Mathematical theory of compressible viscous fluids

Eduard Feireisl and Milan Pokorný

Another advantage of a mathematical statement is that it is so definite that it might be definitely wrong. . . Some verbal statements have not this merit.

F.L.Richardson (1881-1953)

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1 Introduction

Despite the concerted effort of generations of excellent mathematicians, the fundamental problems in partial differential equations related to continuum fluid mechanics remain largely open. Solvability of the Navier-Stokes system describing the motion of an incompressible viscous fluid is one in the sample of millenium problems proposed by Clay Institute, see [3]. In contrast with these apparent theoretical difficulties, the Navier-Stokes system became a well established model serving as a reliable basis of investigation in continuum fluid mechanics, including the problems involving turbulence phenomena. An alternative approach to problems in fluid mechanics is based on the concept of weak solutions. As a matter of fact, the balance laws, expressed in classical fluid mechanics in the form of partial differential equations, have their origin in integral identities that seem to be much closer to the modern weak formulation of these problems. Leray [6] constructed the weak solutions to the incompressible Navier-Stokes system as early as in 1930, and his "turbulent solutions" are still the only ones available for investigating large data and/or problems on large time intervals. Recently, the real breakthrough is the work of Lions citeLI4 who generalized Leray's theory to the case of barotropic compressible viscous fluids (see also Vaigant and Kazhikhov [9]). The quantities playing a crucial role in the description of density oscillations as the effective viscous flux were identified and used in combination with a renormalized version of the equation of continuity to obtain first large data/large time existence results in the framework of compressible viscous fluids.

The main goal of the this lecture series is present the mathematical theory of *compressible barotropic fluids* in the framework of Lions [7], together with the extensions developed in [4]. We focus on the crucial question of *stability* of a family of weak solutions that is the core of the abstract theory, with implications to numerical analysis and the associated real world applications. For the sake of clarity of presentation, we discuss first the case, where the pressure term has sufficient growth for large value of the density yielding sufficiently strong energy bonds. We also start with the simplest geometry of the physical space, here represented by a cube, on the boundary of which the fluid satisfies the slip boundary conditions. As is well-known, such a situation may be reduced to studying the purely spatially periodic case, where the additional difficulties connected with the presence of boundary conditions is entirely eliminated.

2 Mathematical model

As the main goal of this lecture series is the *mathematical theory*, we avoid a detailed derivation of the mathematical model of a compressible viscous fluid. Remaining on the platform of *continuum fluid mechanics*, we suppose that the motion of a compressible barotropic fluid is described by means of two basic *fields*:

the mass density $\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \varrho = \varrho(t, x)$,
the velocity field $\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \mathbf{u} = \mathbf{u}(t, x)$,

functions of the time $t \in R$ and the spatial position $x \in R^3$.

2.1 Mass conservation

Let us recall the classical argument leading to the mathematical formulation of the physical principle of *mass conservation*, see Chorin and Marsden [1]. Consider a volume $B \subset R^3$ containing a fluid of density ϱ. The change of the total mass of the fluid contained in B during a time interval $[t_1, t_2]$, $t_1 < t_2$ is given as

$$\int_B \varrho(t_2, x) \, \mathrm{d}x - \int_B \varrho(t_1, x) \, \mathrm{d}x.$$

One of the basic laws of physics incorporated in continuum mechanics as the *principle of mass conservation* asserts that mass is neither created nor destroyed. Accordingly, the change of the fluid mass in B is only because of the mass *flux* through the boundary ∂B, here represented by $\varrho\mathbf{u} \cdot \mathbf{n}$, where $\mathbf{n}$ denotes the *outer* normal vector to ∂B:

$$\int_B \varrho(t_2, x) \, \mathrm{d}x - \int_B \varrho(t_1, x) \, \mathrm{d}x = -\int_{t_1}^{t_2} \int_{\partial B} \varrho(t, x)\mathbf{u}(t, x) \cdot \mathbf{n}(x) \, \mathrm{dS}_x \, dt. \quad (2.1)$$

One should remember formula (2.1) since it contains *all* relevant piece of information provided by *physics*. The following discussion is based on *mathematical* arguments based on the (unjustified) hypotheses of *smoothness* of all field in question. To begin, apply Gauss-Green theorem to rewrite (2.1) in the form:

$$\int_B \varrho(t_2, x) \, \mathrm{d}x - \int_B \varrho(t_1, x) \, \mathrm{d}x = -\int_{t_1}^{t_2} \int_B \mathrm{div}_x\Big(\varrho(t, x)\mathbf{u}(t, x)\Big) \, \mathrm{d}x \, \mathrm{d}t.$$

Furthermore, fixing $t_1 = t$ and performing the limit $t_2 \to t_1$ we may use the mean value theorem to obtain

$$\int_B \partial_t \varrho(t,x) \ \mathrm{d}x = \lim_{t_2 \to t} \frac{1}{t_2 - t} \int_B \varrho(t_2, x) \ \mathrm{d}x - \int_B \varrho(t,x) \ \mathrm{d}x \qquad (2.2)$$

$$= -\lim_{t_2 \to t} \frac{1}{t_2 - t} \int_{t_1}^{t_2} \int_B \mathrm{div}_x \Big(\varrho(t,x) \mathbf{u}(t,x) \Big) \ \mathrm{d}x \ \mathrm{d}t$$

$$= -\int_B \mathrm{div}_x \Big(\varrho(t,x) \mathbf{u}(t,x) \Big) \ \mathrm{d}x.$$

Finally, as relation (2.2) should hold for *any* volume element B, we may infer that

$$\partial_t \varrho(t,x) + \mathrm{div}_x \Big(\varrho(t,x) \mathbf{u}(t,x) \Big) = 0. \qquad (2.3)$$

Relation (2.3) is a first order partial differential equation called *equation of continuity*.

2.2 Balance of momentum

Using arguments similar to the preceding part, we derive *balance of momentum* in the form

$$\partial_t \Big(\varrho(t,x) \mathbf{u}(t,x) \Big) + \mathrm{div}_x \Big(\varrho(t,x) \mathbf{u}(t,x) \otimes \mathbf{u}(t,x) \Big) = \mathrm{div}_x \mathbb{T}(t,x) + \varrho(t,x) \mathbf{f}(t,x),$$
$$(2.4)$$

or, equivalently (cf. (2.3),

$$\varrho(t,x) \Big[\partial_t \mathbf{u}(t,x) + \mathbf{u}(t,x) \cdot \nabla_x \mathbf{u}(t,x) \Big] = \mathrm{div}_x \mathbb{T}(t,x) + \varrho(t,x) \mathbf{f}(t,x),$$

where the tensor $\mathbb{T}$ is the Cauchy stress and $\mathbf{f}$ denotes the (specific) external force acting on the fluid.

We adopt the standard *mathematical* definition of fluids in the form of *Stokes' law*

$$\mathbb{T} = \mathbb{S} - p\mathbb{I},$$

where tnS is the viscous stress and p is a scalar function termed pressure. In addition, we suppose that the viscous stress is a *linear* function of the velocity gradient, specifically $\mathbb{S}$ obeys *Newton's rheological law*

$$\mathbb{S} = \mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta \mathrm{div}_x \mathbf{u} \mathbb{I}, \qquad (2.5)$$

with the shear viscosity coefficient μ and the bulk viscosity coefficient η, here assumed constant, $\mu > 0$, $\eta \geq 0$.

In order to close the system, we suppose the fluid is *barotropic*, meaning the pressure p is an explicitly given function of the density $p = p(\varrho)$. Accordingly,

$$\mathrm{div}_x \mathbb{T} = \mu \Delta \mathbf{u} + (\lambda + \mu)\nabla_x \mathrm{div}_x \mathbf{u}, \ \mu > 0, \ \lambda \geq -\frac{2}{3}\mu,$$

and equations (2.3), (2.4) can be written in a concise form as

NAVIER-STOKES SYSTEM

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{2.6}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \mu \Delta \mathbf{u} + (\lambda + \mu)\nabla_x \mathrm{div}_x \mathbf{u} + \varrho \mathbf{f}. \tag{2.7}$$

The system of equations (2.6), (2.7) should be compared with a "more famous" incompressible Navier-Stokes system, where the density is constant, say $\varrho \equiv 1$, while (2.6), (2.7) "reduces" to

$$\mathrm{div}_x \mathbf{u} = 0, \tag{2.8}$$

$$\partial_t \mathbf{u} + \mathrm{div}_x(\mathbf{u} \otimes \mathbf{u}) + \nabla_x p = \mu \Delta \mathbf{u} + \mathbf{f}. \tag{2.9}$$

Unlike in (2.7), the pressure p in (2.9) is an unknown function determined (implicitly) by the fluid motion! The pressure in the *incompressible* Navier-Stokes system has non-local character and may depend on the far field behavior of the fluid system.

2.3 Spatial domain and boundary conditions

In the real world applications, the fluid is confined to a bounded spatial domain $\Omega \subset R^3$. The presence of the physical boundary $\partial\Omega$ and the associated problem of fluid-structure interaction represent a source of substantial difficulties in the mathematical analysis of fluids in motion. In order to avoid technicalities, we suppose that the motion is *space-periodic*, specifically,

$$\varrho(t, x) = \varrho(t, x + \mathbf{a}), \ \mathbf{u}(t, x) = \mathbf{u}(t, x + \mathbf{a}) \text{ for all } t, x,$$

where the period vector $\mathbf{a} \in R^3$ is given. Equivalently, we may assume that Ω is a flat *torus*,

$$\Omega = [0, a_1]|_{\{0, a_1\}} \times [0, a_2]|_{\{0, a_2\}} \times [0, a_3]|_{\{0, a_3\}}.$$

The space-periodic boundary conditions have a nice physical interpretation in fluid mechanics, see Ebin [2]. Indeed, if we restrict ourselves to the classes of functions defined on the torus Ω and satisfying the extra geometric restrictions:

$$\varrho(t, x) = \varrho(t, -x), \ \ u_i(t, \cdot, x_i, \cdot) = -u_i(t, \cdot, -x_i, \cdot), \ \ i = 1, 2, 3,$$

$$u_i(t, \cdot, x_j, \cdot) = u_i(t, \cdot, -x_j, \cdot) \text{ for } i \neq j,$$

and, similarly,

$$f_i(t, \cdot, x_i, \cdot) = -f_i(t, \cdot, -x_i, \cdot), \ \ f_i(t, \cdot, x_j, \cdot) = f_i(t, \cdot, -x_j, \cdot) \text{ for } i \neq j,$$

we can check that

- the equations (2.6), (2.7) are invariant with respect to the above transformations;

- the velocity field $\mathbf{u}$ satisfies the so-called *complete slip* conditions

$$\mathbf{u} \cdot \mathbf{n} = 0, \ \ [\mathbb{S}\mathbf{n}] \times \mathbf{n} = 0 \tag{2.10}$$

on the boundary of the spatial block $[0, a_1] \times [0, a_2] \times [0, a_3]$.

We remark that the most commonly used boundary conditions for *viscous* fluids confined to a general spatial domain Ω (not necessarily a flat torus) are the *no-slip*

$$\mathbf{u}|_{\partial\Omega} = 0.$$

As a matter of fact, the problem of the choice of correct boundary conditions in the real world applications is rather complex, some parts of the boundaries may consist of a different fluid in motion, or the fluid domain is not *a priori* known (free boundary problems). The interested reader may consult Priezjev and Troian [8] for relevant discussion.

2.4 Initial conditions

Given the initial state at a reference time t_0, say $t_0 = 0$, the time evolution of the fluid is determined as a solution of the Navier-Stokes system (2.6), (2.7). It is convenient to introduce the initial density

$$\varrho(0, x) = \varrho_0(x), \; x \in \Omega, \tag{2.11}$$

together with the initial distribution of the momentum,

$$(\varrho\mathbf{u})(0, x) = (\varrho\mathbf{u})_0(x), \; x \in \Omega, \tag{2.12}$$

as, strictly speaking, the momentum balance (2.7) is an evolutionary equation for $\varrho\mathbf{u}$ rather than $\mathbf{u}$. Such a difference will become clear in the so-called *weak* formulation of the problem discussed in the forthcoming section.

3 Weak solutions

A vast class of *non-linear* evolutionary problems arising in mathematical fluid mechanics is not known to admit classical (differentiable, smooth) solutions for all choices of data and on an arbitrary time interval. On the other hand, most of the real world problems call for solutions defined in-the-large approached in the numerical simulations. In order to perform a rigorous analysis, we have to introduce a concept of *generalized* or *weak* solutions, for which derivatives are interpreted in the sense of distributions. It rn represented by viscosity should provide a strong regularizing effect. Another motivation, at least in the case of the compressible Navier-Stokes system (2.6), (2.7), is the possibility to study the fluid dynamics emanating from irregular initial state, for instance, the density ϱ_0 may not be continuous. As shown by Hoff [5], the singularities incorporated initially will "survive" in the system at any time; thus the weak solutions are necessary in order to describe the dynamics.

3.1 Equation of continuity - weak formulation

We consider equation (2.6) on the space-time cylinder $(0, T) \times \Omega$, where Ω is the flat torus introduced in Section 2.3. Multiplying (2.6) on $\varphi \in$

$C_c^\infty((0,T) \times \Omega)$, integrating the resulting expression over $(0,T) \times \Omega$, and performing by-parts integration, we obtain

$$\int_0^T \int_\Omega \Big(\varrho(t,x)\partial_t \varphi(t,x) + \varrho(t,x)\mathbf{u}(t,x) \cdot \nabla_x \varphi(t,x) \Big) \, \mathrm{d}x \, \mathrm{d}t = 0. \qquad (3.1)$$

Definition 3.1 *We say that a pair of functions ϱ, $\mathbf{u}$ is a* weak solution *to equation (2.6) in the space-time cylinder $(0,T) \times \Omega$ if ϱ, $\varrho\mathbf{u}$ are locally integrable in $(0,T) \times \Omega$ and the integral identity (3.1) holds for any test function $\varphi \in C_c^\infty((0,T) \times \Omega)$.*

3.1.1 Weak-strong compatibility

It is easy to see that any classical (smooth) solution of equation (2.6) is also a weak solution. Similarly, any weak solution that is continuously differentiable satisfies (2.6) pointwise. Such a property is called *weak-strong compatibility.*

3.1.2 Weak continuity

Up to now, we have left apart the problem of satisfaction of the initial condition (2.11). Obviously, some kind of *weak continuity* is needed for (2.11) to make sense. To this end, we make an extra hypothesis, namely,

$$\varrho\mathbf{u} \in L^1(0,T; L^1(\Omega; R^3)). \qquad (3.2)$$

Taking

$$\varphi(t,x) = \psi(t)\phi(x), \ \ \psi \in C_c^\infty(0,T), \ \ \phi \in C_c^\infty(\Omega)$$

as a test function in (3.1) we may infer, by virtue of (3.2), that the function

$$t \mapsto \int_\Omega \varrho(t,x)\phi(x) \, \mathrm{d}x \text{ is absolutely continuous in } [0,T] \qquad (3.3)$$

for any $\phi \in C_c^\infty(\Omega)$. In particular, the initial condition (2.11) may be satisfied in the sense that

$$\lim_{t \to 0+} \int_\Omega \varrho(t,x)\phi(x) \, \mathrm{d}x = \int_\Omega \varrho_0(x)\phi(x) \, \mathrm{d}x \text{ for any } \phi \in C_c^\infty(\Omega).$$

Now, take

$$\varphi_\varepsilon(t, x) = \psi_\varepsilon(t)\varphi(t, x), \ \varphi \in C_c^\infty([0, T] \times \Omega),$$

where $\psi_\varepsilon \in C_c^\infty(0, \tau)$,

$$0 \le \psi_\varepsilon \le 1, \ \psi_\varepsilon \nearrow 1_{[0,\tau]} \text{ as } \varepsilon \to 0.$$

Taking φ_ε as a test function in (3.1) and letting $\varepsilon \to 0$, we conclude, making use of (3.3), that

$$\int_\Omega \varrho(\tau, x)\varphi(\tau, x) \, \mathrm{d}x - \int_\Omega \varrho_0(x)\varphi(0, x) \, \mathrm{d}x \tag{3.4}$$

$$= \int_0^\tau \int_\Omega \Big(\varrho(t, x)\partial_t\varphi(t, x) + \varrho(t, x)\mathbf{u}(t, x) \cdot \nabla_x\varphi(t, x)\Big) \, \mathrm{d}x \, \mathrm{d}t$$

for any $\tau \in [0, T]$ and any $\varphi \in C_c^\infty([0, T] \times \Omega)$.

Formula (3.4) can be alternatively used a definition of *weak solution* to problem (2.6), (2.11). It is interesting to compare (3.4) with the original integral formulation of the principle of mass conservation stated in (2.1). To this end, we take

$$\varphi_\varepsilon(t, x) = \phi_\varepsilon(x),$$

with $\phi_\varepsilon \in C_c^\infty(B)$ such that

$$0 \le \phi_\varepsilon \le 1, \ \phi_\varepsilon \nearrow 1_B \text{ as } \varepsilon \to 0.$$

It is easy to see that

$$\int_\Omega \varrho(\tau, x)\varphi_\varepsilon(\tau, x) \, \mathrm{d}x - \int_\Omega \varrho_0(x)\varphi_\varepsilon(0, x) \, \mathrm{d}x \to \int_B \varrho(\tau, x) \, \mathrm{d}x - \int_B \varrho_0(x) \, \mathrm{d}x \text{ as } \varepsilon \to 0,$$

which coincides with the expression on the left-hand side of (2.1). Consequently, the right-hand side of (3.4) must posses a limit and we set

$$\int_0^\tau \int_\Omega \varrho(t, x)\mathbf{u}(t, x) \cdot \nabla_x\phi_\varepsilon(x) \, \mathrm{d}x \, \mathrm{d}t \to -\int_0^\tau \int_{\partial B} \varrho(t, x)\mathbf{u}(t, x) \cdot \mathbf{n} \, \mathrm{dS}_x \, \mathrm{d}t.$$

In other words, the weak solutions possess a *normal trace* on the boundary of the cylinder $(0, \tau) \times B$ that satisfies (2.1), see Chen and Frid [?] for more elaborate treatment of the normal traces of solutions to conservation laws.

3.1.3   Total mass conservation

Taking $\varphi = 1$ for $t \in [0, \tau]$ in (3.4) we obtain

$$\int_\Omega \varrho(\tau, x)\ \mathrm{d}x = \int_\Omega \varrho_0(x)\ \mathrm{d}x = M_0 \text{ for any } \tau \geq 0, \qquad (3.5)$$

meaning, the total mass M_0 of the fluid is a constant of motion.

3.2   Balance of momentum - weak formulation

Similarly to the preceding part, we introduce a weak formulation of the balance of momentum (2.7):

Definition 3.2 *The functions ϱ, $\mathbf{u}$ represent a* weak solution *to the momentum equation (2.7) in the set $(0, T) \times \Omega$ if the integral identity*

$$\int_0^T \int_\Omega \Big((\varrho\mathbf{u})(t, x)\partial_t\varphi(t, x) + (\varrho\mathbf{u} \otimes \mathbf{u})(t, x) : \nabla_x\varphi(t, x)$$

$$+ p(\varrho)(t, x)\mathrm{div}_x\varphi(t, x) \Big)\ \mathrm{d}x\ \mathrm{d}t \qquad (3.6)$$

$$= \int_0^T \int_\Omega \Big(\mu\nabla_x\mathbf{u}(t, x) : \nabla_x\varphi(t, x)$$

$$+ (\lambda + \mu)\mathrm{div}_x\mathbf{u}(t, x)\mathrm{div}_x\varphi(t, x) - \varrho(t, x)\mathbf{f}(t, x) \cdot \varphi(t, x) \Big)\ \mathrm{d}x\ \mathrm{d}t$$

is satisfied for any test function $\varphi \in C_c^\infty((0, T) \times \Omega; R^3)$.

   Of course, we have tacitly assume that all quantities appearing in (3.6) are at least locally integrable in $(0, T) \times \Omega$.

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