alpha_\varepsilon(t) \geqslant f(t, \alpha_\varepsilon(t))$ and $\varepsilon\beta_\varepsilon''(t) + k\beta_\varepsilon(t) \leqslant f(t, \beta_\varepsilon(t))$. Denote $h(t, y) = f(t, y) - ky$. By the Taylor theorem we obtain

$$h(t, \alpha_\varepsilon(t)) = h(t, \alpha_\varepsilon(t)) - h(t, u(t)) = \frac{\partial h(t, \theta_\varepsilon(t))}{\partial y}(v_\varepsilon(t) - v_\varepsilon^{(\mathrm{corr})}(t) - \hat{v}_\varepsilon(t) - \Gamma_\varepsilon),$$

where $(t, \theta_\varepsilon(t))$ is a point between $(t, \alpha_\varepsilon(t))$ and $(t, u(t))$, and $(t, \theta_\varepsilon(t)) \in H(u)$ for sufficiently small ε. Hence, from the inequalities $m \leqslant \partial h(t, \theta_\varepsilon(t))/\partial y \leqslant m + 2w$ in $H(u)$ we have

$$\begin{aligned}&\varepsilon\alpha_\varepsilon''(t) - h(t, \alpha_\varepsilon(t)) \geqslant \varepsilon u''(t) + \varepsilon v_\varepsilon''(t)\\ &\quad - \varepsilon v_\varepsilon^{(\mathrm{corr})''}(t) - \varepsilon\hat{v}_\varepsilon''(t) - (m + 2w)v_\varepsilon(t) + m v_\varepsilon^{(\mathrm{corr})}(t) + m\hat{v}_\varepsilon(t) + m\Gamma_\varepsilon.\end{aligned}$$

Because $v_\varepsilon(t) = |v_\varepsilon(t)|$ we have $-\varepsilon v_\varepsilon^{(\mathrm{corr})''}(t) - 2wv_\varepsilon(t) + mv_\varepsilon^{(\mathrm{corr})}(t) = 0$, as follows from DE (3.1), we get

$$\varepsilon\alpha_\varepsilon''(t) - h(t, \alpha_\varepsilon(t)) \geqslant \varepsilon u''(t) + m\Gamma_\varepsilon \geqslant -\varepsilon|u''(t)| + \varepsilon\Delta.$$

For $\beta_\varepsilon(t))$ we have the inequality

$$\begin{aligned}h(t, \beta_\varepsilon(t)) - \varepsilon\beta_\varepsilon''(t) &= \frac{\partial h(t, \tilde{\theta}_\varepsilon(t))}{\partial y}(v_\varepsilon(t) + \hat{v}_\varepsilon(t) + \Gamma_\varepsilon) - \varepsilon\beta_\varepsilon''(t)\\ &= m(v_\varepsilon(t) + \hat{v}_\varepsilon(t) + \Gamma_\varepsilon) - \varepsilon(u''(t) + v_\varepsilon''(t) + \hat{v}_\varepsilon''(t))\\ &\geqslant \varepsilon\Delta - \varepsilon|u''(t)|\end{aligned}$$

where $(t, \tilde{\theta}_\varepsilon(t))$ is a point between $(t, u(t))$ and $(t, \beta_\varepsilon(t))$ and $(t, \tilde{\theta}_\varepsilon(t)) \in H(u)$ for sufficiently small ε.

The case $u(c) - u(a) \leqslant 0$:

The lower solutions

$$\alpha_\varepsilon(t) = u(t) + v_\varepsilon(t) - \hat{v}_\varepsilon(t) - \Gamma_\varepsilon$$

and the upper solutions

$$\beta_\varepsilon(t) = u(t) + v_\varepsilon(t) + v_\varepsilon^{(\mathrm{corr})}(t) + \hat{v}_\varepsilon(t) + \Gamma_\varepsilon$$

satisfy

$$
\begin{aligned}
\varepsilon\alpha_\varepsilon'' - h(t,\alpha_\varepsilon) &= \varepsilon u'' + \varepsilon v_\varepsilon'' - \varepsilon\hat{v}_\varepsilon'' - \frac{\partial h}{\partial y}(v_\varepsilon - \hat{v}_\varepsilon - \Gamma_\varepsilon) \\
&= \varepsilon u'' + \varepsilon v_\varepsilon'' - \varepsilon\hat{v}_\varepsilon'' + \frac{\partial h}{\partial y}(-v_\varepsilon + \hat{v}_\varepsilon + \Gamma_\varepsilon) \\
&\geqslant \varepsilon u'' + \varepsilon v_\varepsilon'' - \varepsilon\hat{v}_\varepsilon'' + m(-v_\varepsilon + \hat{v}_\varepsilon + \Gamma_\varepsilon) \\
&= \varepsilon u'' + \varepsilon\Delta \geqslant \varepsilon\Delta - \varepsilon|u''| \\
h(t,\beta_\varepsilon) - \varepsilon\beta_\varepsilon'' &= \frac{\partial h}{\partial y}(v_\varepsilon + v_\varepsilon^{(\mathrm{corr})} + \hat{v}_\varepsilon + \Gamma_\varepsilon) - \varepsilon u'' - \varepsilon v_\varepsilon'' - \varepsilon v_\varepsilon^{(\mathrm{corr})''} - \varepsilon\hat{v}_\varepsilon'' \\
&\geqslant (m+2w)v_\varepsilon + m(v_\varepsilon^{(\mathrm{corr})} + \hat{v}_\varepsilon + \Gamma_\varepsilon) - \varepsilon u'' - \varepsilon v_\varepsilon'' - \varepsilon v_\varepsilon^{(\mathrm{corr})''} - \varepsilon\hat{v}_\varepsilon'' \\
&= -2w\,|v_\varepsilon| + m v_\varepsilon^{(\mathrm{corr})} - \varepsilon v_\varepsilon^{(\mathrm{corr})''} + \varepsilon\Delta - \varepsilon u'' = \varepsilon\Delta - \varepsilon u'' \\
&\geqslant \varepsilon\Delta - \varepsilon|u''|.
\end{aligned}
$$

Now, if we choose a constant Δ such that $\Delta \geqslant |u''(t)|$, $t \in \langle a,b\rangle$ then $\varepsilon\alpha_\varepsilon''(t) \geqslant h(t,\alpha_\varepsilon(t))$ and $\varepsilon\beta_\varepsilon''(t) \leqslant h(t,\beta_\varepsilon(t))$ in $\langle a,b\rangle$.

The existence of a solution for (1.1), (1.2) satisfying the above inequality follows from Lemma 1.1. The uniqueness of solutions follows from the fact that the function $h(t,y)$ is increasing in the variable y in $H(u)$ (Peano's phenomenon). □

R e m a r k 2. Theorem 2.1 implies that $y_\varepsilon(t) = u(t) + O(\varepsilon)$ on every compact subset of (a,b) and

$$\lim_{\varepsilon\to 0^+} y_\varepsilon(a) = \lim_{\varepsilon\to 0^+} y_\varepsilon(b) = u(c).$$

The boundary layer effect occurs at the point a or/and b in the case when $u(a) \neq u(c)$ or/and $u(b) \neq u(c)$.

E x a m p l e 1. Consider the linear problem

$$\varepsilon y'' - y = t, \quad t \in \langle 0,2\rangle,\ 0 < \varepsilon \ll 1$$

with the boundary condition

$$y(0) = y(1) = y(2).$$

Its unique solution

$$
\begin{aligned}
y_\varepsilon(t) = {} & \frac{\mathrm{e}^{2\sqrt{1/\varepsilon}} - 2\mathrm{e}^{\sqrt{1/\varepsilon}} + 1}{\mathrm{e}^{4\sqrt{1/\varepsilon}} - 2\mathrm{e}^{3\sqrt{1/\varepsilon}} + 2\mathrm{e}^{\sqrt{1/\varepsilon}} - 1} \cdot \mathrm{e}^{\sqrt{1/\varepsilon}t} \\
& + \frac{-\mathrm{e}^{4\sqrt{1/\varepsilon}} + 2\mathrm{e}^{3\sqrt{1/\varepsilon}} - 2\mathrm{e}^{2\sqrt{1/\varepsilon}}}{\mathrm{e}^{4\sqrt{1/\varepsilon}} - 2\mathrm{e}^{3\sqrt{1/\varepsilon}} + 2\mathrm{e}^{\sqrt{1/\varepsilon}} - 1} \cdot \mathrm{e}^{-\sqrt{1/\varepsilon}t} - t
\end{aligned}
$$

converges (by virtue of Theorem 2.1 and Remark 2) to the solution $u(t) = -t$ of the reduced problem as $\varepsilon \to 0^+$ on the interval $(0, 2)$ and

$$\lim_{\varepsilon \to 0^+} y_\varepsilon(0) = \lim_{\varepsilon \to 0^+} y_\varepsilon(2) = u(1) = -1$$

and

$$\lim_{\varepsilon \to 0^+} y'_\varepsilon(0) = \lim_{\varepsilon \to 0^+} y'_\varepsilon(2) = \infty$$

(the boundary layer phenomenon).

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