On the steady equations for compressible radiative gas

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November 9, 2011

Abstract

We study the equations describing the steady flow of a compressible radiative gas with newtonian rheology. Under suitable assumptions on the data which include the physically relevant situations (i.e. the pressure law for monoatomic gas, the heat conductivity growing with square root of the temperature) we show the existence of a variational entropy solution to the corresponding system of partial differential equations. Under additional restrictions we also show the existence of a weak solution to this problem.

Keywords: radiative gas; variational entropy solution; weak solution; compensated compactness
MS Classification: 76N10, 35Q30

1 Introduction

We consider the following system of partial differential equations in a bounded domain $\Omega \subset \mathbb{R}^3$

$$\operatorname{div}(\varrho \mathbf{u}) = 0 \tag{1.1}$$

$$\operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) - \operatorname{div} \mathbb{S} + \nabla p = \varrho \mathbf{f} - \mathbf{s}_F \tag{1.2}$$

$$\operatorname{div}(\varrho E \mathbf{u}) = \varrho \mathbf{f} \cdot \mathbf{u} - \operatorname{div}(p\mathbf{u}) + \operatorname{div}(\mathbb{S}\mathbf{u}) - \operatorname{div} \mathbf{q} - s_E \tag{1.3}$$

together with the steady radiative transport equation in $\Omega \times \mathcal{S}^2 \times (0, \infty)$

$$\lambda I + \omega \cdot \nabla_{\mathbf{x}} I = S. \tag{1.4}$$

*The work of O.K. was supported by P201/11/1304 of Grant Agency of the Czech Republic as a part of the general research programme of the Academy of Sciences of the Czech Republic, Institutional Research Plan AV0Z10190503.

†The work of Š.N. was supported by Grant Agency of the Czech Republic n. 201/08/0012 as a part of the general research programme of the Academy of Sciences of the Czech Republic, Institutional Research Plan AV0Z10190503.

‡The work of M.P. was supported by P201/11/1304 of Grant Agency of the Czech Republic.

Here $\varrho(\mathbf{x}) \geq 0$ denotes the density of the gas, $\mathbf{u}(\mathbf{x})$ the velocity. Next unknowns are absolute temperature $\vartheta(\mathbf{x}) \geq 0$ appearing in the system implicitly and the radiative intensity $I(\mathbf{x}, \omega, \nu) \geq 0$ depending also on the direction vector $\omega \in \mathcal{S}^2$ and the frequency $\nu \in (0, \infty)$. By $\mathcal{S}^2 \subset \mathbb{R}^3$ we denote the unit sphere, λ is a positive constant.

We assume the stress tensor $\mathbb{S}$ to have the following form

$$\mathbb{S} = \mathbb{S}^\alpha(\vartheta, \nabla\mathbf{u}) = \mu^\alpha(\vartheta)\left[\nabla\mathbf{u} + (\nabla\mathbf{u})^T - \frac{2}{3}\mathrm{div}\,\mathbf{u}\mathbb{I}\right] + \xi^\alpha(\vartheta)\mathrm{div}\,\mathbf{u}\mathbb{I}, \tag{1.5}$$

where μ^α, ξ^α are given functions with properties which will be specified later on. The pressure p is given by the generalized law for monoatomic gas (cf. [26], [27]), namely

$$p(\varrho, \vartheta) = (\gamma - 1)\varrho e(\varrho, \vartheta) \tag{1.6}$$

with e being the specific internal energy and $\gamma > 1$ the heat capacity ratio. The external forces are denoted by $\mathbf{f}$. By E we denote the specific total (nonradiative) energy which is given as a sum of specific kinetic and internal energies

$$E(\varrho, \mathbf{u}, \vartheta) = \frac{1}{2}|\mathbf{u}|^2 + e(\varrho, \vartheta). \tag{1.7}$$

The heat flux $\mathbf{q}$ fulfills the Fourier law

$$\mathbf{q}(\vartheta) = -\kappa(\vartheta)\nabla\vartheta \tag{1.8}$$

with a given function κ whose properties will be stated later on. The right-hand side of the steady transport equation (1.4) is given by

$$S(\mathbf{x}, \omega, \nu) = S_a + S_s = \sigma_a(B - I) + \sigma_s(\widetilde{I} - I). \tag{1.9}$$

Here $\sigma_a(\nu, \vartheta), \sigma_s(\nu, \vartheta)$ and $B(\nu, \vartheta)$ are again given functions with properties stated later on. By $\widetilde{I}$ we denote the following integral mean

$$\widetilde{I}(\mathbf{x}, \nu) = \frac{1}{4\pi}\int_{\mathcal{S}^2} I(\cdot, \omega, \cdot)\,\mathrm{d}\omega. \tag{1.10}$$

Finally, the terms describing the effects of radiation are given by

$$s_E(\mathbf{x}) = \int_{\mathcal{S}^2}\int_0^\infty S(\mathbf{x}, \omega, \nu)\,\mathrm{d}\nu\,\mathrm{d}\omega, \tag{1.11}$$

and

$$\mathbf{s}_F(\mathbf{x}) = \frac{1}{c}\int_{\mathcal{S}^2}\int_0^\infty \omega S(\mathbf{x}, \omega, \nu)\,\mathrm{d}\nu\,\mathrm{d}\omega \tag{1.12}$$

with c denoting the speed of light. We will also use the following notation for the radiation energy

$$E_r(\mathbf{x}) = \frac{1}{c}\int_{\mathcal{S}^2}\int_0^\infty I(\mathbf{x}, \omega, \nu)\,\mathrm{d}\nu\,\mathrm{d}\omega. \tag{1.13}$$

The foundations of the previous system have been extensively described by Pomraning [29] and Mihalas and Weibel-Mihalas [23] in the full framework of special relativity (oversimplified in our present considerations). System (1.1)–(1.4) has been recently investigated (in the inviscid case) by Lowrie, Morel and Hittinger in [21], Buet and Després [1] with a special attention to asymptotic regimes, and by Dubroca and Feugeas in [2], Lin in [19] and Lin, Coulombel and Goudon in [20] for various numerical aspects. Concerning the existence of solutions for evolutionary problem, a proof of local-in-time existence and blow-up of solutions (in the inviscid case) has been recently proposed by Zhong and Jiang [30] (see also the recent papers by Jiang and Wang [16] [17] for a 1D related "Euler-Boltzmann" model), moreover, a simplified version of the system has been investigated by Golse and Perthame [14].

In [4]–[6], the authors derived a one-dimensional version of non-steady version of system (1.1)–(1.4) and studied existence, uniqueness and large time behavior of the system.

Let us now return to theormodynamical assumptions of our model. The Gibbs relation

$$Ds(\varrho,\vartheta) = \frac{1}{\vartheta}\left(De(\varrho,\vartheta) + p(\varrho,\vartheta)D\left(\frac{1}{\varrho}\right)\right) \tag{1.14}$$

gives the specific entropy up to an additive constant. Using (1.1)–(1.3) it is easy to verify that the specific entropy formally fulfills the entropy equality

$$\operatorname{div}(\varrho s\mathbf{u}) + \operatorname{div}\left(\frac{\mathbf{q}}{\vartheta}\right) = \frac{\mathbb{S}:\nabla\mathbf{u}}{\vartheta} - \frac{\mathbf{q}\cdot\nabla\vartheta}{\vartheta^2} + \frac{1}{\vartheta}(\mathbf{u}\cdot\mathbf{s}_F - s_E). \tag{1.15}$$

Note further that relation (1.14) is equivalent with the so–called Maxwell relation

$$\frac{\partial e(\varrho,\vartheta)}{\partial\varrho} = \frac{1}{\varrho^2}\left(p(\varrho,\vartheta) - \vartheta\frac{\partial p(\varrho,\vartheta)}{\partial\vartheta}\right) \tag{1.16}$$

assuming p, e are sufficiently smooth. Relations (1.6) and (1.16) yield the following form of the pressure

$$p(\varrho,\vartheta) = \vartheta^{\frac{\gamma}{\gamma-1}} P\left(\frac{\varrho}{\vartheta^{\frac{1}{\gamma-1}}}\right) \tag{1.17}$$

with P being a nonnegative function. We will further assume

$$\begin{gathered} P \in C^1([0,\infty)) \cap C^2((0,\infty)) \\ P(0) = 0, \quad P'(z) > 0\ \forall z \geq 0, \quad P'(0) = p_0 \\ \lim_{z\to\infty}\frac{P(z)}{z^\gamma} = p_\infty > 0 \\ 0 < \frac{\gamma P(z) - zP'(z)}{(\gamma-1)z} \leq c < \infty\ \forall z > 0. \end{gathered} \tag{1.18}$$

We recall some properties of functions $p(\varrho,\vartheta)$, $e(\varrho,\vartheta)$ and $s(\varrho,\vartheta)$ which are consequences of assumptions (1.6), (1.14)–(1.18). We have for a fixed constant K_0

$$\begin{gathered} c_1\varrho\vartheta \leq p(\varrho,\vartheta) \leq c_2\rho\vartheta, \quad \text{for } \varrho \leq K_0\vartheta^{\frac{1}{\gamma-1}} \\ c_3\varrho^\gamma \leq p(\varrho,\vartheta) \leq c_4 \begin{cases} \vartheta^{\frac{\gamma}{\gamma-1}} & \text{for } \varrho \leq K_0\vartheta^{\frac{1}{\gamma-1}} \\ \varrho^\gamma & \text{for } \varrho > K_0\vartheta^{\frac{1}{\gamma-1}}. \end{cases} \end{gathered} \tag{1.19}$$

Moreover,

$$
\frac{\partial p(\varrho,\vartheta)}{\partial\varrho} > 0 \quad \text{in } (0,\infty)^2
$$

$$
p = d\varrho^\gamma + p_m(\varrho,\vartheta), \quad d>0, \quad \text{with } \frac{\partial p_m(\varrho,\vartheta)}{\partial\varrho} \geq 0 \quad \text{in } (0,\infty)^2. \tag{1.20}
$$

For the internal energy defined by (1.6) we obtain

$$
\frac{1}{\gamma-1} p_\infty \varrho^{\gamma-1} \leq e(\varrho,\vartheta) \leq c_5(\varrho^{\gamma-1}+\vartheta) \quad \text{in } (0,\infty)^2
$$

$$
\frac{\partial e(\varrho,\vartheta)}{\partial\varrho}\varrho \leq c_6(\varrho^{\gamma-1}+\vartheta) \quad \text{in } (0,\infty)^2. \tag{1.21}
$$

The specific entropy defined by the Gibbs relation (1.14) satisfies

$$
\frac{\partial s(\varrho,\vartheta)}{\partial\varrho} = \frac{1}{\vartheta}\left(-\frac{p(\varrho,\vartheta)}{\varrho^2} + \frac{\partial e(\varrho,\vartheta)}{\partial\varrho}\right) = -\frac{1}{\varrho^2}\frac{\partial p(\varrho,\vartheta)}{\partial\vartheta}
$$

$$
\frac{\partial s(\varrho,\vartheta)}{\partial\vartheta} = \frac{1}{\vartheta}\frac{\partial e(\varrho,\vartheta)}{\partial\vartheta} = \frac{1}{\gamma-1}\frac{\vartheta^{\frac{1}{\gamma-1}}}{\varrho}\left(\gamma P\left(\frac{\varrho}{\vartheta^{\frac{1}{\gamma-1}}}\right) - \frac{\varrho}{\vartheta^{\frac{1}{\gamma-1}}}P'\left(\frac{\varrho}{\vartheta^{\frac{1}{\gamma-1}}}\right)\right) > 0. \tag{1.22}
$$

Finally, the specific entropy fulfills

$$
\begin{array}{lcll}
|s(\varrho,\vartheta)| & \leq & c_7(1+|\ln\varrho|+|\ln\vartheta|) & \text{in} \quad (0,\infty)^2 \\
|s(\varrho,\vartheta)| & \leq & c_8(1+|\ln\varrho|) & \text{in} \quad (0,\infty)\times(1,\infty) \\
s(\varrho,\vartheta) & \geq & c_9 > 0 & \text{in} \quad (0,1)\times(1,\infty) \\
s(\varrho,\vartheta) & \geq & c_{10}(1+\ln\vartheta) & \text{in} \quad (0,1)\times(0,1).
\end{array} \tag{1.23}
$$

System of equations (1.1)–(1.4) is supplemented with the following boundary conditions at $\partial\Omega$

$$
\mathbf{u} = \mathbf{0} \tag{1.24}
$$

$$
-\mathbf{q}(\vartheta)\cdot\mathbf{n} + L(\vartheta)(\vartheta-\Theta_0) = 0 \tag{1.25}
$$

$$
I = 0 \quad \text{for } \omega\cdot\mathbf{n} \leq 0, \tag{1.26}
$$

where $\mathbf{n}$ denotes the outer unit normal to $\partial\Omega$, $\Theta_0 > 0$ is a given function at the boundary $\partial\Omega$ and $L(\vartheta)$ is a given function with properties specified later. Finally, we prescribe the total mass of the gas

$$
\int_\Omega \varrho \, \mathrm{d}\mathbf{x} = M > 0. \tag{1.27}
$$

We impose the following conditions on functions appearing in the system (1.1)–(1.4). We assume that L and κ are continuous functions with

$$
0 < c_{11} \leq L(\vartheta) \leq c_{12} \tag{1.28}
$$

$$c_{13}(1+\vartheta^m) \le \kappa(\vartheta) \le c_{14}(1+\vartheta^m) \tag{1.29}$$

for some $m > 0$. We pay special attention to dependence of the viscosity coefficients on temperature. We assume that μ^α and ξ^α are continuous functions with

$$\begin{aligned} c_{15}(1+\vartheta^\alpha) \le \mu^\alpha(\vartheta) &\le c_{16}(1+\vartheta^\alpha) \\ 0 \le \xi^\alpha(\vartheta) &\le c_{17}(1+\vartheta^\alpha) \end{aligned} \tag{1.30}$$

for some $\alpha \in (0,1]$. In the special case $\alpha = 1$ we have

$$\begin{aligned} c_{15}(1+\vartheta) \le \mu^1(\vartheta) \equiv \mu(\vartheta) &\le c_{16}(1+\vartheta) \\ 0 \le \xi^1(\vartheta) \equiv \xi(\vartheta) &\le c_{17}(1+\vartheta). \end{aligned} \tag{1.31}$$

Recall that case $\alpha = 1$ (for fluid without radiation) has been usually considered in previous works in this field, i.e. [26], [27]. The case of general $\alpha \in (0,1)$ is more complicated and we have to use a slightly different approach in deriving a priori estimates.

Moreover we assume that σ_a, σ_s and B are continuous functions with

$$0 \le \sigma_a(\nu,\vartheta), \sigma_s(\nu,\vartheta) \le \min\{c_{18}, c_{19}\vartheta\} \tag{1.32}$$

$$\sigma_a(\nu,\vartheta), \sigma_s(\nu,\vartheta) \le h(\nu), \quad h \in L^1(0,\infty) \cap L^\infty(0,\infty) \tag{1.33}$$

$$\sigma_a(\nu,\vartheta), \sigma_s(\nu,\vartheta), B(\nu,\vartheta) \le c_{20} \tag{1.34}$$

for all $\nu \ge 0$, $\vartheta \ge 0$. Relations (1.32)–(1.34) represent a hypothesis neglecting the effects of radiation at large frequencies ν and low temperatures ϑ.

Next we present notions of solutions to system (1.1)–(1.4).

Definition 1.1. The quadruple $(\varrho, \mathbf{u}, \vartheta, I)$ is called a weak solution to system (1.1)–(1.4), if $\varrho \ge 0$ a.e. in Ω, $\varrho \in L^{\gamma\frac{3p}{4p-3}}(\Omega)$, $\int_\Omega \varrho \mathrm{d}\mathbf{x} = M$, $\mathbf{u} \in W_0^{1,p}(\Omega)$ for some $p \in (1,2]$, $\vartheta > 0$ a.e. in Ω, $\vartheta \in W^{1,r}(\Omega) \cap L^{3m}(\Omega)$, $I \in L^\infty(\Omega \times \mathcal{S}^2 \times (0,\infty))$, moreover $\varrho|\mathbf{u}|^2 \in L^{\frac{3p}{4p-3}}(\Omega)$, $\varrho\mathbf{u}\vartheta \in L^1(\Omega)$, $\mathbb{S}^\alpha(\vartheta, \nabla\mathbf{u})\mathbf{u} \in L^1(\Omega)$, $\vartheta^m\nabla\vartheta \in L^1(\Omega)$ and

$$\int_\Omega \varrho\mathbf{u} \cdot \nabla\psi \,\mathrm{d}\mathbf{x} = 0 \quad \forall \psi \in C^\infty(\overline{\Omega}) \tag{1.35}$$

$$\begin{aligned} \int_\Omega \left(-\varrho(\mathbf{u}\otimes\mathbf{u}) : \nabla\boldsymbol{\varphi} - p(\varrho,\vartheta)\mathrm{div}\,\boldsymbol{\varphi} + \mathbb{S}^\alpha(\vartheta,\nabla\mathbf{u}) : \nabla\boldsymbol{\varphi} + \mathbf{s}_F \cdot \boldsymbol{\varphi}\right) \mathrm{d}\mathbf{x} \\ = \int_\Omega \varrho\mathbf{f} \cdot \boldsymbol{\varphi} \,\mathrm{d}\mathbf{x} \quad \forall \boldsymbol{\varphi} \in C_0^\infty(\Omega) \end{aligned} \tag{1.36}$$

$$\begin{aligned} \int_\Omega -\left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \rho e(\varrho,\vartheta)\right)\mathbf{u}\cdot\nabla\psi \,\mathrm{d}\mathbf{x} &= \int_\Omega \left(\varrho\mathbf{f}\cdot\mathbf{u}\psi + p(\varrho,\vartheta)\mathbf{u}\cdot\nabla\psi\right) \mathrm{d}\mathbf{x} \\ &- \int_\Omega \left((\mathbb{S}^\alpha(\vartheta,\nabla\mathbf{u})\mathbf{u})\cdot\nabla\psi + \kappa(\vartheta)\nabla\vartheta\cdot\nabla\psi - s_E\psi\right) \mathrm{d}\mathbf{x} \\ &- \int_{\partial\Omega} L(\vartheta)(\vartheta - \Theta_0)\psi \,\mathrm{d}S \quad \forall \psi \in C^\infty(\overline{\Omega}) \end{aligned} \tag{1.37}$$

$$\lambda I + \omega \cdot \nabla I = S \quad \text{in } \Omega \times \mathcal{S}^2 \times (0,\infty) \text{ in the sense of distributions.} \tag{1.38}$$

Definition 1.2. The quadruple $(\varrho, \mathbf{u}, \vartheta, I)$ is called a variational entropy solution to system (1.1)–(1.4), if $\varrho \geq 0$ a.e. in Ω, $\varrho \in L^\gamma(\Omega)$, $\int_\Omega \varrho \mathrm{d}\mathbf{x} = M$, $\mathbf{u} \in W_0^{1,p}(\Omega)$ for some $p \in (1,2]$, $\vartheta > 0$ a.e. in Ω, $\vartheta \in W^{1,r}(\Omega) \cap L^{3m}(\Omega)$, $I \in L^\infty(\Omega \times \mathcal{S}^2 \times (0,\infty))$, moreover $\varrho |\mathbf{u}|^2 \in L^1(\Omega)$, $\varrho\vartheta \in L^1(\Omega)$, $\vartheta^{-1}\mathbb{S}^\alpha(\vartheta, \nabla \mathbf{u})\mathbf{u} \in L^1(\Omega)$, $\vartheta^m \frac{\nabla\vartheta}{\vartheta} \in L^1(\Omega)$, $\vartheta^{-1} \in L^1(\partial\Omega)$, the equalities (1.35), (1.36) and (1.38) are satisfied in the same sense as in Definition 1.1 and instead of (1.37) we have the entropy inequality

$$
\begin{aligned}
&\int_\Omega \Big(\frac{\mathbb{S}^\alpha(\vartheta, \nabla\mathbf{u}) : \nabla\mathbf{u}}{\vartheta} + \kappa(\vartheta)\frac{|\nabla\vartheta|^2}{\vartheta^2}\Big)\psi \,\mathrm{d}\mathbf{x} + \int_{\partial\Omega} \frac{L(\vartheta)}{\vartheta}\Theta_0 \psi \,\mathrm{d}S \\
&\leq \int_\Omega \Big(\kappa(\vartheta)\frac{\nabla\vartheta : \nabla\psi}{\vartheta} - \varrho s(\varrho,\vartheta)\mathbf{u}\cdot\nabla\psi - \frac{1}{\vartheta}(\mathbf{u}\cdot\mathbf{s}_F - s_E)\psi\Big)\,\mathrm{d}\mathbf{x} \\
&\qquad + \int_{\partial\Omega} L(\vartheta)\psi \,\mathrm{d}S
\end{aligned} \tag{1.39}
$$

for all nonnegative $\psi \in C^\infty(\overline{\Omega})$ together with the global total energy balance

$$
\int_{\partial\Omega} L(\vartheta)(\vartheta - \Theta_0)\,\mathrm{d}S + \lambda c \int_\Omega E_r \,\mathrm{d}\mathbf{x} + \int_{\partial\Omega,\, \omega\cdot\mathbf{n}>0} \int_{\mathcal{S}^2} \int_0^\infty I\omega\cdot\mathbf{n}\,\mathrm{d}S = \int_\Omega \varrho\mathbf{f}\cdot\mathbf{u}\,\mathrm{d}\mathbf{x}. \tag{1.40}
$$

We introduce an important notion of the renormalized solution to the continuity equation

Definition 1.3. Let $\mathbf{u} \in W^{1,p}_{loc}(\mathbb{R}^3)$ and $\varrho \in L^{\frac{p}{p-1}}_{loc}(\mathbb{R}^3)$ solve

$$
\operatorname{div}(\varrho\mathbf{u}) = 0 \text{ in } \mathcal{D}'(\mathbb{R}^3).
$$

Then the pair $(\varrho, \mathbf{u})$ is called a renormalized solution to the continuity equation, if

$$
\operatorname{div}(b(\varrho)\mathbf{u}) + \big(\varrho b'(\varrho) - b(\varrho)\big)\operatorname{div}\mathbf{u} = 0 \text{ in } \mathcal{D}'(\mathbb{R}^3) \tag{1.41}
$$

for all $b \in C^1([0,\infty)) \cap W^{1,\infty}(0,\infty)$ with $zb'(z) \in L^\infty(0,\infty)$.

The main result of this paper is

Theorem 1.1. *Let $\Omega \in C^2$ be a bounded domain in $\mathbb{R}^3$, $\mathbf{f} \in L^\infty(\Omega)$, $\Theta_0 \geq K_0 > 0$ a.e. at $\partial\Omega$, $\Theta_0 \in L^1(\Omega)$, $M > 0$. Moreover, let*

$$
\alpha \in (0,1]
$$

$$
\begin{aligned}
&\gamma > \max\Big\{\frac{3}{2}, 1 + \frac{1-\alpha}{6\alpha} + \frac{1}{2}\sqrt{\frac{4(1-\alpha)}{3\alpha} + \frac{(1-\alpha)^2}{9\alpha^2}}\Big\} \\
&m > \max\Big\{1-\alpha, \frac{1+\alpha}{3}, \frac{\gamma(1-\alpha)}{2\gamma-3}, \frac{\gamma(1-\alpha)^2}{3(\gamma-1)^2\alpha - \gamma(1-\alpha)}, \\
&\qquad \frac{1-\alpha}{6(\gamma-1)\alpha - 1}, \frac{1+\alpha+\gamma(1-\alpha)}{3(\gamma-1)}\Big\}.
\end{aligned} \tag{1.42}
$$

Then there exists a variational entropy solution to system (1.1)–(1.34) in the sense of Definition 1.2. Moreover, the pair $(\varrho, \mathbf{u})$ is a renormalized solution to the continuity equation in the sense of Definition 1.3.

If additionally

$$
(1.43)\qquad \begin{aligned} &\gamma > \max\Big\{\frac{5}{3}, \frac{2+\alpha}{3\alpha}\Big\} \\ &m > \max\Big\{1, \frac{(3\gamma-1)(1-\alpha)}{3\gamma-5}, \frac{(3\gamma-1)(1-\alpha)+2}{3(\gamma-1)}, \frac{(1-\alpha)(\gamma(2-3\alpha)+\alpha)}{\alpha(6\gamma^2-9\gamma+5)-2\gamma}\Big\}, \end{aligned}
$$

then this solution is a weak solution in the sense of Definition 1.1.

Recall that system (1.1)–(1.4) (without the radiation terms s_E and $\mathbf{s}_F$, but with additional term modelling the radiation $\sim \vartheta^4$ in the pressure), i.e. the heat conducting compressible Newtonian fluid, has been recently considered in [11] for $\gamma = 5/3$, i.e. for the model of the monoatomic gas. The same problem as in this paper was considered in the evolutionary case in [3]; unfortunately, even though the authors consider the evolutionary model for the radiative transport equation in the form similar to (1.4), due to technical reasons, they had to assume the pressure law of the form $p(\varrho,\vartheta) = p_0(\varrho,\vartheta)+c\vartheta^4$, where p_0 is the pressure from our paper and the last term represents another model of radiation. In our paper for the steady system, we may remove this term and get the corresponding result only for the radiation as in (1.4).

Similarly as the results in the steady case below, the approach to treat the compressible equations goes back to the pioneering seminal work of P.L. Lions [22]. The reader may consult also [9] or [28] and references quoted there for more details.

In the steady isentropic case (i.e. $p = p(\varrho) \sim \varrho^\gamma$), the first result can be found in [22]; there, the existence of a weak solution was shown for $\gamma > \frac{5}{3}$. Based on the method developed by E. Feireisl, an alternative proof is given in [28] and introduces a technique allowing to treat also the case $\gamma < \frac{5}{3}$ provided the a priori estimates are available. After a series of improvements of the a priori estimates, in [12] the authors gave an existence result for the homogeneous Dirichlet conditions for $\gamma > 4/3$. A new technique, which improved the existence for any $\gamma > 1$ in case of space-periodic boundary conditions was introduced in [18]; a generalization of this method allowing to treat also the slip boundary conditions can be found in [15].

In the last few years, a significant progress has been also done in the steady problem for the heat conducting fluid. The first result for large data goes again back to P.L. Lions [22], however, only under additional a priori assumption that ϱ is bounded in $L^p(\Omega)$ for p sufficiently large. The heat conducting fluid with only $\varrho \in L^1(\Omega)$ a priori was studied for the first time in [24] for $p(\varrho,\vartheta) = \varrho^\gamma + \varrho\vartheta$ with $\gamma > 3$ and $m > \frac{3\gamma-1}{3\gamma-7}$, with Navier (slip) boundary conditions for the velocity. In this case, one can get $\varrho \in L^\infty(\Omega)$ and $\mathbf{u}$, $\vartheta \in W^{1,p}(\Omega)$, $\forall p < \infty$. In the next paper [25], the authors showed the existence of a weak solution for $\gamma > \frac{7}{3}$ with both slip and no-slip boundary conditions for velocity. In these two papers, the viscosities were independent of the temperature which corresponds to $\alpha = 0$. In [26], [27] the authors observed that for $\alpha = 1$ much better a priori estimates are available and showed the existence of an entropy variational solution for $\gamma > \frac{3}{2}$, $m > \max\big\{\frac{2}{3}, \frac{2}{3(\gamma-1)}\big\}$ (based on the estimates for the density via Bogovskii-type estimates) and for $\gamma > \frac{3+\sqrt{41}}{8}$, $m > \max\big\{\frac{2}{3}, \frac{2}{3(\gamma-1)}, \frac{2}{9}\frac{\gamma(4\gamma-1)}{4\gamma^2-3\gamma-2}\big\}$ (for estimates of the density based on the technique of Frehse, Steinhauer and Weigant). These solutions are weak ones (i.e. fulfill also the weak formulation of the total energy) provided $\gamma > \frac{5}{3}$ and $m > 1$ (Bogovskii-type estimates) and $\gamma > \frac{4}{3}$, $m > \max\{1, \frac{2}{3}\frac{\gamma}{3\gamma-4}\}$ (Frehse-Steinahuer-Weigant-type estimates). Note that in our paper, for $\alpha < 1$, we are able to use only the Bogovskii-type estimates. The problem how to adapt the technique of the local estimates of the pressure to the case $\alpha < 1$ is one of the interesting open problems.

2 Approximative system

We will use four-level approximative system with parameters $N \to \infty$ (denoting the dimension of space of the Galerkin approximations), $\eta \to 0^+$ (denoting the mollification and truncation of the stress tensor), $\varepsilon \to 0^+$ (denoting the elliptic regularization of the continuity equation) and $\delta \to 0^+$ (denoting the artificial pressure constant). This kind of approximation is standard in this area. Let us first recall that we have (see e.g. [26, Lemma 2])

Lemma 2.1. *Let $\mathbf{u} \in W_0^{1,2}(\Omega)$, $\vartheta > 0$ and $\mathbb{S}^\alpha(\vartheta, \nabla\mathbf{u})$ satisfies (1.5) with (1.30). Then*

$$\int_\Omega \mathbb{S}^\alpha(\vartheta, \mathbf{u}) : \nabla\mathbf{u}\,\mathrm{dx} \geq C\,\|\mathbf{u}\|_{1,2}^2 . \tag{2.1}$$

We proceed with introducing the approximative scheme and prove existence of solutions to this system. For this reason, let us fix $N \in \mathbb{N}$ and $\eta, \varepsilon, \delta > 0$, and denote

$$X^N = \mathrm{span}\left\{\mathbf{w}^1, ..., \mathbf{w}^N\right\} \subset W_0^{1,2}(\Omega) \tag{2.2}$$

with $\{\mathbf{w}^i\}_{i=1}^\infty$ being an orthonormal basis of $W_0^{1,2}(\Omega)$ such that $\mathbf{w}^i \in W^{2,q}(\Omega)$ for all $q \in [1,\infty)$. We look for $(\varrho, \mathbf{u}, \vartheta, I)$[1] such that $\varrho \in W^{2,q}(\Omega)$, $\mathbf{u} \in X^N$, $\vartheta \in W^{2,q}(\Omega)$ and $I \in L^\infty(\Omega \times \mathcal{S}^2 \times (0,\infty))$ with $q \in [1,\infty)$ arbitrary, where the following set of equations holds:

$$\begin{aligned} &\int_\Omega \Big(\frac{1}{2}\varrho(\mathbf{u}\cdot\nabla\mathbf{u})\cdot\mathbf{w}^i - \frac{1}{2}\varrho(\mathbf{u}\otimes\mathbf{u}) : \nabla\mathbf{w}^i + \mathbb{S}^\alpha_\eta(\vartheta, \nabla\mathbf{u}) : \nabla\mathbf{w}^i\Big)\,\mathrm{dx} \\ &- \int_\Omega \big(p(\varrho,\vartheta) + \delta(\varrho^\beta + \varrho^2)\big)\,\mathrm{div}\,\mathbf{w}^i\,\mathrm{dx} = \int_\Omega \big(\varrho\mathbf{f}\cdot\mathbf{w}^i - \mathbf{s}_F\cdot\mathbf{w}^i\big)\,\mathrm{dx} \end{aligned} \tag{2.3}$$

for all $i = 1, ..., N$,

$$\varepsilon\varrho - \varepsilon\Delta\varrho + \mathrm{div}\,(\varrho\mathbf{u}) = \varepsilon h \quad \text{a.e. in } \Omega \tag{2.4}$$

$$\begin{aligned} &-\mathrm{div}\,\Big(\big(\kappa_\eta(\vartheta) + \delta\vartheta^B + \delta\vartheta^{-1}\big)\frac{\varepsilon+\vartheta}{\vartheta}\nabla\vartheta\Big) + \mathrm{div}\,(\varrho e(\varrho,\vartheta)\mathbf{u}) \\ &= \mathbb{S}^\alpha_\eta(\vartheta, \nabla\mathbf{u}) : \nabla\mathbf{u} + \delta\vartheta^{-1} - p(\varrho,\vartheta)\mathrm{div}\,\mathbf{u} + \delta\varepsilon\,|\nabla\varrho|^2\,(\beta\varrho^{\beta-2} + 2) \\ &\quad -(s_E - \mathbf{s}_F\cdot\mathbf{u}) \quad \text{a.e. in } \Omega \end{aligned} \tag{2.5}$$

and

$$\lambda I + \omega\cdot\nabla I = S, \tag{2.6}$$

[1]For simplicity of notation we skip denoting dependence on parameters $N, \eta, \varepsilon, \delta$.

with β and B large enough,

$$\mathbb{S}_\eta^\alpha(\vartheta,\nabla\mathbf{u}) = \frac{\mu_\eta^\alpha(\vartheta)}{1+\eta\vartheta^\alpha}\left[\nabla\mathbf{u}+(\nabla\mathbf{u})^T-\frac{2}{3}\mathrm{div}\,\mathbf{u}\mathbb{I}\right]+\frac{\xi_\eta^\alpha(\vartheta)}{1+\eta\vartheta^\alpha}\mathrm{div}\,\mathbf{u}\mathbb{I}, \tag{2.7}$$

and where $h=\frac{M}{|\Omega|}$, μ_η^α, ξ_η^α and κ_η are suitable regularizations of μ^α, ξ^α and κ, that conserve properties (1.29) and (1.30) and converge uniformly on compact subsets of $[0,\infty)$ to μ^α, ξ^α and κ, respectively. We add to system (2.3)–(2.6) boundary conditions at $\partial\Omega$

$$\frac{\partial\varrho}{\partial\mathbf{n}}=0 \tag{2.8}$$

$$(\kappa_\eta(\vartheta)+\delta\vartheta^B+\delta\vartheta^{-1})\frac{\varepsilon+\vartheta}{\vartheta}\frac{\partial\vartheta}{\partial\mathbf{n}}+(L+\delta\vartheta^{B-1})(\vartheta-\Theta_0^\eta)+\varepsilon\ln\vartheta=0. \tag{2.9}$$

Here Θ_0^η is a strictly positive smooth approximation of Θ_0.

Theorem 2.2. *Let $N\in\mathbb{N}$, $\eta,\varepsilon,\delta>0$, let moreover β and B be large enough, ε sufficiently small with respect to δ. Then under assumptions of Theorem 1.1 and assumptions made above in this section, there exists a solution to system (2.3)–(2.9) such that $\varrho\in W^{2,q}(\Omega),\forall q<\infty$, $\varrho\geq 0$ in Ω, $\int_\Omega\varrho\,\mathrm{d}\mathbf{x}=M$, $\mathbf{u}\in X_N$, $\vartheta\in W^{2,q}(\Omega),\forall q<\infty$, $\vartheta>C(N)>0$ and $I\in L^\infty(\Omega\times\mathcal{S}^2\times(0,\infty))$.*

The proof is basically analogous to proof of a similar theorem in [26], thus we will present only the main ideas and give more details in the proof of a priori estimates and solvability to (2.6). We consider mapping

$$\mathcal{T}:\ X_N\times W^{2,q}(\Omega)\to X_N\times W^{2,q}(\Omega), \tag{2.10}$$

where

$$\mathcal{T}(\mathbf{v},z)=(\mathbf{u},r), \tag{2.11}$$

defined in the following way. For a given $\mathbf{v}$ we first find ϱ as a (unique) solution to

$$\begin{aligned}\varepsilon\varrho-\varepsilon\Delta\varrho+\mathrm{div}\,(\varrho\mathbf{v})&=\varepsilon h\quad\text{in }\Omega\\ \frac{\partial\varrho}{\partial\mathbf{n}}&=0\quad\text{at }\partial\Omega\end{aligned} \tag{2.12}$$

and for given z we find I as a solution to

$$\begin{aligned}&\lambda I+(\sigma_a(\nu,\mathrm{e}^z)+\sigma_s(\nu,\mathrm{e}^z))I+\omega\cdot\nabla I=\\ &=\sigma_a(\nu,\mathrm{e}^z)B(\nu,\mathrm{e}^z)+\sigma_s(\nu,\mathrm{e}^z)\widetilde{I}\quad\text{in }\Omega\times\mathcal{S}^2\times(0,\infty)\\ &I=0\quad\text{at }\partial\Omega,\omega\cdot\mathbf{n}\leq 0.\end{aligned} \tag{2.13}$$

Finally, we find $\mathbf{u}$ as a solution to

$$\begin{aligned}\int_\Omega\mathbb{S}_\eta^\alpha(\mathrm{e}^z,\nabla\mathbf{u}):\nabla\mathbf{w}^i\,\mathrm{d}\mathbf{x}&=\int_\Omega\Big(\frac{1}{2}\varrho(\mathbf{v}\otimes\mathbf{v}):\nabla\mathbf{w}^i-\frac{1}{2}\varrho(\mathbf{v}\cdot\nabla\mathbf{v})\cdot\mathbf{w}^i\Big)\,\mathrm{d}\mathbf{x}\\ &+\int_\Omega\big((p(\varrho,\mathrm{e}^z)+\delta(\varrho^\beta+\varrho^2))\mathrm{div}\,\mathbf{w}^i+\varrho\mathbf{f}\cdot\mathbf{w}^i-\mathbf{s}_F\cdot\mathbf{w}^i\big)\,\mathrm{d}\mathbf{x}\end{aligned} \tag{2.14}$$

for all $i = 1, ..., N$, and r as a solution to

$$(2.15)\qquad \begin{aligned} &-\mathrm{div}\,\big((\kappa_\eta(\mathrm{e}^z) + \delta \mathrm{e}^{zB} + \delta \mathrm{e}^{-z})(\varepsilon + \mathrm{e}^z)\nabla r\big) = -\mathrm{div}\,(\varrho e(\varrho, \mathrm{e}^z)\mathbf{v}) \\ &+\mathbb{S}^\alpha_\eta(\mathrm{e}^z, \nabla \mathbf{v}) : \nabla \mathbf{v} + \delta \mathrm{e}^{-z} - p(\varrho, \mathrm{e}^z)\mathrm{div}\,\mathbf{v} + \delta\varepsilon\, |\nabla \varrho|^2 \,(\beta \varrho^{\beta-2} + 2) \\ &\qquad -(s_E - \mathbf{s}_F \cdot \mathbf{v}) \quad \text{a.e. in } \Omega \end{aligned}$$

with boundary condition at $\partial\Omega$

$$(2.16)\qquad (\kappa_\eta(\mathrm{e}^z) + \delta \mathrm{e}^{zB} + \delta \mathrm{e}^{-z})(\varepsilon + \mathrm{e}^z)\frac{\partial r}{\partial \mathbf{n}} + (L + \delta \mathrm{e}^{z(B-1)})(\mathrm{e}^z - \Theta^\eta_0) + \varepsilon r = 0.$$

Note that possible fixed points of $\mathcal{T}$ correspond to $r = \ln \vartheta$ in the approximative system (2.3)–(2.6).

We use the following version of the Schauder fixed point theorem to prove existence of fixed points of $\mathcal{T}$.

Theorem 2.3. *Let $\mathcal{T}\colon X \to X$ be a continuous, compact mapping and let X be a Banach space. Let for any $t \in [0, 1]$ the fixed points $t\mathcal{T}u = u$ be bounded. Then $\mathcal{T}$ possesses at least one fixed point in X.*

Proof. The proof can be found e.g. in [8, Theorem 9.2.4]. □

For fixed $\mathbf{v} \in X_N$, we can find unique solution to the approximative continuity equation (2.12). We have

Lemma 2.4. *Let $\varepsilon > 0$, $h = \frac{M}{|\Omega|}$. Let $\mathbf{v} \in X_N$. Then there exists unique solution to (2.12) such that $\varrho \in W^{2,p}(\Omega)$ for all $p < \infty$, $\int_\Omega \varrho\,\mathrm{d}\mathbf{x} = M$ and $\varrho \geq 0$ in Ω. Moreover, the mapping $S\colon \mathbf{v} \mapsto \varrho$ is continuous and compact from X_N to $W^{2,p}(\Omega)$.*

Proof. For proof of this lemma see e.g. [28]. □

One of the parts which deserve more detailed explanation is the radiative transport equation. First we need some compactness of the averages over sphere $\mathcal{S}^2$ to pass to the limit in the radiative terms. For that purpose we recall the result of Golse et al. [13, Theorem 4].

Lemma 2.5. *Let $I \in L^p(\Omega \times \mathcal{S}^2 \times (0, \infty))$ and $\omega \cdot \nabla_x I \in L^p(\Omega \times \mathcal{S}^2 \times (0, \infty))$ for some $1 < p < \infty$. Then*

$$(2.17)\qquad \widetilde{I} \equiv \frac{1}{4\pi}\int_{\mathcal{S}^2} I(\cdot, \omega, \cdot)\,\mathrm{d}\omega$$

belongs to the space $L^p((0, \infty); W^{s,p}(\Omega))$ for any $0 < s < \min\{\frac{1}{p}, 1 - \frac{1}{p}\}$, and

$$(2.18)\qquad \|\widetilde{I}(\cdot, \nu)\|_{W^{s,p}(\Omega)} \leq C(\|I(\cdot, \cdot, \nu)\|_{L^p(\Omega\times\mathcal{S}^2)} + \|\omega \cdot \nabla_x I(\cdot, \cdot, \nu)\|_{L^p(\Omega\times\mathcal{S}^2)}).$$

We have

Lemma 2.6. *Let $z \in L^p(\Omega)$ for some $p \geq 1$. Then there exists a solution to (2.13) such that $I \in L^\infty(\Omega \times \mathcal{S}^2 \times (0,\infty))$. Moreover, $I \geq 0$, and*

$$\begin{aligned}\lambda \int_\Omega \int_{\mathcal{S}^2} I\psi \,\mathrm{d}\omega \,\mathrm{d}\mathbf{x} - \int_\Omega \int_{\mathcal{S}^2} I\omega \cdot \nabla\psi \,\mathrm{d}\omega \,\mathrm{d}\mathbf{x} + \int_{\{\mathbf{x}\in\partial\Omega;\omega\cdot\mathbf{n}>0\}} \int_{\mathcal{S}^2} I\omega \cdot \mathbf{n}\psi \,\mathrm{d}\omega \,\mathrm{d}S \\ = \int_\Omega \int_{\mathcal{S}^2} S\psi \,\mathrm{d}\omega \,\mathrm{d}\mathbf{x}\end{aligned} \tag{2.19}$$

for any $\psi \in C^1(\overline{\Omega})$.

Proof. The existence of a solution is achieved by elliptic approximation, i.e. we add term $-\alpha\Delta I$ to the left-hand side of the equation with a Neumann boundary condition to get

$$\begin{aligned}\lambda I_\alpha + (\sigma_a(\nu, \mathrm{e}^z) + \sigma_s(\nu, \mathrm{e}^z))I_\alpha + \omega \cdot \nabla I_\alpha - \alpha\Delta I_\alpha \\ = \sigma_a(\nu, \mathrm{e}^z)B(\nu, \mathrm{e}^z) + \sigma_s(\nu, \mathrm{e}^z)\widetilde{I}_\alpha \quad \text{in } \Omega \times \mathcal{S}^2 \times (0,\infty) \\ I_\alpha = 0 \quad \text{at } \partial\Omega, \omega \cdot \mathbf{n} \leq 0 \\ \frac{\partial I_\alpha}{\partial n} = 0 \quad \text{at } \partial\Omega, \omega \cdot \mathbf{n} > 0.\end{aligned} \tag{2.20}$$

Standard elliptic theory (more precisely, the Fredholm theory) yields the existence of a solution to the approximated radiative transfer equation (2.20), at least for a sequence $\alpha_n \to 0^+$. Next step is deriving a priori estimates independent of α and passing with α to zero. We proceed as follows. Multiplying $(2.20)_1$ by I_α^{b-1} with $b > 1$, integrating over $\mathcal{S}^2$ and using Hölder inequality we get

$$\lambda \int_{\mathcal{S}^2} I_\alpha^b \,\mathrm{d}\omega + \sigma_a \int_{\mathcal{S}^2} I_\alpha^b \,\mathrm{d}\omega + \frac{1}{b} \int_{\mathcal{S}^2} \omega \cdot \nabla(I_\alpha^b) \,\mathrm{d}\omega - \alpha \int_{\mathcal{S}^2} I_\alpha^{b-1} \Delta I_\alpha \,\mathrm{d}\omega \leq \sigma_a B \int_{\mathcal{S}^2} I_\alpha^{b-1} \,\mathrm{d}\omega. \tag{2.21}$$

Next we integrate over Ω to get

$$\begin{aligned}\lambda \int_\Omega \int_{\mathcal{S}^2} I_\alpha^b \,\mathrm{d}\omega \,\mathrm{d}\mathbf{x} + \int_\Omega \int_{\mathcal{S}^2} \sigma_a I_\alpha^b \,\mathrm{d}\omega \,\mathrm{d}\mathbf{x} + \alpha \frac{4(b-1)}{b^2} \int_\Omega \int_{\mathcal{S}^2} \left|\nabla(I_\alpha^{\frac{b}{2}})\right|^2 \mathrm{d}\omega \,\mathrm{d}\mathbf{x} \\ \leq \int_\Omega \int_{\mathcal{S}^2} \sigma_a B I_\alpha^{b-1} \,\mathrm{d}\omega \,\mathrm{d}\mathbf{x},\end{aligned} \tag{2.22}$$

where we have used also integration by parts and boundary conditions $(2.20)_2$ and $(2.20)_3$. Finally, integrating $\mathrm{d}\nu$ over $(0,\infty)$ and using (1.33), Hölder's and Young's inequalities we get

$$\|I_\alpha\|_{L^b(\Omega\times\mathcal{S}^2\times(0,\infty))}^b + \alpha \frac{4(b-1)}{b^2} \left\|\nabla I_\alpha^{\frac{b}{2}}\right\|_{L^2(\Omega\times\mathcal{S}^2\times(0,\infty))}^2 \leq C(\lambda, \Omega) \|h\|_{L^b(0,\infty)}^b \tag{2.23}$$

and thus in particular

$$\|I_\alpha\|_{L^b(\Omega\times\mathcal{S}^2\times(0,\infty))} \leq C(\lambda, \Omega, h) \tag{2.24}$$

with the constant C independent of b; whence

$$\|I_\alpha\|_{L^\infty(\Omega\times\mathcal{S}^2\times(0,\infty))} \leq C. \tag{2.25}$$

To pass to the limit with $\alpha_n \to 0^+$ in equation (2.20) it is enough to use $b = 2$. For the limit I we also have

$$\|I\|_{L^\infty(\Omega\times\mathcal{S}^2\times(0,\infty))} \leq C, \tag{2.26}$$

and moreover, using (1.32)–(1.34), we get

$$\|s_E\|_{L^\infty(\Omega)} + \|\mathbf{s}_F\|_{L^\infty(\Omega)} \leq C, \tag{2.27}$$

and

$$\Big\|\frac{1}{\vartheta}s_E\Big\|_{L^\infty(\Omega)} + \Big\|\frac{1}{\vartheta}\mathbf{s}_F\Big\|_{L^\infty(\Omega)} \leq C. \tag{2.28}$$

Note that the constants C in (2.25)–(2.28) are independent of any approximation parameters.

The non-negativity of I is a direct consequence of the maximum principle for (2.20), the uniqueness follows from the linearity and limit passage in (2.22). Finally, to show (2.19), we multiply $(2.20)_1$ by a smooth function $\psi \in C^1(\overline{\Omega})$, perform the integration by parts and pass with $\alpha_n \to 0^+$. At this step we apply Lemma 2.5 to ensure the existence of the trace of $\widetilde{I}$ at $\partial\Omega$. The lemma is proved. □

For the operator $\mathcal{T}$ it holds

Lemma 2.7. *Under the assumptions of Theorem 2.2, for $q > 3$, the operator $\mathcal{T}$ is a continuous and compact operator from $X_N \times W^{2,p}(\Omega)$ into itself.*

Proof. The proof is a straightforward application of a Lax–Milgram theorem as the right-hand sides of the equations (2.14), (2.15) as well as the boundary terms in (2.16) are sufficiently smooth and of lower order. The continuity of the operator is standard. □

Finally we have to prove the following

Lemma 2.8. *Let assumptions of Theorem 2.2 be satisfied. Let $q > 3$. Then there exists $C > 0$ such that all solutions to*

$$t\mathcal{T}(\mathbf{u}, r) = (\mathbf{u}, r) \tag{2.29}$$

fulfill

$$\|\mathbf{u}\|_{1,2} + \|r\|_{2,q} + \|\vartheta\|_{2,q} \leq C, \tag{2.30}$$

where $\vartheta = \mathrm{e}^r$ and C is independent of $t \in [0, 1]$.

As the proof is analogous to the one presented in [26] in the case of model without radiation we just summarize the basic ideas and point out the differences related to radiative terms in the equations. We proceed as follows:

- Test the approximative momentum equation (2.3) by $\mathbf{u}$ which is a suitable combination of $\mathbf{w}^i$:

$$\int_\Omega \mathbb{S}_\eta^\alpha(\vartheta, \mathbf{u}) : \nabla\mathbf{u}\,\mathrm{d}\mathbf{x} = t\int_\Omega \Big(\big(p(\varrho,\vartheta) + \delta(\varrho^\beta + \varrho^2)\big)\operatorname{div}\mathbf{u} + \varrho\mathbf{f} - \mathbf{s}_F\cdot\mathbf{u}\Big)\,\mathrm{d}\mathbf{x}. \tag{2.31}$$

- Integrate the approximative internal energy balance (2.5) over Ω and use boundary condition (2.9) as well as (2.31):

$$
\begin{aligned}
&\int_{\partial\Omega} \Big(t\big(L+\delta\vartheta^{B-1}\big)(\vartheta-\Theta_0^\eta)+\varepsilon\ln\vartheta\Big)\,\mathrm{d}S \\
&+(1-t)\int_\Omega \mathbb{S}_\eta(\vartheta,\mathbf{u}):\nabla\mathbf{u}\,\mathrm{d}\mathbf{x}+\varepsilon\delta t\int_\Omega\Big(\frac{\beta}{\beta-1}\varrho^\beta+2\varrho^2\Big)\,\mathrm{d}\mathbf{x} \\
&=t\int_\Omega\Big(\varrho\mathbf{f}\cdot\mathbf{u}+\varepsilon\delta\frac{\beta}{\beta-1}h\varrho^{\beta-1}+2\varepsilon\delta h\varrho-s_E+\delta\vartheta^{-1}\Big)\,\mathrm{d}\mathbf{x}.
\end{aligned}
\tag{2.32}
$$

- Derive the entropy version of the approximative energy balance, i.e. divide (2.5) by ϑ, and integrate over Ω:

(2.33)
$$
\begin{aligned}
&\int_\Omega\big(\kappa_\eta(\vartheta)+\delta\vartheta^B+\delta\vartheta^{-1}\big)\frac{\varepsilon+\vartheta}{\vartheta}\frac{|\nabla\vartheta|^2}{\vartheta^2}\,\mathrm{d}\mathbf{x}+t\int\Big(\frac{1}{\vartheta}\mathbb{S}_\eta(\vartheta,\mathbf{u}):\nabla\mathbf{u}+\delta\vartheta^{-2}\Big)\,\mathrm{d}\mathbf{x} \\
&+\int_{\partial\Omega}\frac{1}{\vartheta}\Big(t\big(L+\delta\vartheta^{B-1}\big)\Theta_0^\eta-\varepsilon\ln\vartheta\Big)\,\mathrm{d}S+t\varepsilon\delta\int_\Omega\frac{1}{\vartheta}|\nabla\varrho|^2\big(\beta\varrho^{\beta-2}+2\big)\,\mathrm{d}\mathbf{x} \\
&\le t\int_{\partial\Omega}\big(L+\delta\vartheta^{B-1}\big)\,\mathrm{d}S+t\frac{\varepsilon}{2}\frac{\beta}{\beta-1}\int_\Omega\big(\varrho^\beta+\vartheta^{-1}(s_E-\mathbf{s}_F\cdot\mathbf{u})\big)\,\mathrm{d}\mathbf{x}+Ct\varepsilon.
\end{aligned}
$$

- Combine arising identities together with the approximative continuity equation (2.4) (tested by suitable powers of ϱ) and properties of $p(\varrho,\vartheta), e(\varrho,\vartheta)$ and $s(\varrho,\vartheta)$. To handle the radiative terms we use (2.27) and (2.28) and we end up with:

(2.34)
$$
\begin{aligned}
&\int_\Omega\big(\kappa_\eta(\vartheta)+\delta\vartheta^B+\delta\vartheta^{-1}\big)\frac{\varepsilon+\vartheta}{\vartheta}\frac{|\nabla\vartheta|^2}{\vartheta^2}\,\mathrm{d}\mathbf{x}+t\int_\Omega\Big(\frac{1}{\vartheta}\mathbb{S}_\eta(\vartheta,\mathbf{u}):\nabla\mathbf{u}+\delta\vartheta^{-2}\Big)\,\mathrm{d}\mathbf{x} \\
&\quad+(1-t)\int_\Omega\mathbb{S}_\eta(\vartheta,\mathbf{u}):\nabla\mathbf{u}\,\mathrm{d}\mathbf{x}+\frac{1}{2}\varepsilon\delta t\int_\Omega\Big(\frac{\beta}{\beta-1}\varrho^\beta+2\varrho^2\Big)\,\mathrm{d}\mathbf{x} \\
&\quad+t\varepsilon\delta\int_\Omega\frac{1}{\vartheta}|\nabla\varrho|^2(\beta\varrho^{\beta-2}+2)\,\mathrm{d}\mathbf{x} \\
&+\int_{\partial\Omega}\Big(t\big(L\vartheta+\delta\vartheta^B\big)+\varepsilon|\ln\vartheta|+t\frac{\Theta_0^\eta}{\vartheta}L\Big)\,\mathrm{d}S\le Ct\Big(1+\Big|\int_\Omega(\varrho\mathbf{f}\cdot\mathbf{u}+|\mathbf{u}|)\,\mathrm{d}\mathbf{x}\Big|\Big).
\end{aligned}
$$

- Estimate (2.34) immediately yields:

$$
\|\mathbf{u}\|_{1,2}+\|\vartheta\|_{3B}+\|\vartheta\|_{1,2}+\|\varrho\|_\beta+\|I\|_\infty\le C
\tag{2.35}
$$

with the constant C independent of t (and of N and η).

- Use properties of space X_N and regularity results for elliptic equations to get:

$$
\|\mathbf{u}\|_{2,q}+\|\varrho\|_{2,q}\le C(N).
\tag{2.36}
$$

- Finally use Kirchhoff transform in (2.5) to end up with:

$$
\|r\|_{2,q}+\|\vartheta\|_{2,q}\le C(N).
\tag{2.37}
$$

3 First limit passages

We pass subsequently with $N \to \infty$, $\eta \to 0^+$ and $\varepsilon \to 0^+$. As these limit passages (except for the strong convergence of the density in the last case) are relatively easy, we will only shortly comment on certain difficulties and rather concentrate ourselves on the more complex passage $\delta \to 0^+$ in the last two sections.

3.1 Limit passage $N \to \infty$

From (2.34), (2.25) and the standard regularity results for the elliptic problem (2.4) we get uniform estimates

$$
\begin{aligned}
&\|\mathbf{u}_N\|_{1,2} + \|\varrho_N\|_\beta + \|\vartheta_N\|_{3B} + \|\vartheta_N\|_{1,2} + \left\|\vartheta_N^{-2}\right\|_1 + \left\|\vartheta_N^{-1}\right\|_{1,\partial\Omega} \\
&\quad + \left\|\vartheta_N^{-4}\left|\nabla\vartheta_N\right|^2\right\|_1 + \|\varrho_N\|_{2,2} + \|I_N\|_{\infty,\Omega\times\mathcal{S}^2\times(0,\infty)} \le C(\varepsilon,\delta).
\end{aligned} \tag{3.1}
$$

Therefore we can extract subsequences from ϱ_{N_k}, $\mathbf{u}_{N_k}$, ϑ_{N_k} and I_{N_k} converging weakly to ϱ, $\mathbf{u}$, ϑ and I, respectively, in spaces given by estimates (3.1). This allows us to pass to the limit with $N \to \infty$ in the approximative system and get

$$
\begin{aligned}
&\int_\Omega \Big(\frac{1}{2}\varrho(\mathbf{u}\cdot\nabla\mathbf{u})\cdot\boldsymbol{\varphi} - \frac{1}{2}\varrho(\mathbf{u}\otimes\mathbf{u}) : \nabla\boldsymbol{\varphi} + \mathbb{S}^\alpha_\eta(\vartheta,\mathbf{u}) : \nabla\boldsymbol{\varphi}\Big)\,\mathrm{d}\mathbf{x} \\
&- \int_\Omega \big(p(\varrho,\vartheta) + \delta(\varrho^\beta + \varrho^2)\big)\mathrm{div}\,\boldsymbol{\varphi}\,\mathrm{d}\mathbf{x} = \int_\Omega \left(\varrho\mathbf{f}\cdot\boldsymbol{\varphi} - \mathbf{s}_F\cdot\boldsymbol{\varphi}\right)\,\mathrm{d}\mathbf{x}
\end{aligned} \tag{3.2}
$$

for all $\boldsymbol{\varphi} \in W^{1,2}_0(\Omega)$,

$$
\varepsilon\varrho - \varepsilon\Delta\varrho + \mathrm{div}\,(\varrho\mathbf{u}) = \varepsilon h \quad \text{a.e. in } \Omega, \tag{3.3}
$$

with boundary condition

$$
\frac{\partial\varrho}{\partial\mathbf{n}} = 0 \quad \text{at } \partial\Omega, \tag{3.4}
$$

$$
\begin{aligned}
&\int_\Omega \Big(\big(\kappa_\eta(\vartheta) + \delta\vartheta^B + \delta\vartheta^{-1}\big)\frac{\varepsilon+\vartheta}{\vartheta}\nabla\vartheta : \nabla\psi - \varrho e(\varrho,\vartheta)\mathbf{u}\cdot\nabla\psi\Big)\,\mathrm{d}\mathbf{x} \\
&\qquad + \int_{\partial\Omega} \big((L + \delta\vartheta^{B-1})(\vartheta - \Theta_0^\eta) + \varepsilon\ln\vartheta\big)\,\psi\,\mathrm{d}S \\
&= \int_\Omega \big(\mathbb{S}^\alpha_\eta(\vartheta,\mathbf{u}) : \nabla\mathbf{u} + \delta\vartheta^{-1} - p(\varrho,\vartheta)\mathrm{div}\,\mathbf{u} + \delta\varepsilon\left|\nabla\varrho\right|^2(\beta\varrho^{\beta-2} + 2)\big)\psi\,\mathrm{d}\mathbf{x} \\
&\qquad - \int_\Omega (s_E - \mathbf{s}_F\cdot\mathbf{u})\psi\,\mathrm{d}\mathbf{x}
\end{aligned} \tag{3.5}
$$

for all $\psi \in C^1(\overline{\Omega})$, and

$$
\lambda I + \omega\cdot\nabla I = S \quad \text{in } \Omega\times\mathcal{S}^2\times(0,\infty) \text{ in the sense of distributions.} \tag{3.6}
$$

The most difficult step in this limit passage is to show that $\mathbb{S}^\alpha_\eta(\vartheta_N,\mathbf{u}_N) : \nabla\mathbf{u}_N \to \mathbb{S}^\alpha_\eta(\vartheta,\mathbf{u}) : \nabla\mathbf{u}$ in $L^1(\Omega)$. This is a consequence of the fact that we are able to use $\mathbf{u}$ as a test function in (3.2) and $\mathbf{u}_N$ as a test function in (2.3). Combining these two facts and using the Vitali theorem we also get the strong convergence of $\nabla \mathbf{u}_N$ in $L^2(\Omega)$. We apply Lemma 2.5 to pass to the limit in the radiative transport equation and in the radiative terms in the momentum and internal energy equations. Indeed, we have

$$\int_0^\infty \int_{\mathcal{S}^2} I_N(\cdot,\omega,\nu)\,\mathrm{d}\omega\,\mathrm{d}\nu \to \int_0^\infty \int_{\mathcal{S}^2} I(\cdot,\omega,\nu)\,\mathrm{d}\omega\,\mathrm{d}\nu \quad \text{strongly in } L^p(\Omega) \tag{3.7}$$

and

$$\int_0^\infty \int_{\mathcal{S}^2} \omega I_N(\cdot,\omega,\nu)\,\mathrm{d}\omega\,\mathrm{d}\nu \to \int_0^\infty \int_{\mathcal{S}^2} \omega I(\cdot,\omega,\nu)\,\mathrm{d}\omega\,\mathrm{d}\nu \quad \text{strongly in } L^p(\Omega) \tag{3.8}$$

for any $p < \infty$. Even though to pass to the limit in equalities above we needed only weak convergence of s_E and $\mathbf{s}_F$, to get also the entropy inequality we need the strong convergence. Using also (3.5) and (3.6) we show

$$\begin{aligned}
&\int_\Omega \Big(\frac{1}{\vartheta}\mathbb{S}^\alpha_\eta(\vartheta,\mathbf{u}) : \nabla\mathbf{u} + \delta\vartheta^{-2} + (\kappa_\eta(\vartheta) + \delta\vartheta^B + \delta\vartheta^{-1})\frac{\varepsilon+\vartheta}{\vartheta}\frac{|\nabla\vartheta|^2}{\vartheta^2}\Big)\psi\,\mathrm{d}\mathbf{x}\\
&\le \int_\Omega \Big((\kappa_\eta(\vartheta) + \delta\vartheta^B + \delta\vartheta^{-1})\frac{\varepsilon+\vartheta}{\vartheta}\frac{\nabla\vartheta : \nabla\psi}{\vartheta} - \varrho s(\varrho,\vartheta)\mathbf{u}\cdot\nabla\psi\Big)\,\mathrm{d}\mathbf{x}\\
&\quad + \int_{\partial\Omega} \left(\frac{L+\delta\vartheta^{B-1}}{\vartheta}(\vartheta - \Theta_0^\eta) + \varepsilon\ln\vartheta\right)\psi\,\mathrm{d}S\\
&\quad + \int_\Omega \frac{1}{\vartheta}(s_E - \mathbf{s}_F\cdot\mathbf{u})\psi\,\mathrm{d}\mathbf{x} + F_\varepsilon
\end{aligned} \tag{3.9}$$

for all nonnegative $\psi \in C^1(\overline{\Omega})$, where $F_\varepsilon = o(\varepsilon)$ as $\varepsilon \to 0^+$.

3.2 Limit passage $\eta \to 0^+$

As estimates (3.1) remain valid, we can also use them to pass to the limit with $\eta \to 0^+$. We again find subsequences converging in proper spaces and pass to the limit in the approximative momentum equation to get

$$\begin{aligned}
&\int_\Omega \Big(\frac{1}{2}\varrho(\mathbf{u}\cdot\nabla\mathbf{u})\cdot\boldsymbol{\varphi} - \frac{1}{2}\varrho(\mathbf{u}\otimes\mathbf{u}) : \nabla\boldsymbol{\varphi} + \mathbb{S}^\alpha(\vartheta,\mathbf{u}) : \nabla\boldsymbol{\varphi}\Big)\,\mathrm{d}\mathbf{x}\\
&- \int_\Omega \big(p(\varrho,\vartheta) + \delta(\varrho^\beta + \varrho^2)\big)\operatorname{div}\boldsymbol{\varphi}\,\mathrm{d}\mathbf{x} = \int_\Omega (\varrho\mathbf{f}\cdot\boldsymbol{\varphi} - \mathbf{s}_F\cdot\boldsymbol{\varphi})\,\mathrm{d}\mathbf{x}
\end{aligned} \tag{3.10}$$

for all $\boldsymbol{\varphi} \in W_0^{1,\frac{6B}{3B-2}}(\Omega)$. We also pass to the limit in the approximative continuity equation and get

$$\varepsilon \int_\Omega (\varrho\psi + \nabla\varrho\cdot\nabla\psi)\,\mathrm{d}\mathbf{x} - \int_\Omega \varrho\mathbf{u}\cdot\nabla\psi\,\mathrm{d}\mathbf{x} = \varepsilon h \int_\Omega \psi\,\mathrm{d}\mathbf{x} \tag{3.11}$$

for all $\psi \in W^{1,\frac{6}{5}}(\Omega)$. Next, passing to the limit in the entropy inequality reads

$$
\begin{aligned}
&\int_\Omega \Big(\frac{1}{\vartheta}\mathbb{S}^\alpha(\vartheta,\mathbf{u}) : \nabla\mathbf{u} + \delta\vartheta^{-2} + (\kappa(\vartheta) + \delta\vartheta^B + \delta\vartheta^{-1})\frac{\varepsilon+\vartheta}{\vartheta}\frac{|\nabla\vartheta|^2}{\vartheta^2}\Big)\psi \,\mathrm{d}\mathbf{x} \\
&\quad\le \int_\Omega \Big((\kappa(\vartheta) + \delta\vartheta^B + \delta\vartheta^{-1})\frac{\varepsilon+\vartheta}{\vartheta}\frac{\nabla\vartheta : \nabla\psi}{\vartheta} - \varrho s(\varrho,\vartheta)\mathbf{u}\cdot\nabla\psi\Big)\,\mathrm{d}\mathbf{x} \\
&\qquad + \int_{\partial\Omega}\Big(\frac{L+\delta\vartheta^{B-1}}{\vartheta}(\vartheta-\Theta_0) + \varepsilon\ln\vartheta\Big)\psi\,\mathrm{d}S \\
&\qquad + \int_\Omega \frac{1}{\vartheta}(s_E - \mathbf{s}_F\cdot\mathbf{u})\psi\,\mathrm{d}\mathbf{x} + F_\varepsilon
\end{aligned} \tag{3.12}
$$

for all nonnegative $\psi \in C^1(\overline{\Omega})$ with F_ε having the same properties as above. It is also easy to pass to the limit in the radiative transfer equation

$$
\lambda I + \omega\cdot\nabla I = S \quad \text{in } \Omega\times\mathcal{S}^2\times(0,\infty) \text{ in the sense of distributions.} \tag{3.13}
$$

However, the situation in the energy equation is more complicated. We are not able to recover strong convergence of the velocity gradients and therefore we cannot pass to the limit in the internal energy balance. Therefore we switch to the total energy balance, which we get by summing the approximative internal energy balance and the approximative momentum equation tested by $\mathbf{u}_\eta\psi$ with smooth ψ. Note that this was not possible in the previous step as $\mathbf{u}_\eta\psi$ was not a proper test function. Summing above mentioned equations helps us to get rid of the problematic term $\int_\Omega \mathbb{S}^\alpha_\eta(\vartheta_\eta,\mathbf{u}_\eta) : \nabla\mathbf{u}_\vartheta\psi\,\mathrm{d}\mathbf{x}$ which is now replaced by $\int_\Omega \mathbb{S}^\alpha_\eta(\vartheta_\eta,\mathbf{u}_\eta)\mathbf{u}_\eta\cdot\nabla\psi\,\mathrm{d}\mathbf{x}$. Altogether we end up with

$$
\begin{aligned}
&\int_\Omega \Big((\kappa(\vartheta) + \delta\vartheta^B + \delta\vartheta^{-1})\frac{\varepsilon+\vartheta}{\vartheta}\nabla\vartheta : \nabla\psi - (\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta))\mathbf{u}\cdot\nabla\psi\Big)\,\mathrm{d}\mathbf{x} \\
&\quad + \int_{\partial\Omega}\big((L+\delta\vartheta^{B-1})(\vartheta-\Theta_0) + \varepsilon\ln\vartheta\big)\,\psi\,\mathrm{d}S = \int_\Omega \varrho\mathbf{f}\cdot\mathbf{u}\psi\,\mathrm{d}\mathbf{x} \\
&\quad + \int_\Omega \big((-\mathbb{S}^\alpha(\vartheta,\mathbf{u})\mathbf{u} + p(\varrho,\vartheta)\mathbf{u} + \delta(\varrho^\beta + \varrho^2)\mathbf{u})\cdot\nabla\psi + \delta\vartheta^{-1}\psi\big)\,\mathrm{d}\mathbf{x} \\
&\quad + \delta\int_\Omega \frac{1}{\beta-1}\left(\varepsilon\beta h\varrho^{\beta-1}\psi + \varrho^\beta\mathbf{u}\cdot\nabla\psi - \varepsilon\beta\varrho^\beta\psi\right)\,\mathrm{d}\mathbf{x} \\
&\quad + \delta\int_\Omega \left(2\varepsilon h\varrho\psi + \varrho^2\mathbf{u}\cdot\nabla\psi - 2\varepsilon\varrho^2\psi\right)\,\mathrm{d}\mathbf{x} - \int_\Omega s_E\psi\,\mathrm{d}\mathbf{x}
\end{aligned} \tag{3.14}
$$

for all $\psi \in C^1(\overline{\Omega})$.

3.3 Limit passage $\varepsilon \to 0^+$

Using the entropy inequality (3.14) and the estimate for the radiative transport equation (2.26) we can deduce

(3.15)

$$
\begin{aligned}
&\int_\Omega \frac{1}{\vartheta}\mathbb{S}^\alpha(\vartheta_\varepsilon,\mathbf{u}_\varepsilon) : \nabla\mathbf{u}_\varepsilon\,\mathrm{d}\mathbf{x} + \|\vartheta_\varepsilon\|_{3B}^B + \|\vartheta_\varepsilon\|_{1,2}^2 + \left\|\nabla(\vartheta_\varepsilon^{-\frac{1}{2}})\right\|_2^2 + \left\|\vartheta_\varepsilon^{-2}\right\|_1 \\
&+ \left\|\vartheta_\varepsilon^{-1}\right\|_{1,\partial\Omega} + \|\vartheta_\varepsilon\|_{B,\partial\Omega}^B + \|I_\varepsilon\|_{\infty,\Omega\times\mathcal{S}^2\times(0,\infty)} \le C\Big(1 + \Big|\int_\Omega \varrho_\varepsilon\mathbf{f}\cdot\mathbf{u}_\varepsilon\,\mathrm{d}\mathbf{x}\Big| + \|\mathbf{u}_\varepsilon\|_1\Big).
\end{aligned}
$$

Here we have to deal with the fact that we cannot use Korn's inequality on the first term on the left-hand side and thus we do not control $W^{1,2}$-norm of $\mathbf{u}_\varepsilon$ anymore. However, using procedure which is in detail described in the limit passage $\delta \to 0^+$ we are able to control at least $W^{1,p}$ norm of $\mathbf{u}_\varepsilon$ for some $p < 2$. Indeed, for

$$p' = \frac{6B}{3B+1-\alpha} \tag{3.16}$$

we have

$$\begin{aligned} \|\mathbf{u}_\varepsilon\|_{1,p'}^{p'} &= \int_\Omega |\nabla \mathbf{u}_\varepsilon|^{p'} \,\mathrm{d}\mathbf{x} \le \Big(\int_\Omega \vartheta_\varepsilon^{\alpha-1} |\nabla \mathbf{u}_\varepsilon|^2 \,\mathrm{d}\mathbf{x}\Big)^{\frac{p'}{2}} \|\vartheta_\varepsilon\|_{3B}^{\frac{3B(2-p')}{2}} \\ &\le C\Big(1 + \Big|\int_\Omega \varrho_\varepsilon \mathbf{f}\cdot \mathbf{u}_\varepsilon \,\mathrm{d}\mathbf{x}\Big| + \|\mathbf{u}_\varepsilon\|_1\Big)^{3-p'} \le C\Big(1 + \|\nabla \mathbf{u}_\varepsilon\|_{p'}^{3-p'} \big(1 + \|\varrho_\varepsilon\|_{\frac{3p'}{4p'-3}}^{3-p'}\big)\Big) \end{aligned} \tag{3.17}$$

which implies

$$\|\nabla \mathbf{u}_\varepsilon\|_{p'} \le C\big(1 + \|\varrho_\varepsilon\|_{\frac{3p'}{4p'-3}}^{\frac{3-p'}{2p'-3}}\big) \tag{3.18}$$

and consequently

$$\begin{aligned} \|\vartheta_\varepsilon\|_{3B}^B + \|\vartheta_\varepsilon\|_{1,2}^2 + \Big\|\nabla(\vartheta_\varepsilon^{-\frac12})\Big\|_2^2 + \big\|\vartheta_\varepsilon^{-2}\big\|_1 + \big\|\vartheta_\varepsilon^{-1}\big\|_{1,\partial\Omega} + \|\vartheta_\varepsilon\|_{B,\partial\Omega}^B \\ + \|I_\varepsilon\|_{\infty,\Omega\times\mathcal{S}^2\times(0,\infty)} \le C\big(1 + \|\varrho_\varepsilon\|_{\frac{3p'}{4p'-3}}^{\frac{p'}{2p'-3}}\big). \end{aligned} \tag{3.19}$$

Note that $p' > \frac32$ and thus the procedure works.

About density, we do not have any information independent of ε, except for the L^1-norm. Thus we proceed in a standard way and use Bogovskii–operator type estimates, i.e. we use in the momentum equation (3.10) test function $\mathbf{\Phi}$ such that

$$\begin{aligned} \operatorname{div} \mathbf{\Phi} = \varrho_\varepsilon^{(s-1)\beta} - \frac{1}{|\Omega|}\int_\Omega \varrho_\varepsilon^{(s-1)\beta}\,\mathrm{d}\mathbf{x} \quad \text{in } \Omega \\ \mathbf{\Phi} = \mathbf{0} \quad \text{at } \partial\Omega \end{aligned} \tag{3.20}$$

with

$$\|\mathbf{\Phi}\|_{1,\frac{s}{s-1}}^{\frac{s}{s-1}} \le C\,\|\varrho_\varepsilon\|_{s\beta}^{s\beta} \quad 1 < s < \infty. \tag{3.21}$$

We skip the details now as similar procedure will be used later on, see also [26]. We end up with (i.e. we may take $s = \frac53$)

$$\|\varrho_\varepsilon\|_{\frac53\beta} \le C. \tag{3.22}$$

Now we are able to choose subsequences (denoted again $(\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon, I_\varepsilon)$) such that

$$\begin{array}{llll} \mathbf{u}_\varepsilon \rightharpoonup \mathbf{u} & \text{in } W_0^{1,p'}(\Omega), & \mathbf{u}_\varepsilon \to \mathbf{u} & \text{in } L^q(\Omega),\ q < \dfrac{3p'}{3-p'}, \\ \varrho_\varepsilon \rightharpoonup \varrho & \text{in } L^{\frac53\beta}(\Omega), & \varepsilon\nabla\varrho_\varepsilon \to 0 & \text{in } L^2(\Omega), \\ \vartheta_\varepsilon \rightharpoonup \vartheta & \text{in } W^{1,2}(\Omega), & \vartheta_\varepsilon \to \vartheta & \text{in } L^q(\Omega),\ q < 3B, \\ \vartheta_\varepsilon \to \vartheta & \text{in } L^q(\partial\Omega),\ q < 2B, & \ln\vartheta_\varepsilon \rightharpoonup \ln\vartheta & \text{in } W^{1,2}(\Omega), \\ \ln\vartheta_\varepsilon \to \ln\vartheta & \text{in } L^q(\Omega),\ q < 6, & \ln\vartheta_\varepsilon \to \ln\vartheta & \text{in } L^q(\partial\Omega),\ q < 4, \\ \vartheta_\varepsilon^{-\frac12} \to \vartheta^{-\frac12} & \text{in } L^q(\Omega),\ q < 6, & \vartheta_\varepsilon^{-\frac12} \to \vartheta^{-\frac12} & \text{in } L^q(\partial\Omega),\ q < 4, \\ I_\varepsilon \rightharpoonup^* I & \text{in } L^\infty(\Omega\times\mathcal{S}^2\times(0,\infty)). & & \end{array} \tag{3.23}$$

Now we pass to the limit with $\varepsilon \to 0^+$. We have the continuity equation

$$\int_\Omega \varrho \mathbf{u} \cdot \nabla \psi \, \mathrm{d}\mathbf{x} = 0 \tag{3.24}$$

for all $\psi \in W^{1,\frac{15p'\beta}{20p'\beta-15\beta-9p'}}(\Omega)$. Note that the estimates above do not guarantee the strong convergence of $\varrho_\varepsilon \to \varrho$ in $L^1(\Omega)$. However, using a similar procedure as in the case $\delta \to 0^+$ (we comment on the simplifications for $\varepsilon \to 0^+$ at the corresponding places in Section 5) we can show that $\varrho_\varepsilon \to \varrho$ in $L^1(\Omega)$, hence $\varrho_\varepsilon \to \varrho$ in $L^q(\Omega)$ for all $q < \frac{5}{3}\beta$. Thus the limit passage in the momentum equation yields

$$\begin{aligned} \int_\Omega \big(-\varrho(\mathbf{u}\otimes\mathbf{u}) : \nabla\boldsymbol{\varphi} + \mathbb{S}^\alpha(\vartheta,\mathbf{u}) : \nabla\boldsymbol{\varphi} - (p(\varrho,\vartheta) + \delta(\varrho^\beta + \varrho^2))\mathrm{div}\,\boldsymbol{\varphi}\big)\,\mathrm{d}\mathbf{x} \\ = \int_\Omega (\varrho\mathbf{f}\cdot\boldsymbol{\varphi} - \mathbf{s}_F\cdot\boldsymbol{\varphi})\,\mathrm{d}\mathbf{x} \end{aligned} \tag{3.25}$$

for all $\boldsymbol{\varphi} \in W_0^{1,\frac{5}{2}}(\Omega)$, the total energy balance reads

$$\begin{aligned} &\int_\Omega \Big((\kappa(\vartheta) + \delta\vartheta^B + \delta\vartheta^{-1})\nabla\vartheta : \nabla\psi - (\frac{1}{2}\rho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta))\mathbf{u}\cdot\nabla\psi\Big)\,\mathrm{d}\mathbf{x} \\ &\quad + \int_{\partial\Omega} (L + \delta\vartheta^{B-1})(\vartheta - \Theta_0)\psi\,\mathrm{d}S = \int_\Omega \varrho\mathbf{f}\cdot\mathbf{u}\psi\,\mathrm{d}\mathbf{x} \\ &\quad + \int_\Omega \big(\big(-\mathbb{S}^\alpha(\vartheta,\mathbf{u})\mathbf{u} + (p(\varrho,\vartheta) + \delta(\varrho^\beta + \varrho^2))\mathbf{u}\big)\cdot\nabla\psi + \delta\vartheta^{-1}\psi\big)\,\mathrm{d}\mathbf{x} \\ &\quad + \int_\Omega \Big(\delta\Big(\frac{1}{\beta-1}\varrho^\beta + \varrho^2\Big)\mathbf{u}\cdot\nabla\psi - s_E\psi\Big)\,\mathrm{d}\mathbf{x} \end{aligned} \tag{3.26}$$

for all $\psi \in C^1(\overline{\Omega})$. We may also pass to the limit in the entropy inequality to get

$$\begin{aligned} &\int_\Omega \Big(\frac{1}{\vartheta}\mathbb{S}^\alpha(\vartheta,\mathbf{u}) : \nabla\mathbf{u} + \delta\vartheta^{-2} + (\kappa(\vartheta) + \delta\vartheta^B + \delta\vartheta^{-1})\frac{|\nabla\vartheta|^2}{\vartheta^2}\Big)\psi\,\mathrm{d}\mathbf{x} \\ &\leq \int_\Omega \Big((\kappa(\vartheta) + \delta\vartheta^B + \delta\vartheta^{-1})\frac{\nabla\vartheta : \nabla\psi}{\vartheta} - \varrho s(\varrho,\vartheta)\mathbf{u}\cdot\nabla\psi\Big)\,\mathrm{d}\mathbf{x} \\ &+ \int_{\partial\Omega} \Big(\frac{L + \delta\vartheta^{B-1}}{\vartheta}(\vartheta - \Theta_0)\Big)\psi\,\mathrm{d}S + \int_\Omega \frac{1}{\vartheta}(s_E - \mathbf{s}_F\cdot\mathbf{u})\psi\,\mathrm{d}\mathbf{x} \end{aligned} \tag{3.27}$$

for all nonnegative $\psi \in C^1(\overline{\Omega})$. Finally we also easily recover the radiative transfer equation

$$\lambda I + \omega\cdot\nabla I = S \quad \text{in } \Omega\times\mathcal{S}^2\times(0,\infty) \text{ in the sense of distributions.} \tag{3.28}$$

4 A priori estimates independent of δ

We start with the entropy inequality (3.27), where we use $\psi \equiv 1$ as a test function. We get

$$\begin{aligned} &\int_\Omega \Big(\frac{1}{\vartheta_\delta}\mathbb{S}^\alpha(\vartheta_\delta,\mathbf{u}_\delta) : \nabla\mathbf{u}_\delta + \delta\vartheta_\delta^{-2} + (\kappa(\vartheta_\delta) + \delta\vartheta_\delta^B + \delta\vartheta_\delta^{-1})\frac{|\nabla\vartheta_\delta|^2}{\vartheta_\delta^2}\Big)\,\mathrm{d}\mathbf{x} \\ &+ \int_{\partial\Omega} \frac{L + \delta\vartheta_\delta^{B-1}}{\vartheta_\delta}\Theta_0\,\mathrm{d}S \leq \int_{\partial\Omega} (L + \delta\vartheta_\delta^{B-1})\mathrm{d}S + \int_\Omega \frac{1}{\vartheta_\delta}(s_E - \mathbf{s}_F\cdot\mathbf{u}_\delta)\,\mathrm{d}\mathbf{x}. \end{aligned} \tag{4.1}$$

We also use the total energy balance (3.26) with the same test function

$$\int_{\partial\Omega} (L\vartheta_\delta + \delta\vartheta_\delta^B)\,\mathrm{d}S = \int_{\partial\Omega} (L + \delta\vartheta_\delta^{B-1})\Theta_0\,\mathrm{d}S + \int_\Omega \big(\varrho_\delta \mathbf{f}\cdot\mathbf{u}_\delta + \delta\vartheta_\delta^{-1} - s_E\big)\,\mathrm{d}\mathbf{x}. \tag{4.2}$$

Estimating the right-hand side of (4.2) we get

$$\|\vartheta_\delta\|_{1,\partial\Omega} + \delta\,\|\vartheta_\delta\|_{B,\partial\Omega}^B \le C\Big(1 + \delta\int_\Omega \vartheta_\delta^{-1}\,\mathrm{d}\mathbf{x} + \Big|\int_\Omega \varrho_\delta\mathbf{f}\cdot\mathbf{u}_\delta\,\mathrm{d}\mathbf{x}\Big|\Big). \tag{4.3}$$

Summing (4.1) with a version of (4.3) reads

$$\begin{aligned}\int_\Omega \Big(\frac{1}{\vartheta_\delta}\mathbb{S}^\alpha(\vartheta_\delta,\mathbf{u}_\delta):\nabla\mathbf{u}_\delta + \delta\vartheta_\delta^{-2} + (\kappa(\vartheta_\delta) + \delta\vartheta_\delta^B + \delta\vartheta_\delta^{-1})\frac{|\nabla\vartheta_\delta|^2}{\vartheta_\delta^2}\Big)\,\mathrm{d}\mathbf{x}\\ + \int_{\partial\Omega} \frac{L + \delta\vartheta_\delta^{B-1}}{\vartheta_\delta}\Theta_0\,\mathrm{d}S + \delta^{\frac{1}{B}}\,\|\vartheta_\delta\|_{1,\partial\Omega}^{\frac{B-1}{B}} + \delta\,\|\vartheta_\delta\|_{B,\partial\Omega}^{B-1}\\ \le \int_{\partial\Omega} (L + \delta\vartheta_\delta^{B-1})\,\mathrm{d}S + \int_\Omega \frac{1}{\vartheta_\delta}(s_E - \mathbf{s}_F\cdot\mathbf{u}_\delta)\,\mathrm{d}\mathbf{x}\\ + C\Big(1 + \delta\Big(\int_\Omega \vartheta_\delta^{-1}\,\mathrm{d}\mathbf{x}\Big)^{\frac{B-1}{B}} + \delta^{\frac{1}{B}}\Big|\int_\Omega \varrho_\delta\mathbf{f}\cdot\mathbf{u}_\delta\mathrm{d}\mathbf{x}\Big|^{\frac{B-1}{B}}\Big).\end{aligned} \tag{4.4}$$

We can estimate easily all terms except for the last one, so we have (recall (1.31))

$$\begin{aligned}\int_\Omega \Big(\frac{1}{\vartheta_\delta}\mathbb{S}^\alpha(\vartheta_\delta,\mathbf{u}_\delta):\nabla\mathbf{u}_\delta + \delta\vartheta_\delta^{-2} + (\kappa(\vartheta_\delta) + \delta\vartheta_\delta^B + \delta\vartheta_\delta^{-1})\frac{|\nabla\vartheta_\delta|^2}{\vartheta_\delta^2}\Big)\,\mathrm{d}\mathbf{x}\\ + \int_{\partial\Omega} \frac{L + \delta\vartheta_\delta^{B-1}}{\vartheta_\delta}\Theta_0\,\mathrm{d}S + \delta^{\frac{1}{B}}\,\|\vartheta_\delta\|_{1,\partial\Omega}^{\frac{B-1}{B}} + \delta\,\|\vartheta_\delta\|_{B,\partial\Omega}^{B-1}\\ \le C\Big(1 + \|\mathbf{u}_\delta\|_1 + \delta^{\frac{1}{B}}\Big|\int_\Omega \varrho_\delta\mathbf{f}\cdot\mathbf{u}_\delta\,\mathrm{d}\mathbf{x}\Big|^{\frac{B-1}{B}}\Big).\end{aligned} \tag{4.5}$$

The last term can be estimated

$$\begin{aligned}\Big|\int_\Omega \varrho_\delta\mathbf{f}\cdot\mathbf{u}_\delta\,\mathrm{d}\mathbf{x}\Big|^{\frac{B-1}{B}} \le C\,\|\mathbf{u}_\delta\|_{1,p}^{\frac{B-1}{B}}\,\|\varrho_\delta\|_{\frac{3p}{4p-3}}^{\frac{B-1}{B}}\\ \le C\,\|\varrho_\delta\|_{\frac{3p}{4p-3}}^{\frac{B-1}{B}}\Big(\int_\Omega \frac{|\nabla\mathbf{u}_\delta|^2}{\vartheta_\delta^{1-\alpha}}\,\mathrm{d}\mathbf{x}\Big)^{\frac{B-1}{2B}}\,\big\|\vartheta_\delta^{1-\alpha}\big\|_{\frac{p}{2-p}}^{\frac{B-1}{2B}}.\end{aligned} \tag{4.6}$$

We have information about the middle term on the left-hand side of (4.5), Young's inequality then yields

$$\begin{aligned}\int_\Omega \Big(\frac{1}{\vartheta_\delta}\mathbb{S}^\alpha(\vartheta_\delta,\mathbf{u}_\delta):\nabla\mathbf{u}_\delta + \delta\vartheta_\delta^{-2} + (\kappa(\vartheta_\delta) + \delta\vartheta_\delta^B + \delta\vartheta_\delta^{-1})\frac{|\nabla\vartheta_\delta|^2}{\vartheta_\delta^2}\Big)\,\mathrm{d}\mathbf{x}\\ + \int_{\partial\Omega} \frac{L + \delta\vartheta_\delta^{B-1}}{\vartheta_\delta}\Theta_0\,\mathrm{d}S + \delta^{\frac{1}{B}}\,\|\vartheta_\delta\|_{1,\partial\Omega}^{\frac{B-1}{B}} + \delta\,\|\vartheta_\delta\|_{B,\partial\Omega}^{B-1}\\ \le C\Big(1 + \|\mathbf{u}_\delta\|_1 + \delta^{\frac{2}{B+1}}\,\|\varrho_\delta\|_{\frac{3p}{4p-3}}^{\frac{2(B-1)}{B+1}}\,\big\|\vartheta_\delta^{1-\alpha}\big\|_{\frac{p}{2-p}}^{\frac{B-1}{B+1}}\Big)\\ \le C\Big(1 + \|\mathbf{u}_\delta\|_1 + \delta^{\frac{2}{B+1}}\,\|\varrho_\delta\|_{\frac{3p}{4p-3}}^{\frac{2(B-1)}{B+1}}\,\|\vartheta_\delta\|_{3B}^{\frac{(B-1)(1-\alpha)}{B+1}}\Big),\end{aligned} \tag{4.7}$$

for $B > \frac{p}{2-p}\frac{1-\alpha}{3}$. Friedrichs' inequality implies

$$\|\vartheta_\delta\|_{3B} \leq C\Big(\|\vartheta_\delta\|_{B,\partial\Omega} + \Big\|\nabla \vartheta_\delta^{\frac{B}{2}}\Big\|_2^{\frac{2}{B}} \Big) \tag{4.8}$$

and thus going back to (4.7)

$$\begin{aligned} &\int_\Omega \Big(\frac{1}{\vartheta_\delta}\mathbb{S}^\alpha(\vartheta_\delta,\mathbf{u}_\delta) : \nabla \mathbf{u}_\delta + \delta\vartheta_\delta^{-2} + (\kappa(\vartheta_\delta) + \delta\vartheta_\delta^B + \delta\vartheta_\delta^{-1})\frac{|\nabla\vartheta_\delta|^2}{\vartheta_\delta^2}\Big)\,\mathrm{d}\mathbf{x} \\ &\quad + \int_{\partial\Omega} \frac{L+\delta\vartheta_\delta^{B-1}}{\vartheta_\delta}\Theta_0\,\mathrm{d}S + \delta^{\frac{1}{B}}\|\vartheta_\delta\|_{1,\partial\Omega}^{\frac{B-1}{B}} + \delta\|\vartheta_\delta\|_{B,\partial\Omega}^{B-1} \\ &\leq C\Big(1 + \|\mathbf{u}_\delta\|_1 + \delta^{\frac{2}{B+1}}\|\varrho_\delta\|_{\frac{3p}{4p-3}}^{\frac{2(B-1)}{B+1}}\Big(\|\vartheta_\delta\|_{B,\partial\Omega}^{\frac{(B-1)(1-\alpha)}{B+1}} + \Big\|\nabla\vartheta_\delta^{\frac{B}{2}}\Big\|_2^{\frac{2(B-1)(1-\alpha)}{B(B+1)}}\Big)\Big), \end{aligned} \tag{4.9}$$

and finally

$$\begin{aligned} &\int_\Omega \Big(\frac{1}{\vartheta_\delta}\mathbb{S}^\alpha(\vartheta_\delta,\mathbf{u}_\delta) : \nabla \mathbf{u}_\delta + \delta\vartheta_\delta^{-2} + (\kappa(\vartheta_\delta) + \delta\vartheta_\delta^B + \delta\vartheta_\delta^{-1})\frac{|\nabla\vartheta_\delta|^2}{\vartheta_\delta^2}\Big)\,\mathrm{d}\mathbf{x} \\ &\quad + \int_{\partial\Omega} \frac{L+\delta\vartheta_\delta^{B-1}}{\vartheta_\delta}\Theta_0\,\mathrm{d}S + \delta^{\frac{1}{B}}\|\vartheta_\delta\|_{1,\partial\Omega}^{\frac{B-1}{B}} + \delta\|\vartheta_\delta\|_{B,\partial\Omega}^{B-1} \\ &\leq C\Big(1 + \|\mathbf{u}_\delta\|_1 + \delta^{\frac{1+\alpha}{B+\alpha}}\|\varrho_\delta\|_{\frac{3p}{4p-3}}^{\frac{2(B-1)}{B+\alpha}} + \delta^{\frac{B(1+\alpha)+1-\alpha}{B(B+\alpha)+1-\alpha}}\|\varrho_\delta\|_{\frac{3p}{4p-3}}^{\frac{2B(B-1)}{B(B+\alpha)+1-\alpha}}\Big) \\ &=: C\big(1 + \|\mathbf{u}_\delta\|_1 + F(\varrho_\delta,\delta)\big) =: A(\mathbf{u}_\delta,\varrho_\delta,\delta). \end{aligned} \tag{4.10}$$

We use estimate (4.10) to derive the bound on the L^p norm of the velocity. Denote

$$p = \frac{6m}{3m+1-\alpha}, \quad \text{i.e. } 3m = \frac{(1-\alpha)p}{2-p}, \tag{4.11}$$

and therefore

$$\begin{aligned} \|\nabla\mathbf{u}_\delta\|_p^p = \int_\Omega |\nabla\mathbf{u}_\delta|^p\,\mathrm{d}\mathbf{x} &\leq \Big(\int_\Omega \vartheta_\delta^{\alpha-1}|\nabla\mathbf{u}_\delta|^2\,\mathrm{d}\mathbf{x}\Big)^{\frac{p}{2}}\Big(\int_\Omega \vartheta_\delta^{3m}\mathrm{d}\mathbf{x}\Big)^{\frac{2-p}{2}} \\ &\leq A(\mathbf{u}_\delta,\varrho_\delta,\delta)^{\frac{p}{2}}\|\vartheta_\delta\|_{3m}^{\frac{3m(2-p)}{2}}. \end{aligned} \tag{4.12}$$

We have

$$\begin{aligned} \|\vartheta_\delta\|_{3m} &\leq C\Big(\|\vartheta_\delta\|_{1,\partial\Omega} + \Big\|\nabla\vartheta_\delta^{\frac{m}{2}}\Big\|_2^{\frac{2}{m}}\Big) \\ &\leq C\Big(\Big|\int_\Omega \varrho_\delta\mathbf{u}_\delta\cdot\mathbf{f}\,\mathrm{d}\mathbf{x}\Big| + \delta^{\frac{1}{2}}A(\mathbf{u}_\delta,\varrho_\delta,\delta)^{\frac{1}{2}} + A(\mathbf{u}_\delta,\varrho_\delta,\delta)^{\frac{1}{m}}\Big). \end{aligned} \tag{4.13}$$

We now distinguish two cases. First, let $m \geq 2$. Plugging (4.13) into (4.12) and using (4.11)

$$\|\nabla\mathbf{u}_\delta\|_p \leq A(\mathbf{u}_\delta,\varrho_\delta,\delta)^{\frac{1}{2}}\big(1 + A(\mathbf{u}_\delta,\varrho_\delta,\delta)^{\frac{1}{2}} + \|\nabla\mathbf{u}_\delta\|_p\|\varrho_\delta\|_{\frac{3p}{4p-3}}\big)^{\frac{1-\alpha}{2}} \tag{4.14}$$

and thus using $\|\mathbf{u}_\delta\|_1 \leq C\|\nabla\mathbf{u}_\delta\|_p$ and (4.10)

$$\|\nabla\mathbf{u}_\delta\|_p \leq C\big(1 + \|\varrho_\delta\|_{\frac{3p}{4p-3}}\big)^{\frac{1-\alpha}{\alpha}}\big(1 + F(\varrho_\delta,\delta)\big)^{\frac{1}{1+\alpha}} \tag{4.15}$$

and consequently

$$\|\vartheta_\delta\|_{3m} \leq C\big(1 + \|\varrho_\delta\|_{\frac{3p}{4p-3}}\big)^{\frac{1}{\alpha}}\big(1 + F(\varrho_\delta,\delta)\big)^{\frac{1}{1+\alpha}}. \tag{4.16}$$

For $m < 2$ we proceed similarly. We have

$$\|\nabla \mathbf{u}_\delta\|_p \leq A(\mathbf{u}_\delta,\varrho_\delta,\delta)^{\frac{1}{2}}\big(1 + A(\mathbf{u}_\delta,\varrho_\delta,\delta)^{\frac{1}{m}} + \|\nabla \mathbf{u}_\delta\|_p\,\|\varrho_\delta\|_{\frac{3p}{4p-3}}\big)^{\frac{1-\alpha}{2}} \tag{4.17}$$

and thus we immediately get the restriction

$$m > 1 - \alpha. \tag{4.18}$$

Proceeding as above we get instead of (4.15) and (4.16)

$$\|\nabla \mathbf{u}_\delta\|_p \leq C\Big(1 + \|\varrho_\delta\|_{\frac{3p}{4p-3}}^{\frac{1-\alpha}{\alpha}} + F(\varrho_\delta,\delta)\Big), \tag{4.19}$$

and

$$\|\vartheta_\delta\|_{3m} \leq C\Big(1 + \|\varrho_\delta\|_{\frac{3p}{4p-3}}^{\frac{1}{\alpha}} + F(\varrho_\delta,\delta)^{\frac{1}{m}} + F(\varrho_\delta,\delta)\,\|\varrho_\delta\|_{\frac{3p}{4p-3}}\Big). \tag{4.20}$$

Next we need to control the dependence of norms of ϱ_δ on δ. Therefore we return to the momentum equation (3.25) and use as test function

$$\begin{aligned} \operatorname{div}\mathbf{\Phi} &= \varrho_\delta^{(s-1)\beta} - \frac{1}{|\Omega|}\int_\Omega \varrho_\delta^{(s-1)\beta}\,\mathrm{d}\mathbf{x} \quad \text{a.e. in } \Omega \\ \mathbf{\Phi} &= \mathbf{0} \quad \text{at } \partial\Omega \end{aligned} \tag{4.21}$$

with

$$\|\mathbf{\Phi}\|_{1,\frac{s}{s-1}}^{\frac{s}{s-1}} \leq C\,\|\varrho_\delta\|_{s\beta}^{s\beta}; \tag{4.22}$$

especially we choose $s - 1 = \frac{1}{\beta}$, i.e.

$$\|\mathbf{\Phi}\|_{1,\beta+1}^{\beta+1} \leq C\,\|\varrho_\delta\|_{\beta+1}^{\beta+1}. \tag{4.23}$$

This yields

$$\begin{aligned} &\int_\Omega p(\varrho_\delta,\vartheta_\delta)\varrho_\delta\,\mathrm{d}\mathbf{x} + \delta\int_\Omega(\varrho_\delta^{\beta+1} + \varrho_\delta^3)\,\mathrm{d}\mathbf{x} = -\int_\Omega \varrho_\delta(\mathbf{u}_\delta\otimes\mathbf{u}_\delta):\nabla\mathbf{\Phi}\,\mathrm{d}\mathbf{x} \\ &\quad + \int_\Omega \mathbb{S}^\alpha(\vartheta_\delta,\mathbf{u}_\delta):\nabla\mathbf{\Phi}\,\mathrm{d}\mathbf{x} - \int_\Omega \varrho_\delta\mathbf{f}\cdot\mathbf{\Phi}\,\mathrm{d}\mathbf{x} + \int_\Omega \mathbf{s}_F\cdot\mathbf{\Phi}\,\mathrm{d}\mathbf{x} \\ &\quad + \frac{1}{|\Omega|}\int_\Omega \big(p(\varrho_\delta,\vartheta_\delta) + \delta(\varrho_\delta^\beta + \varrho_\delta^2)\big)\,\mathrm{d}\mathbf{x}\int_\Omega \varrho_\delta\,\mathrm{d}\mathbf{x} = I_1 + \ldots + I_5. \end{aligned} \tag{4.24}$$

Before estimating the terms on the right hand side we present the key interpolation inequality ($C = C(M)$, see (1.27))

$$\|\varrho_\delta\|_{\frac{3p}{4p-3}} \leq C\,\|\varrho_\delta\|_{\beta+1}^{\frac{\beta+1}{\beta}\frac{3-p}{3p}}. \tag{4.25}$$

We estimate terms on the right hand side of (4.24). First, for $m \geq 2$ we have

$$
\begin{aligned}
|I_1| &\leq \|\varrho_\delta\|_{\frac{3p(\beta+1)}{2p+5p\beta-6(\beta+1)}} \|\nabla \mathbf{u}_\delta\|_p^2 \|\varrho_\delta\|_{\beta+1} \\
&\leq C \|\varrho_\delta\|_{\beta+1}^{\frac{(\beta+1)(6+p)}{3p\beta}} \big(1 + \|\varrho_\delta\|_{\beta+1}^{\frac{2(1-\alpha)}{\alpha}\frac{\beta+1}{\beta}\frac{3-p}{3p}}\big)\big(1 + F(\varrho_\delta, \delta)^{\frac{2}{1+\alpha}}\big),
\end{aligned} \tag{4.26}
$$

while for $m < 2$

$$
|I_1| \leq C \|\varrho_\delta\|_{\beta+1}^{\frac{(\beta+1)(6+p)}{3p\beta}} \big(1 + \|\varrho_\delta\|_{\beta+1}^{\frac{2(1-\alpha)}{\alpha}\frac{\beta+1}{\beta}\frac{3-p}{3p}} + F(\varrho_\delta, \delta)^2\big). \tag{4.27}
$$

Next for $\frac{3mp}{3m-p} \leq \beta + 1$ and $m \geq 2$

$$
\begin{aligned}
|I_2| &\leq \|\vartheta_\delta\|_{3m}^\alpha \|\nabla \mathbf{u}\|_p \|\nabla \mathbf{\Phi}\|_{\beta+1} \\
&\leq C \|\varrho_\delta\|_{\beta+1} \big(1 + \|\varrho_\delta\|_{\beta+1}^{\frac{1}{\alpha}\frac{\beta+1}{\beta}\frac{3-p}{3p}}\big)\big(1 + F(\varrho_\delta, \delta)\big)
\end{aligned} \tag{4.28}
$$

while for $m < 2$, additionally assuming

$$
m > \frac{1+\alpha}{3}, \tag{4.29}
$$

we get

$$
|I_2| \leq C \|\varrho_\delta\|_{\beta+1} \big(1 + \|\varrho_\delta\|_{\beta+1}^{\frac{1}{\alpha}\frac{\beta+1}{\beta}\frac{3-p}{3p}} + F(\varrho_\delta, \delta)^{1+\frac{\alpha}{m}}\big). \tag{4.30}
$$

Easily

$$
|I_3| + |I_4| \leq C \|\varrho_\delta\|_{\beta+1}. \tag{4.31}
$$

Finally we divide the last term into two parts. First term yields

$$
\delta \int_\Omega \varrho_\delta \,\mathrm{d}\mathbf{x} \int_\Omega (\varrho_\delta^\beta + \varrho_\delta^2) \,\mathrm{d}\mathbf{x} \leq C\delta \left(\int_\Omega \varrho_\delta^{\beta+1}\right)^{1-\eta} \tag{4.32}
$$

for certain $\eta \in (0,1)$, so it can be absorbed into the left hand side. The second term gives

$$
\begin{aligned}
\int_\Omega p(\varrho_\delta, \vartheta_\delta) \,\mathrm{d}\mathbf{x} &\leq C\left(\int_\Omega \varrho_\delta^\gamma \,\mathrm{d}\mathbf{x} + \int_\Omega \varrho_\delta \vartheta_\delta \,\mathrm{d}\mathbf{x}\right) \\
&\leq C\big(\|\varrho_\delta\|_{\gamma+1}^{\gamma-\frac{1}{\gamma}} + \|\varrho_\delta^\gamma \vartheta_\delta\|_1^{\frac{1}{\gamma}} \|\vartheta_\delta\|_1^{\frac{\gamma-1}{\gamma}}\big) \leq \varepsilon \int_\Omega p(\varrho_\delta, \vartheta_\delta) \,\mathrm{d}\mathbf{x} + C(1 + \|\vartheta_\delta\|_{3m}).
\end{aligned} \tag{4.33}
$$

The first term can be absorbed into the left hand side and the other is estimated using (4.16). We collect all the estimates and search for the largest exponent of $\|\varrho_\delta\|_{\beta+1}$ on the right-hand side. It is not difficult to see that for β and B sufficiently large we get

$$
\delta \|\varrho_\delta\|_{\beta+1}^{\beta+1-\beta_0(\alpha)} \leq C, \tag{4.34}
$$

where $\beta_0(\alpha) = O(1)$ for $\beta \to \infty$. Hence we return to (4.10) and conclude that for suitable B

$$
F(\varrho_\delta, \delta) \leq C, \tag{4.35}
$$

so from (4.15) and (4.19)

$$\|\nabla \mathbf{u}_\delta\|_p \le C\big(1 + \|\varrho_\delta\|_{\frac{3p}{4p-3}}^{\frac{1-\alpha}{\alpha}}\big) \tag{4.36}$$

and from (4.16) and (4.20)

$$\|\vartheta_\delta\|_{3m} \le C\big(1 + \|\varrho_\delta\|_{\frac{3p}{4p-3}}^{\frac{1}{\alpha}}\big), \tag{4.37}$$

where $m > \max\{1-\alpha, \frac{1+\alpha}{3}\}$.

Interpolating between $L^1(\Omega)$ and $L^{s\gamma}(\Omega)$ we get for $s\gamma > \frac{3p}{4p-3}$

$$\|\mathbf{u}_\delta\|_{1,p} \le C\big(1 + \|\varrho_\delta\|_{s\gamma}^{\frac{3-p}{3p}\frac{s\gamma}{s\gamma-1}\frac{1-\alpha}{\alpha}}\big). \tag{4.38}$$

We need a priori estimates of the density independent of δ. Therefore similarly as before we use in the momentum equation (3.25) test function $\mathbf{\Phi}$ such that

$$\begin{gathered} \operatorname{div} \mathbf{\Phi} = \varrho_\delta^{(s-1)\gamma} - \frac{1}{|\Omega|}\int_\Omega \varrho_\delta^{(s-1)\gamma}\,\mathrm{d}\mathbf{x} \quad \text{a.e. in } \Omega \\ \mathbf{\Phi} = \mathbf{0} \quad \text{at } \partial\Omega \end{gathered} \tag{4.39}$$

with

$$\|\mathbf{\Phi}\|_{1,\frac{s}{s-1}}^{\frac{s}{s-1}} \le C\,\|\varrho_\delta\|_{s\gamma}^{s\gamma}. \tag{4.40}$$

We get

(4.41)

$$\begin{aligned} \int_\Omega p(\varrho_\delta,\vartheta_\delta)\varrho_\delta^{(s-1)\gamma}\,\mathrm{d}\mathbf{x} + \delta\int_\Omega \Big(\varrho_\delta^{\beta+(s-1)\gamma} + \varrho_\delta^{2+(s-1)\gamma}\Big)\,\mathrm{d}\mathbf{x} = -\int_\Omega \varrho_\delta(\mathbf{u}_\delta\otimes\mathbf{u}_\delta) : \nabla\mathbf{\Phi}\,\mathrm{d}\mathbf{x} \\ + \int_\Omega \mathbb{S}^\alpha(\vartheta_\delta,\mathbf{u}_\delta) : \nabla\mathbf{\Phi}\,\mathrm{d}\mathbf{x} + \frac{1}{|\Omega|}\int_\Omega \varrho_\delta^{(s-1)\gamma}\,\mathrm{d}\mathbf{x}\int_\Omega (p(\varrho_\delta,\vartheta_\delta) + \delta(\varrho_\delta^\beta + \varrho_\delta^2))\,\mathrm{d}\mathbf{x} \\ - \int_\Omega \varrho_\delta\mathbf{f}\cdot\mathbf{\Phi}\,\mathrm{d}\mathbf{x} + \int_\Omega \mathbf{s}_F\cdot\mathbf{\Phi}\,\mathrm{d}\mathbf{x} = J_1 + J_2 + J_3 + J_4 + J_5. \end{aligned}$$

We estimate all terms on the right hand side. Starting with J_1

$$\begin{aligned} |J_1| &\le C\,\|\nabla\mathbf{\Phi}\|_{\frac{s}{s-1}}\,\|\varrho_\delta\|_{s\gamma}\,\|\mathbf{u}_\delta\|_{1,p}^2 \\ &\le C\,\|\varrho_\delta\|_{s\gamma}^{(s-1)\gamma+1}\,\big(1 + \|\varrho_\delta\|_{s\gamma}^{2\frac{3-p}{3p}\frac{s\gamma}{s\gamma-1}\frac{1-\alpha}{\alpha}}\big), \end{aligned} \tag{4.42}$$

where

$$\frac{1}{s\gamma} + \frac{2}{p} \le \frac{1}{s} + \frac{2}{3}. \tag{4.43}$$

Assuming

$$2\frac{3-p}{3p}\frac{s\gamma}{s\gamma-1}\frac{1-\alpha}{\alpha} < \gamma - 1 \tag{4.44}$$

we end up with

$$|J_1| \leq \varepsilon \, \|\varrho_\delta\|_{s\gamma}^{s\gamma} + C(\varepsilon) \tag{4.45}$$

and the first term can be absorbed in the left hand side. Next we proceed with the second term

$$\begin{aligned} |J_2| &\leq C \, \|\nabla \mathbf{\Phi}\|_{\frac{s}{s-1}} \|\vartheta_\delta\|_{3m}^{\alpha} \|\mathbf{u}_\delta\|_{1,p} \\ &\leq C \, \|\varrho_\delta\|_{s\gamma}^{(s-1)\gamma} \big(1 + \|\varrho_\delta\|_{s\gamma}^{\frac{3-p}{3p}\frac{s\gamma}{s\gamma-1}\frac{1-\alpha}{\alpha}}\big)\big(1 + \|\varrho_\delta\|_{s\gamma}^{\frac{3-p}{3p}\frac{s\gamma}{s\gamma-1}\frac{1}{\alpha}}\big)^{\alpha} \\ &\leq C \, \|\varrho_\delta\|_{s\gamma}^{(s-1)\gamma} \big(1 + \|\varrho_\delta\|_{s\gamma}^{\frac{3-p}{3p}\frac{s\gamma}{s\gamma-1}\frac{1}{\alpha}}\big) \end{aligned} \tag{4.46}$$

with

$$\frac{\alpha}{3m} + \frac{1}{p} \leq \frac{1}{s}. \tag{4.47}$$

Thus assuming

$$\frac{3-p}{3p}\frac{s\gamma}{s\gamma-1}\frac{1}{\alpha} < \gamma \tag{4.48}$$

we can again use Young's inequality to get

$$|J_2| \leq \varepsilon \, \|\varrho_\delta\|_{s\gamma}^{s\gamma} + C(\varepsilon) \tag{4.49}$$

and absorb the first term in the left-hand side. The second part of the integral J_3 can be easily estimated using interpolation between $L^1(\Omega)$ and $L^{\beta+(s-1)\gamma}(\Omega)$

$$\delta \int_\Omega \varrho_\delta^{(s-1)\gamma} \, \mathrm{d}\mathbf{x} \int_\Omega (\varrho_\delta^\beta + \varrho_\delta^2) \, \mathrm{d}\mathbf{x} \leq C\delta \Big(\int_\Omega \varrho_\delta^{\beta+(s-1)\gamma} \, \mathrm{d}\mathbf{x} \Big)^{1-\eta} \tag{4.50}$$

for some $\eta \in (0,1)$. The first part is slightly more complicated. However, assuming $(s-1)\gamma \leq 1$, i.e $\int_\Omega \varrho_\delta^{(s-1)\gamma} \, \mathrm{d}\mathbf{x} < C$ we have[2]

$$\begin{aligned} C \int_\Omega p(\varrho_\delta, \vartheta_\delta) \, \mathrm{d}\mathbf{x} &\leq C \Big(\int_\Omega \varrho_\delta^{s\gamma} \, \mathrm{d}\mathbf{x} \Big)^{\frac{1}{s}} \\ + C \int_{\{\varrho_\delta < K_0 \vartheta_\delta^{\frac{1}{\gamma-1}}\}} \big(\varrho_\delta^{1+(s-1)\gamma} \vartheta_\delta\big)^{\frac{1}{1+(s-1)\gamma}} \vartheta_\delta^{\frac{(s-1)\gamma}{1+(s-1)\gamma}} \, \mathrm{d}\mathbf{x} &\leq C \Big(\Big(\int_\Omega \varrho_\delta^{s\gamma} \, \mathrm{d}\mathbf{x} \Big)^{\frac{1}{s}} \\ + \Big(\int_{\{\varrho_\delta < K_0 \vartheta_\delta^{\frac{1}{\gamma-1}}\}} \varrho_\delta^{1+(s-1)\gamma} \vartheta_\delta \, \mathrm{d}\mathbf{x} \Big)^{\frac{1}{1+(s-1)\gamma}} &\times \Big(\int_{\{\varrho_\delta < K_0 \vartheta_\delta^{\frac{1}{\gamma-1}}\}} \vartheta_\delta \, \mathrm{d}\mathbf{x} \Big)^{\frac{(s-1)\gamma}{1+(s-1)\gamma}} \Big) \\ &\leq \varepsilon \int_\Omega \varrho_\delta^{(s-1)\gamma} p(\varrho_\delta, \vartheta_\delta) \, \mathrm{d}\mathbf{x} + C(\varepsilon) \int_\Omega \vartheta_\delta \, \mathrm{d}\mathbf{x}. \end{aligned} \tag{4.51}$$

However,

$$\int_\Omega \vartheta_\delta \, \mathrm{d}\mathbf{x} \leq C \, \|\vartheta_\delta\|_{3m} \leq C \big(1 + \|\varrho_\delta\|_{s\gamma}^{\frac{3-p}{3p}\frac{s\gamma}{s\gamma-1}\frac{1}{\alpha}}\big), \tag{4.52}$$

[2]It is also possible to consider the case $(s-1)\gamma > 1$ which plays a role for γ large, but we will not do it here.

so assuming

$$\frac{3-p}{3p}\frac{s\gamma}{s\gamma-1}\frac{1}{\alpha}<s\gamma \tag{4.53}$$

we can proceed as above. The fourth and the fifth term are easy to estimate,

$$|J_4|\le C\,\|\mathbf{\Phi}\|_{\frac{s}{s-1}}\,\|\varrho_\delta\|_{s\gamma}\le C\,\|\varrho_\delta\|_{s\gamma}^{(s-1)\gamma+1}\le\varepsilon\,\|\varrho_\delta\|_{s\gamma}^{s\gamma}+C(\varepsilon), \tag{4.54}$$

and

$$|J_5|\le C\,\|\mathbf{\Phi}\|_{\frac{s}{s-1}}\,\|\mathbf{s}_F\|_\infty\le C\,\|\varrho_\delta\|_{s\gamma}^{(s-1)\gamma}\le\varepsilon\,\|\varrho_\delta\|_{s\gamma}^{s\gamma}+C(\varepsilon). \tag{4.55}$$

Summing up all the estimates we finally get

$$\|\varrho_\delta\|_{s\gamma}\le C. \tag{4.56}$$

We now summarize our conditions on m, γ, s and α. First recall that we have

$$m>\max\Big\{1-\alpha,\frac{1+\alpha}{3}\Big\}. \tag{4.57}$$

The other most restrictive conditions are (4.43), (4.44), (4.47) and (4.48). Condition (4.47) leads to $1<s<2$ and $m>\frac{1+\alpha}{3}$. The other conditions can be rewritten as

$$\frac{1}{3}+\frac{1-\alpha}{3m}\le\frac{1}{s}\frac{\gamma-1}{\gamma}, \tag{4.58}$$

$$\frac{m+1-\alpha}{6m}\frac{s\gamma}{s\gamma-1}\frac{1}{\alpha}<\gamma, \tag{4.59}$$

$$\frac{m+1-\alpha}{3m}\frac{s\gamma}{s\gamma-1}\frac{1-\alpha}{\alpha}<\gamma-1. \tag{4.60}$$

Note that (4.58) is less restrictive for s as small as possible ($s\to1$), while the other ones for s as large as possible ($s\to2$). To optimize the value of s is technically difficult and it does not lead to much better results than those with s formally equal to 1. Thus we analyze (4.58)–(4.60) with $s=1$ and strict inequalities, as well as with (4.57). Passing with $m\to\infty$ we get

$$\gamma>\frac{3}{2},\qquad\gamma>1+\frac{1}{6\alpha},\qquad\gamma>\frac{1+5\alpha+\sqrt{1+10\alpha-11\alpha^2}}{6\alpha}.$$

This leads to restrictions

$$\begin{aligned}\gamma&>\frac{3}{2}&&\text{for }\frac{2}{3}\le\alpha\le1\\ \gamma&>\frac{1+5\alpha+\sqrt{1+10\alpha-11\alpha^2}}{6\alpha}&&\text{for }0<\alpha<\frac{2}{3}.\end{aligned} \tag{4.61}$$

Returning to (4.58)–(4.60) we get, in addition to (4.57),

$$m > \frac{\gamma(1-\alpha)}{2\gamma-3}, \qquad m > \frac{(1-\alpha)}{6\alpha(\gamma-1)-1}, \qquad m > \frac{(1-\alpha)^2\gamma}{3\alpha(\gamma-1)^2-\gamma(1-\alpha)}. \tag{4.62}$$

Note that in the physically relevant case $\alpha = \frac{1}{2}$ we have

$$\gamma > \frac{7+\sqrt{13}}{6} \sim 1.768 \tag{4.63}$$

and

$$m \geq \max\Big\{\frac{1}{2}, \frac{\gamma}{2(2\gamma-3)}, \frac{\gamma}{6\gamma^2-14\gamma+6}\Big\}. \tag{4.64}$$

Now we can extract suitable subsequences to get

$$\begin{array}{llll} \mathbf{u}_\delta \rightharpoonup \mathbf{u} & \text{in } W_0^{1,p}(\Omega), & \mathbf{u}_\delta \to \mathbf{u} & \text{in } L^q(\Omega),\ q < \dfrac{3p}{3-p}, \\ \varrho_\delta \rightharpoonup \varrho & \text{in } L^{s\gamma}(\Omega), & & \\ \vartheta_\delta \rightharpoonup \vartheta & \text{in } W^{1,r}(\Omega),\ r = \min\left\{2, \dfrac{3m}{m+1}\right\}, & & \\ \vartheta_\delta \to \vartheta & \text{in } L^q(\Omega), q < 3m, & \vartheta_\delta \to \vartheta & \text{in } L^q(\partial\Omega),\ q < 2m, \\ I_\delta \rightharpoonup^* I & \text{in } L^\infty(\Omega \times \mathcal{S}^2 \times (0,\infty)) & & \end{array} \tag{4.65}$$

with p defined in (4.11).

At this moment we pass to the limit with $\delta \to 0^+$ and by $\overline{g(\varrho,\mathbf{u},\vartheta)}$ we denote the weak limit of sequence $g(\varrho_\delta, \mathbf{u}_\delta, \vartheta_\delta)$. We have the continuity equation (however, at this moment, not in the renormalized sense)

$$\int_\Omega \varrho\mathbf{u}\cdot\nabla\psi\,\mathrm{d}\mathbf{x} = 0 \tag{4.66}$$

for all $\psi \in C^1(\overline{\Omega})$. The limit passage in the momentum equation yields

$$\begin{aligned} \int_\Omega \Big(-\varrho(\mathbf{u}\otimes\mathbf{u}) : \nabla\boldsymbol{\varphi} + \mathbb{S}^\alpha(\vartheta,\mathbf{u}) : \nabla\boldsymbol{\varphi} - \overline{p(\varrho,\vartheta)}\mathrm{div}\,\boldsymbol{\varphi}\Big)\,\mathrm{d}\mathbf{x} \\ = \int_\Omega (\varrho\mathbf{f}\cdot\boldsymbol{\varphi} - \mathbf{s}_F\cdot\boldsymbol{\varphi})\,\mathrm{d}\mathbf{x} \end{aligned} \tag{4.67}$$

for all $\boldsymbol{\varphi} \in C^1(\overline{\Omega})$, $\boldsymbol{\varphi} = \mathbf{0}$ at $\partial\Omega$. Here we use also the fact that

$$\lim_{\delta\to 0^+} \delta\,\|\varrho_\delta\|_\beta^\beta = 0, \tag{4.68}$$

which is a consequence of

$$\delta\,\|\varrho_\delta\|_{\beta+(s-1)\gamma}^{\beta+(s-1)\gamma} \leq C. \tag{4.69}$$

We pass to the limit in the entropy inequality to get

$$
\begin{aligned}
&\int_\Omega \left(\frac{1}{\vartheta}\mathbb{S}^\alpha(\vartheta,\mathbf{u}) : \nabla\mathbf{u} + \kappa(\vartheta)\frac{|\nabla\vartheta|^2}{\vartheta^2}\right)\psi\,\mathrm{d}\mathbf{x} \\
&\leq \int_\Omega \left(\kappa(\vartheta)\frac{\nabla\vartheta : \nabla\psi}{\vartheta} - \overline{\varrho s(\varrho,\vartheta)}\mathbf{u}\cdot\nabla\psi\right)\mathrm{d}\mathbf{x} \\
&+ \int_{\partial\Omega}\left(\frac{L}{\vartheta}(\vartheta-\Theta_0)\right)\psi\,\mathrm{d}S + \int_\Omega \frac{1}{\vartheta}(s_E - \mathbf{s}_F\cdot\mathbf{u})\psi\,\mathrm{d}\mathbf{x}
\end{aligned} \tag{4.70}
$$

for all nonnegative $\psi \in C^1(\overline{\Omega})$. On the left-hand side we use nonnegativity of some terms, while on the right hand side we use derived a priori estimates and interpolation inequalities. We can easily pass to the limit in the radiative transfer equation

$$
\lambda I + \omega\cdot\nabla I = S \quad \text{in } \Omega\times\mathcal{S}^2\times(0,\infty) \text{ in the sense of distributions,} \tag{4.71}
$$

and in the global balance of total energy (i.e. (3.26) with special choice $\psi = 1$)

$$
\int_{\partial\Omega} L(\vartheta-\Theta_0)\,\mathrm{d}S = \int_\Omega (\varrho\mathbf{f}\cdot\mathbf{u} - s_E)\,\mathrm{d}\mathbf{x}. \tag{4.72}
$$

However, to pass to the limit also in the energy balance, information (4.65) is not sufficient. There are three additional terms which we have to control:

1) The convective term

$$
\int_\Omega \varrho_\delta\,|\mathbf{u}_\delta|^2\,\mathbf{u}_\delta\cdot\nabla\psi\,\mathrm{d}\mathbf{x} \tag{4.73}
$$

Here we need $\varrho_\delta \rightharpoonup \varrho$ in $L^q(\Omega)$ for a certain $q > \frac{p}{2p-3}$, i.e. $s\gamma > \frac{p}{2p-3} = \frac{2m}{m+\alpha-1}$.

2) Stress tensor

$$
\int_\Omega \mathbb{S}^\alpha(\vartheta_\delta,\mathbf{u}_\delta)\mathbf{u}_\delta\cdot\nabla\psi\,\mathrm{d}\mathbf{x} \tag{4.74}
$$

This leads (after some computations) to restriction $m > 1$.

3) Pressure and energy

$$
\int_\Omega (\varrho_\delta^\gamma + \varrho_\delta\vartheta_\delta)\mathbf{u}_\delta\,\mathrm{d}\mathbf{x} \tag{4.75}
$$

The first part gives restriction $s > \frac{3p}{4p-3} = \frac{6m}{5m-1+\alpha}$, while the second part yields $s\gamma > q$ for $\frac{1}{q} + \frac{1}{3m} + \frac{3-p}{3p} = 1$. This leads to $s\gamma > \frac{6m}{5m-3+\alpha}$ which is less restrictive for $m > 1$ than the condition from the convective term.

Passing with $m \to \infty$ we have two conditions, namely $s > \frac{6}{5}$ and $s\gamma > 2$. Plugging these conditions into (4.58)–(4.60) (recall discussion below (4.60)) we get additionally to (4.61)

$$
\gamma > \frac{5}{3}, \qquad \gamma > \frac{2+\alpha}{3\alpha} \tag{4.76}
$$

which come from (4.58) for $s > \frac{6}{5}$ and (4.60) for $s\gamma > 2$. The other conditions are less restrictive. Next we take m finite and similarly as above get in addition to (4.62)

$$
\begin{gathered}
m > 1, \qquad m > \frac{(3\gamma-1)(1-\alpha)+2}{3(\gamma-1)}, \\
m > \frac{(3\gamma-1)(1-\alpha)}{3\gamma-5}, \qquad m > \frac{(1-\alpha)(\gamma(2-3\alpha)+\alpha)}{\alpha(6\gamma^2-9\gamma+5)-2\gamma},
\end{gathered} \tag{4.77}
$$

where the second condition comes from (4.47) with condition on $s\gamma$, the third from (4.58) with the condition on s and the last one from (4.60) with the condition on s. The other conditions are less restrictive. Note that for $\alpha = \frac{1}{2}$ we have the restriction as before while for m we get additional restrictions from (4.77).

To finish the proof, we need to prove strong convergence of the density.

5 Strong convergence of the density for $\delta \to 0^+$

Before starting to deal with the strong convergence, we recall several basic results which will be used throughout the proof. We have (see [28, Lemma 3.3])

Lemma 5.1 (Renormalized continuity equation)**.** *Assume that*

$$
\begin{gathered}
b \in C([0,\infty)) \cap C^1((0,\infty)), \\
\lim_{s\to 0^+} (sb'(s) - b(s)) \in \mathbb{R}, \\
|b'(s)| \le Cs^\lambda, \quad s \in (1,\infty), \quad \lambda \le \frac{a}{2} - 1.
\end{gathered} \tag{5.1}
$$

Let $\mathbf{u} \in W_0^{1,p}(\Omega)$, $\varrho \in L^a(\Omega)$, $a \ge \frac{p}{p-1}$, $\varrho \ge 0$ *a.e. in* Ω, *be such that*

$$
\int_{\mathbb{R}^3} \varrho \mathbf{u} \cdot \nabla \psi \, \mathrm{d}\mathbf{x} = 0
$$

for all $\psi \in C_0^\infty(\mathbb{R}^3)$ *with* ϱ, $\mathbf{u}$ *extended by zero outside of* Ω. *Then the pair* $(\varrho, \mathbf{u})$ *is a renormalized solution to the continuity equation, i.e. we have for all* $b(\cdot)$ *as specified in (5.1)*

$$
\int_{\mathbb{R}^3} \big(- b(\varrho)\mathbf{u} \cdot \nabla \psi + (\varrho b'(\varrho) - b(\varrho)) \operatorname{div} \mathbf{u} \psi \big) \, \mathrm{d}\mathbf{x} = 0 \tag{5.2}
$$

for all $\psi \in C_0^\infty(\mathbb{R}^3)$.

We introduce the operators

$$
\begin{gathered}
\nabla \Delta^{-1} v \equiv \mathcal{F}^{-1}\Big[\frac{\mathrm{i}\xi}{|\xi|^2} \mathcal{F}(v)(\xi) \Big], \\
(\mathcal{R}[v])_{ij} \equiv (\nabla \otimes \nabla \Delta^{-1})_{ij} v = \mathcal{F}^{-1}\Big[\frac{\xi_i \xi_j}{|\xi|^2} \mathcal{F}(v)(\xi) \Big]
\end{gathered} \tag{5.3}
$$

with $\mathcal{F}$ the Fourier transform, and denote

$$(\mathcal{R}[\mathbf{v}])_i = \mathcal{F}^{-1}\Big[\frac{\xi_i\xi_j}{|\xi|^2}\mathcal{F}(v_j)(\xi)\Big],$$
$$\mathcal{R}:\mathbf{A} = \mathcal{F}^{-1}\Big[\frac{\xi_i\xi_j}{|\xi|^2}\mathcal{F}(A_{ij})(\xi)\Big]. \tag{5.4}$$

We recall some properties of these operators which will be used later on. For the proof, see [11, Theorem 10.26]

Lemma 5.2 (Continuity properties of $\nabla\otimes\nabla\Delta^{-1}$ and $\nabla\Delta^{-1}$)**.** *Operator $\mathcal{R}$ is a continuous operator from $L^p(\mathbb{R}^3)$ to $L^p(\mathbb{R}^3)$ for any $1 < p < \infty$.*

Operator $\nabla\Delta^{-1}$ is a continuous linear operator from the space $L^1(\mathbb{R}^3)\cap L^2(\mathbb{R}^3)$ to $L^2(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$ and from $L^p(\mathbb{R}^3)$ to $L^{\frac{3p}{3-p}}(\mathbb{R}^3)$ for any $1 < p < 3$.

Next we recall two results on commutators, the proof can be found in [11, Theorems 10.27–10.28]:

Lemma 5.3 (Commutators I)**.** *Let $\mathbf{U}_\varepsilon \rightharpoonup \mathbf{U}$ in $L^p(\mathbb{R}^3)$, $v_\varepsilon \rightharpoonup v$ in $L^q(\mathbb{R}^3)$, where*

$$\frac{1}{p}+\frac{1}{q}=\frac{1}{s}<1. \tag{5.5}$$

Then

$$v_\varepsilon\mathcal{R}[\mathbf{U}_\varepsilon]-\mathcal{R}[v_\varepsilon]\mathbf{U}_\varepsilon \rightharpoonup v\mathcal{R}[\mathbf{U}]-\mathcal{R}[v]\mathbf{U} \tag{5.6}$$

in $L^s(\mathbb{R}^3)$.

Lemma 5.4 (Commutators II)**.** *Let $w\in W^{1,r}(\mathbb{R}^3)$, $\mathbf{z}\in L^p(\mathbb{R}^3)$, $1<r<3$, $1<p<\infty$, $\frac{1}{r}+\frac{1}{p}-\frac{1}{3}<\frac{1}{s}<1$. Then for all such s we have*

$$\|\mathcal{R}[w\mathbf{z}]-w\mathcal{R}[\mathbf{z}]\|_{a,s,\mathbb{R}^3}\le C\,\|w\|_{1,r,\mathbb{R}^3}\,\|\mathbf{z}\|_{p,\mathbb{R}^3}\,, \tag{5.7}$$

where $\frac{a}{3}=\frac{1}{s}+\frac{1}{3}-\frac{1}{p}-\frac{1}{r}$.

Here the spaces $W^{a,s}(\Omega)$ for a noninteger are the Sobolev–Slobodetskii spaces.

We introduce cut-off functions

$$T_k(z)=kT\Big(\frac{z}{k}\Big),\quad z\ge 0, k\in\mathbb{N}, \tag{5.8}$$

where $T\in C^\infty([0,\infty))$ is function with following properties

$$T(z)=\begin{cases} z \text{ for } 0\le z\le 1,\\ 2 \text{ for } z\ge 3,\\ \text{concave on } (0,\infty).\end{cases} \tag{5.9}$$

Our first aim is to prove the following identity

$$\begin{aligned}&\overline{p(\varrho,\vartheta)T_k(\varrho)}-\Big(\frac{4}{3}\mu^\alpha(\vartheta)+\xi^\alpha(\vartheta)\Big)\overline{T_k(\varrho)\mathrm{div}\,\mathbf{u}}\\ &=\overline{p(\varrho,\vartheta)}\,\overline{T_k(\varrho)}-\Big(\frac{4}{3}\mu^\alpha(\vartheta)+\xi^\alpha(\vartheta)\Big)\overline{T_k(\varrho)}\mathrm{div}\,\mathbf{u}.\end{aligned} \tag{5.10}$$

Lemma 5.5. *Under the assumptions on α, γ and m made above, equality (5.10) holds for any $k \in \mathbb{N}$.*

Proof. We follow an analogous procedure as for the case $\alpha = 1$ without radiation, see [26]. In the momentum equation (3.25) we use as a test function

$$\boldsymbol{\varphi}(\mathbf{x}) = \zeta \nabla \Delta^{-1} \left[1_\Omega T_k(\varrho_\delta)\right] \tag{5.11}$$

and in the limit equation (4.67)

$$\boldsymbol{\varphi}(\mathbf{x}) = \zeta \nabla \Delta^{-1} \left[1_\Omega \overline{T_k(\varrho)}\right], \tag{5.12}$$

with $\zeta \in C_0^\infty(\Omega)$. After routine computations we get

$$\begin{aligned}
&\int_\Omega \zeta(\mathbf{x}) \Big(\big(p(\varrho_\delta, \vartheta_\delta) + \delta \varrho_\delta^\beta + \delta \varrho_\delta^2\big) T_k(\varrho_\delta) - \mathbb{S}^\alpha(\vartheta_\delta, \mathbf{u}_\delta) : \mathcal{R}[1_\Omega T_k(\varrho_\delta)] \Big) \,\mathrm{d}\mathbf{x} \\
&= \int_\Omega \zeta(\mathbf{x}) \Big(\mathbf{u}_\delta \cdot \Big(\mathcal{R}[1_\Omega \varrho_\delta \mathbf{u}_\delta] T_k(\varrho_\delta) - \mathcal{R}[1_\Omega T_k(\varrho_\delta)] \varrho_\delta \mathbf{u}_\delta \Big) \Big) \,\mathrm{d}\mathbf{x} \\
&- \int_\Omega \zeta(\mathbf{x}) \varrho_\delta \mathbf{f} \cdot \nabla \Delta^{-1}[1_\Omega T_k(\varrho_\delta)] \,\mathrm{d}\mathbf{x} + \int_\Omega \zeta(\mathbf{x}) \mathbf{s}_F \cdot \nabla \Delta^{-1}[1_\Omega T_k(\varrho_\delta)] \,\mathrm{d}\mathbf{x} \\
&- \int_\Omega \big(p(\varrho_\delta, \vartheta_\delta) + \delta \varrho_\delta^\beta + \delta \varrho_\delta^2\big) \nabla \zeta(\mathbf{x}) \cdot \nabla \Delta^{-1}[1_\Omega T_k(\varrho_\delta)] \,\mathrm{d}\mathbf{x} \\
&+ \int_\Omega \mathbb{S}^\alpha(\vartheta_\delta, \mathbf{u}_\delta) : \nabla \zeta(\mathbf{x}) \otimes \nabla \Delta^{-1}[1_\Omega T_k(\varrho_\delta)] \,\mathrm{d}\mathbf{x} \\
&- \int_\Omega \varrho_\delta (\mathbf{u}_\delta \otimes \mathbf{u}_\delta) : \nabla \zeta(\mathbf{x}) \otimes \nabla \Delta^{-1}[1_\Omega T_k(\varrho_\delta)] \,\mathrm{d}\mathbf{x},
\end{aligned} \tag{5.13}$$

and

$$\begin{aligned}
&\int_\Omega \zeta(\mathbf{x}) \Big(\overline{p(\varrho, \vartheta)}\, \overline{T_k(\varrho)} - \mathbb{S}^\alpha(\vartheta, \mathbf{u}) : \mathcal{R}[1_\Omega \overline{T_k(\varrho)}] \Big) \,\mathrm{d}\mathbf{x} \\
&= \int_\Omega \zeta(\mathbf{x}) \Big(\mathbf{u} \cdot \Big(\mathcal{R}[1_\Omega \varrho \mathbf{u}] \overline{T_k(\varrho)} - \mathcal{R}[1_\Omega \overline{T_k(\varrho)}] \varrho \mathbf{u} \Big) \Big) \,\mathrm{d}\mathbf{x} \\
&- \int_\Omega \zeta(\mathbf{x}) \varrho \mathbf{f} \cdot \nabla \Delta^{-1}[1_\Omega \overline{T_k(\varrho)}] \,\mathrm{d}\mathbf{x} + \int_\Omega \zeta(\mathbf{x}) \mathbf{s}_F \cdot \nabla \Delta^{-1}[1_\Omega \overline{T_k(\varrho)}] \,\mathrm{d}\mathbf{x} \\
&- \int_\Omega \overline{p(\varrho, \vartheta)} \nabla \zeta(\mathbf{x}) \cdot \nabla \Delta^{-1}[1_\Omega \overline{T_k(\varrho)}] \,\mathrm{d}\mathbf{x} \\
&+ \int_\Omega \mathbb{S}^\alpha(\vartheta, \mathbf{u}) : \nabla \zeta(\mathbf{x}) \otimes \nabla \Delta^{-1}[1_\Omega \overline{T_k(\varrho)}] \,\mathrm{d}\mathbf{x} \\
&- \int_\Omega \varrho (\mathbf{u} \otimes \mathbf{u}) : \nabla \zeta(\mathbf{x}) \otimes \nabla \Delta^{-1}[1_\Omega \overline{T_k(\varrho)}] \,\mathrm{d}\mathbf{x},
\end{aligned} \tag{5.14}$$

where we have used $\mathcal{R}[1_\Omega \varrho_\delta \mathbf{u}_\delta] = \mathcal{R}[1_\Omega \varrho \mathbf{u}] = 0$ as a consequence of $\operatorname{div}(\varrho_\delta \mathbf{u}_\delta) = \operatorname{div}(\varrho \mathbf{u}) = 0$ in $\mathcal{D}'(\Omega)$. Using Lemma 5.3 with

$$\begin{aligned}
v_\delta = T_k(\varrho_\delta) &\rightharpoonup \overline{T_k(\varrho)} \quad \text{in } L^{t_1}(\Omega), t_1 < \infty \\
\mathbf{U}_\delta = \varrho_\delta \mathbf{u}_\delta &\rightharpoonup \varrho \mathbf{u} \quad \text{in } L^{t_2}(\Omega), t_2 < \frac{3ps\gamma}{3p + s\gamma(3-p)},
\end{aligned} \tag{5.15}$$

we have

$$\mathcal{R}[1_\Omega\varrho_\delta\mathbf{u}_\delta]T_k(\varrho_\delta) - \mathcal{R}[1_\Omega T_k(\varrho_\delta)]\varrho_\delta\mathbf{u}_\delta \rightharpoonup \mathcal{R}[1_\Omega\varrho\mathbf{u}]\overline{T_k(\varrho)} - \mathcal{R}[1_\Omega\overline{T_k(\varrho)}]\varrho\mathbf{u} \tag{5.16}$$

in $L^q(\Omega)$ for $q < t_2$. For $\gamma > \frac{3}{2}$ and $m > \frac{(1-\alpha)\gamma}{2\gamma-3}$ we can find $s > 1$ such that $q > \frac{3p}{4p-3}$ and since $\mathbf{u}_\delta \to \mathbf{u}$ in L^t for $t < \frac{3p}{3-p}$, we verify

$$\begin{aligned}&\int_\Omega \zeta(\mathbf{x})\Big(\mathbf{u}_\delta\cdot\big(\mathcal{R}[1_\Omega\varrho_\delta\mathbf{u}_\delta]T_k(\varrho_\delta) - \mathcal{R}[1_\Omega T_k(\varrho_\delta)]\varrho_\delta\mathbf{u}_\delta\big)\Big)\,\mathrm{d}\mathbf{x}\\ &\quad\to \int_\Omega \zeta(\mathbf{x})\Big(\mathbf{u}\cdot\big(\mathcal{R}[1_\Omega\varrho\mathbf{u}]\overline{T_k(\varrho)} - \mathcal{R}[1_\Omega\overline{T_k(\varrho)}]\varrho\mathbf{u}\big)\Big)\,\mathrm{d}\mathbf{x}.\end{aligned} \tag{5.17}$$

Comparing (5.13) and (5.14) we easily end up with

$$\begin{aligned}&\int_\Omega \zeta(\mathbf{x})\left(\overline{p(\varrho,\vartheta)T_k(\varrho)} - \overline{p(\varrho,\vartheta)}\,\overline{T_k(\varrho)}\right)\mathrm{d}\mathbf{x}\\ &= \int_\Omega \zeta(\mathbf{x})\left(\overline{\mathbb{S}^\alpha(\vartheta,\mathbf{u}) : \mathcal{R}[1_\Omega T_k(\varrho)]} - \mathbb{S}^\alpha(\vartheta,\mathbf{u}) : \mathcal{R}[1_\Omega\overline{T_k(\varrho)}]\right)\mathrm{d}\mathbf{x}.\end{aligned} \tag{5.18}$$

Next we have

$$\begin{aligned}\int_\Omega \zeta(\mathbf{x})\overline{\mathbb{S}^\alpha(\vartheta,\mathbf{u}) : \mathcal{R}[1_\Omega T_k(\varrho)]}\,\mathrm{d}\mathbf{x} &= \lim_{\delta\to0^+}\int_\Omega \zeta(\mathbf{x})\Big(\frac{4}{3}\mu^\alpha(\vartheta_\delta) + \xi^\alpha(\vartheta_\delta)\Big)\mathrm{div}\,\mathbf{u}_\delta T_k(\varrho_\delta)\,\mathrm{d}\mathbf{x}\\ &+ \lim_{\delta\to0^+}\int_\Omega T_k(\varrho_\delta)\Big(\mathcal{R}\big[\zeta(\mathbf{x})\mu^\alpha(\vartheta_\delta)(\nabla\mathbf{u}_\delta + (\nabla\mathbf{u}_\delta)^T)\big]\\ &\qquad -\zeta(\mathbf{x})\mu^\alpha(\vartheta_\delta)\mathcal{R} : \big[\nabla\mathbf{u}_\delta + (\nabla\mathbf{u}_\delta)^T\big]\Big)\,\mathrm{d}\mathbf{x}\end{aligned} \tag{5.19}$$

as well as a similar expression for the limit term. We employ Lemma 5.4 with

$$\begin{aligned}w = \zeta(\mathbf{x})\mu^\alpha(\vartheta_\delta) \sim 1 + \vartheta_\delta^\alpha, \quad & w \in W^{1,r}(\Omega), r = \min\Big\{2, \frac{3m}{m+\alpha}\Big\}\\ z_i = \partial_j(u_\delta)_i + \partial_i(u_\delta)_j, j = 1,2,3, \quad & \mathbf{z} \in L^p(\Omega)\end{aligned} \tag{5.20}$$

and conclude that

$$\mathcal{R}\left[\zeta(\mathbf{x})\mu^\alpha(\vartheta_\delta)(\nabla\mathbf{u}_\delta + (\nabla\mathbf{u}_\delta)^T)\right] - \zeta(\mathbf{x})\mu^\alpha(\vartheta_\delta)\mathcal{R} : \left[\nabla\mathbf{u}_\delta + (\nabla\mathbf{u}_\delta)^T\right] \tag{5.21}$$

is bounded in $W^{a,s}(\Omega)$ with $s < \frac{3rp}{3p+3r-pr}$[3] and $a = 3(\frac{1}{s} + \frac{1}{3} - \frac{1}{p} - \frac{1}{r})$. Thus the expression in (5.21) converges strongly to

$$\mathcal{R}\left[\zeta(\mathbf{x})\mu^\alpha(\vartheta)(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)\right] - \zeta(\mathbf{x})\mu^\alpha(\vartheta)\mathcal{R} : \left[\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right] \tag{5.22}$$

in some $L^q(\Omega)$, $q > 1$. Since $T_k(\varrho_\delta) \rightharpoonup \overline{T_k(\varrho)}$ in all $L^p(\Omega)$, $p < \infty$, the proof of Lemma 5.5 is finished. □

Remark 5.1. Note that a similar procedure is used to get the strong convergence of the density in the limit passage $\varepsilon \to 0^+$; however, due to higher integrability we may take ϱ_ε instead of $T_k(\varrho_\varepsilon)$ and the proof is much simpler. Moreover, we get for free that the renormalized continuity equation is fulfilled (see Lemma 5.1) for the limit pair $(\mathbf{u}, \varrho)$ which also significantly simplifies the following steps.

[3]Note that $s > 1$ for $m > \frac{1+\alpha}{3}$.

Recall that we are not able to apply Lemma 5.1. Therefore we introduce the oscillation defect measure

$$\mathbf{osc_q}[\varrho_\delta \to \varrho](Q) = \sup_{k>1} \Big(\limsup_{\delta\to 0^+} \int_Q |T_k(\varrho_\delta) - T_k(\varrho)|^q \, \mathrm{d}\mathbf{x} \Big). \tag{5.23}$$

We have

Lemma 5.6. *Let $\Omega \subset \mathbb{R}^3$ be an open set and let*

$$\begin{gathered} \varrho_\delta \rightharpoonup \varrho \quad \text{in } L^1(\Omega) \\ \mathbf{u}_\delta \rightharpoonup \mathbf{u} \quad \text{in } L^p(\Omega) \\ \nabla\mathbf{u}_\delta \rightharpoonup \nabla\mathbf{u} \text{ in } L^p(\Omega), \quad p>1. \end{gathered} \tag{5.24}$$

Let moreover

$$\mathbf{osc_q}[\varrho_\delta \to \varrho](\Omega) < \infty \tag{5.25}$$

for $q > \frac{p}{p-1}$, where $(\varrho_\delta, \mathbf{u}_\delta)$ solve the renormalized continuity equation (5.2). Then the limit functions are renormalized solutions to the continuity equation in the sense of Definition 1.3.

Proof. See [11, Lemma 3.8] in the evolutionary case, the adaptation to the steady case is easy. □

To apply Lemma 5.6, we need to show (5.25); all the other assumptions are satisfied. First, we recall [26, Lemma 18] proved in the case $\alpha = 1$; generalization for $\alpha \in (0,1)$ is straightforward

Lemma 5.7. *Under assumptions made in Section 1 it holds*

$$\begin{gathered} \limsup_{\delta\to 0^+} \int_\Omega d\,|T_k(\varrho_\delta) - T_k(\varrho)|^{\gamma+1} \,\mathrm{d}\mathbf{x} \le \int_\Omega \Big(\overline{p(\varrho,\vartheta)T_k(\varrho)} - \overline{p(\varrho,\vartheta)}\,\overline{T_k(\varrho)} \Big) \,\mathrm{d}\mathbf{x} \\ \limsup_{\delta\to 0^+} \int_\Omega \frac{d}{1+\vartheta_\delta^\alpha} |T_k(\varrho_\delta) - T_k(\varrho)|^{\gamma+1} \,\mathrm{d}\mathbf{x} \le \int_\Omega \frac{1}{1+\vartheta^\alpha} \Big(\overline{p(\varrho,\vartheta)T_k(\varrho)} - \overline{p(\varrho,\vartheta)}\,\overline{T_k(\varrho)} \Big) \,\mathrm{d}\mathbf{x}. \end{gathered} \tag{5.26}$$

We have

Lemma 5.8. *Let $(\varrho_\delta, \mathbf{u}_\delta, \vartheta_\delta)$ be as above and moreover let $m > \frac{1+\alpha+\gamma(1-\alpha)}{3(\gamma-1)}$. Then there exists $q > \frac{p}{p-1}$ such that (5.25) holds.*

Proof. Denoting

$$G_k(t,\mathbf{x},z) = d\,|T_k(z) - T_k(\varrho(t,\mathbf{x}))|^{\gamma+1}, \tag{5.27}$$

we apply Lemma 5.26

$$\overline{G_k(\cdot,\cdot,\varrho)} \le \overline{p(\varrho,\vartheta)T_k(\varrho)} - \overline{p(\varrho,\vartheta)}\,\overline{T_k(\varrho)} \tag{5.28}$$

and using (5.10)

$$\overline{G_k(\cdot,\cdot,\varrho)} \le \Big(\frac{4}{3}\mu^\alpha(\vartheta) + \xi^\alpha(\vartheta)\Big) \Big(\overline{T_k(\varrho)\operatorname{div}\mathbf{u}} - \overline{T_k(\varrho)}\operatorname{div}\mathbf{u} \Big) \tag{5.29}$$

for all $k \in \mathbb{N}$. Now easily

$$\begin{aligned}(5.30)\qquad \int_\Omega (1+\vartheta^\alpha)^{-1}\overline{G_k(t,\mathbf{x},\varrho)}\,\mathrm{d}\mathbf{x} &\leq C\sup_{\delta>0}\|\mathrm{div}\,\mathbf{u}_\delta\|_p \limsup_{\delta\to 0^+}\|T_k(\varrho_\delta)-T_k(\varrho)\|_{\frac{p}{p-1}}\\ &\leq C\limsup_{\delta\to 0^+}\|T_k(\varrho_\delta)-T_k(\varrho)\|_{\frac{p}{p-1}}.\end{aligned}$$

Finally, we have

$$\begin{aligned}(5.31)\qquad \int_\Omega |T_k(\varrho_\delta)-T_k(\varrho)|^q\,\mathrm{d}\mathbf{x} &\leq \int_\Omega |T_k(\varrho_\delta)-T_k(\varrho)|^q (1+\vartheta^\alpha)^{-\frac{q}{\gamma+1}}(1+\vartheta^\alpha)^{\frac{q}{\gamma+1}}\,\mathrm{d}\mathbf{x}\\ &\leq C\int_\Omega |T_k(\varrho_\delta)-T_k(\varrho)|^{\gamma+1}(1+\vartheta^\alpha)^{-1}\,\mathrm{d}\mathbf{x} + C\int_\Omega (1+\vartheta^\alpha)^{\frac{q}{\gamma+1-q}}\,\mathrm{d}\mathbf{x}.\end{aligned}$$

We can control the second integral if

$$(5.32)\qquad \frac{\alpha q}{\gamma+1-q}\leq 3m, \quad \text{i.e. } q\leq \frac{3m(\gamma+1)}{3m+\alpha}$$

and as we need $q > \frac{p}{p-1}$ we get in view of (4.11)

$$(5.33)\qquad m > \frac{1+\alpha+\gamma(1-\alpha)}{3(\gamma-1)},$$

cf. [26]. The proof is finished. □

Now we are in position to finish the proof of the strong convergence of ϱ_δ. Using the renormalized continuity equation (5.2) with

$$(5.34)\qquad b(\varrho) = \varrho\int_1^\varrho \frac{T_k(z)}{z^2}\,\mathrm{d}z$$

we get

$$(5.35)\qquad \int_\Omega T_k(\varrho)\mathrm{div}\,\mathbf{u}\,\mathrm{d}\mathbf{x} = 0.$$

Since we also have

$$(5.36)\qquad \int_\Omega T_k(\varrho_\delta)\mathrm{div}\,\mathbf{u}_\delta\,\mathrm{d}\mathbf{x} = 0,$$

i.e.

$$(5.37)\qquad \int_\Omega \overline{T_k(\varrho)\mathrm{div}\,\mathbf{u}}\,\mathrm{d}\mathbf{x} = 0,$$

identity (5.10) yields

$$(5.38)\qquad \int_\Omega \frac{1}{\frac{4}{3}\mu^\alpha(\vartheta)+\xi^\alpha(\vartheta)}\Big(\overline{p(\varrho,\vartheta)T_k(\varrho)} - \overline{p(\varrho,\vartheta)}\,\overline{T_k(\varrho)}\Big)\,\mathrm{d}\mathbf{x} = \int_\Omega \Big(T_k(\varrho)-\overline{T_k(\varrho)}\Big)\,\mathrm{div}\,\mathbf{u}\,\mathrm{d}\mathbf{x}.$$

As $\lim_{k\to\infty} \|T_k(\varrho) - \varrho\|_1 = \lim_{k\to\infty} \|\overline{T_k(\varrho)} - \varrho\|_1 = 0$, we have

$$\lim_{k\to\infty} \|T_k(\varrho) - \overline{T_k(\varrho)}\|_1 = 0. \tag{5.39}$$

Therefore

$$\lim_{k\to\infty} \int_\Omega \frac{1}{1+\vartheta^\alpha} \Big(\overline{p(\varrho,\vartheta) T_k(\varrho)} - \overline{p(\varrho,\vartheta)}\, \overline{T_k(\varrho)} \Big) \, \mathrm{d}\mathbf{x} = 0. \tag{5.40}$$

Using Lemma 5.7

$$\lim_{k\to\infty} \limsup_{\delta\to 0^+} \int_\Omega \frac{1}{1+\vartheta_\delta^\alpha} |T_k(\varrho_\delta) - T_k(\varrho)|^{\gamma+1} \, \mathrm{d}\mathbf{x} = 0; \tag{5.41}$$

whence

$$\lim_{k\to\infty} \limsup_{\delta\to 0^+} \int_\Omega |T_k(\varrho_\delta) - T_k(\varrho)|^q \, \mathrm{d}\mathbf{x} = 0, \tag{5.42}$$

where q is the same exponent as in Lemma 5.8. Since

$$\|\varrho_\delta - \varrho\|_1 \le \|\varrho_\delta - T_k(\varrho_\delta)\|_1 + \|T_k(\varrho_\delta) - T_k(\varrho)\|_1 + \|T_k(\varrho) - \varrho\|_1 \,, \tag{5.43}$$

we finally conclude

$$\begin{array}{lll} \varrho_\delta \to \varrho & \text{in } L^1(\Omega), & \text{and thus} \\ \varrho_\delta \to \varrho & \text{in } L^q(\Omega) & \forall q < s\gamma. \end{array} \tag{5.44}$$

This finishes the proof of Theorem 1.1.

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