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From the Finite Element Method toward the Isogeometric Analysis in an Object Oriented Computing Environment

Presentation Outline

- **Motivation**
- **B-spline basis**
- **T-splines = NURBS + PB-splines**
- **Principles of OO design**
- **OOFEM**
- **OO design of IGA module**
- **Numerical example**
- **Summary**

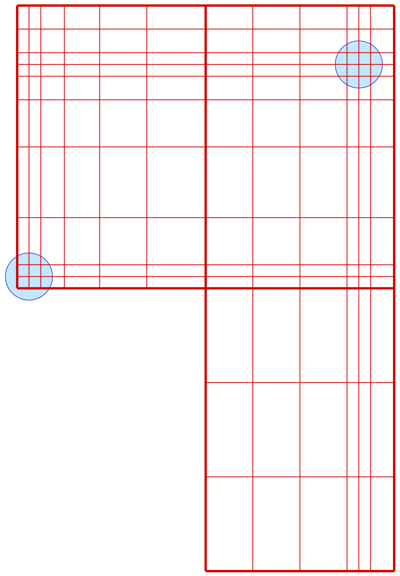
Motivation

Isogeometric Analysis

- recently introduced alternative to the FEM
- employs the same functions for the description of geometry and for the approximation of the solution on that geometry
  - eliminates costly FE mesh generation
  - geometric preprocessing still required
- outperforms classical FEM in various aspects
- still many open issues

  (trimmed geometry, boundary conditions, integration, efficiency issues, implementation, performance . . .

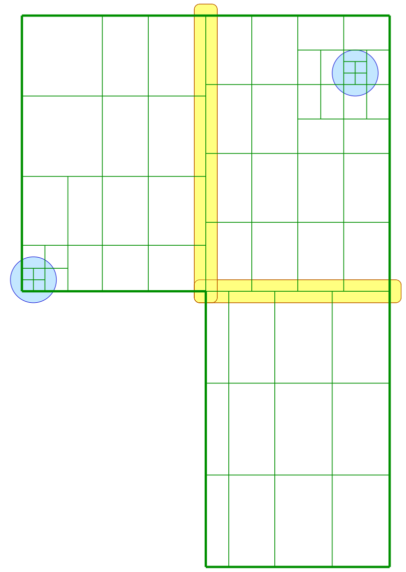
Motivation



- **IGA originally developed for NURBS**
  - **convenient for free-form surface modelling**
  - **exact representation of quadric surfaces**
  - **stable and efficient algorithms available**
  - **present in most CAD systems**

  - **gaps and overlaps cannot be avoided**
  - **trimmed NURBS not handled by IGA**
  - **generally only C^0 continuity across patch boundaries**
  - **tensor product structure of NURBS not efficient for representation of local features and for connection of adjacent surfaces**
  - **most shapes cannot be represented as a single watter-tight NURBS**

Motivation



T-spline Based Isogeometric Analysis

- **generalization of NURBS technology**
  - **inherits geometrical flexibility of NURBS**
  - **allows efficient local refinement**
  - **allows watter-tight merging of adjacent NURBS**
  - **T-splines are forward and backward compatible with NURBS**
  - **trimmed NURBS can be represented as T-spline**

  - **non-straightforward refinement around extraordinary points**
  - **non-trivial representation of solids preserving exactly boundary surface geometry**
  - **limited availability in commercial CAD (Maya, Rhino, SolidWorks)**

Motivation

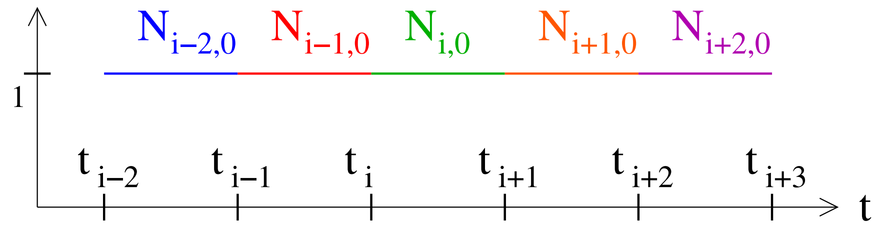
Implementation

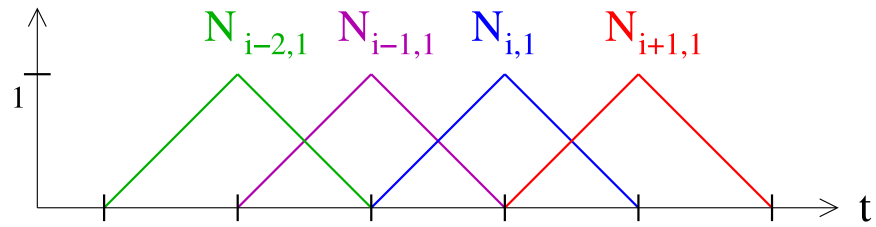
- many similar features between FEM and IGA
- no need to start implementation from scratch
- most of the FE codes can be reused
- object oriented design recognized as very appropriate
  - proved to be a viable concept significantly enhancing modularity, extensibility, maintainability, and robustness of the code without sacrificing its performance
  - supports team work, allows further developments without participation of original authors

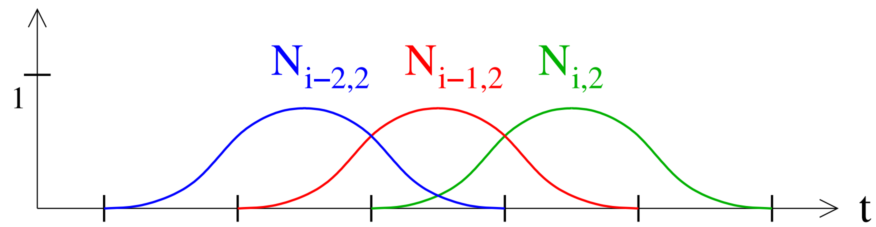
Univariate B-spline basis functions

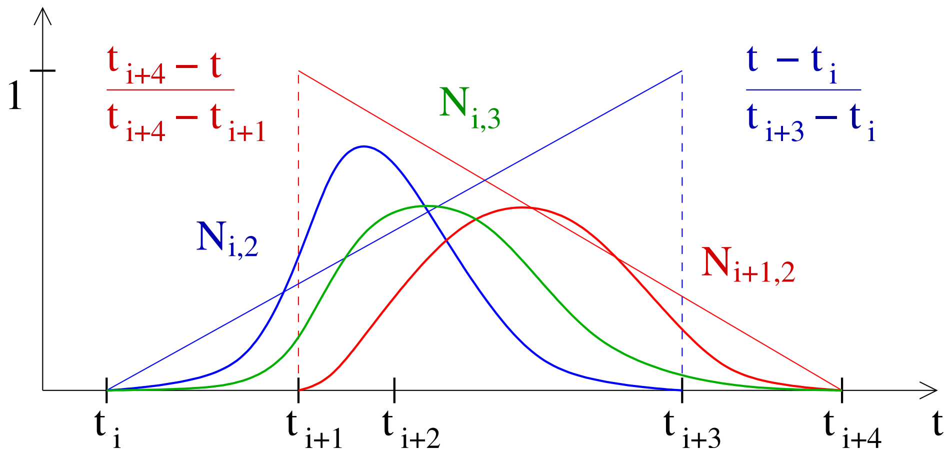
$$N_{i,p}(t) = \frac{t - t_i}{t_{i+p} - t_i} N_{i,p-1}(t) + \frac{t_{i+p+1} - t}{t_{i+p+1} - t_{i+1}} N_{i+1,p-1}(t) \qquad \text{for } p > 0$$

$$N_{i,0}(t) = \begin{cases} 1 & \text{if } t_i \leq t < t_{i+1} \\ 0 & \text{otherwise} \end{cases}$$



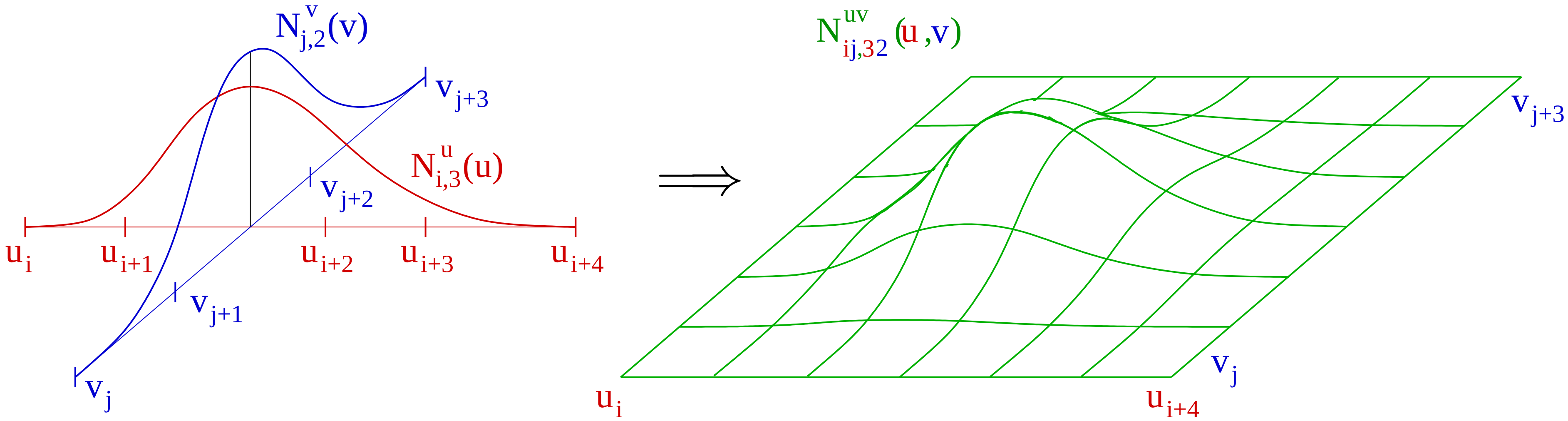






Bivariate B-spline basis functions

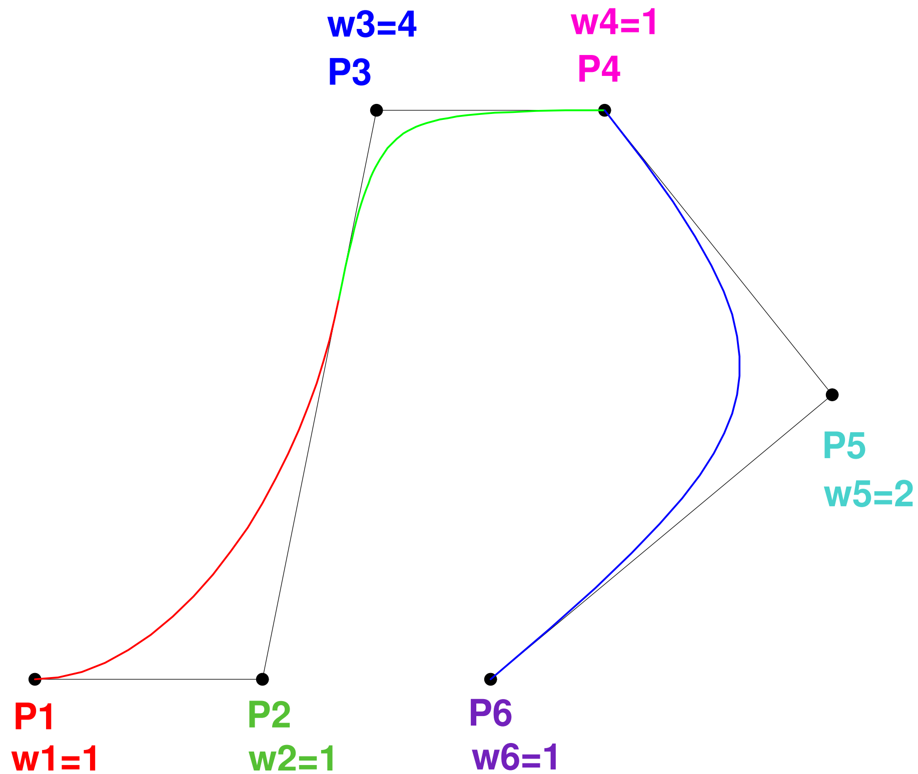
$$N^{uv}_{ij,pq}(u,v) = N^{u}_{i,p}(u)N^{v}_{j,q}(v) = N_k(u,v)$$



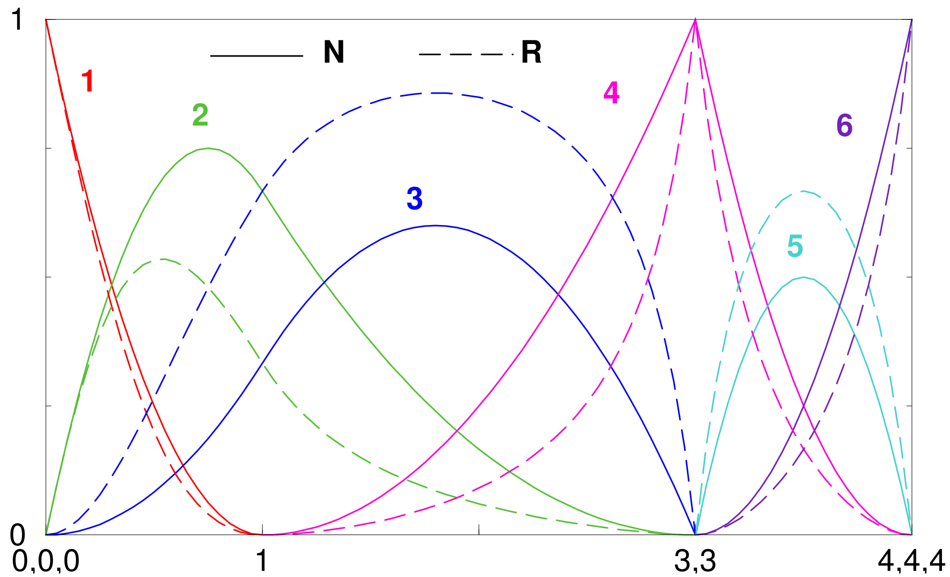
Rational bivariate B-spline basis functions

$$R_k(t) = \frac{N_k(u,v)w_k}{\sum_{m=1}^{n} N_m(u,v)w_m} \qquad k = 1, 2, \ldots, n \qquad w_k > 0$$

Quadratic NURBS curve



$$\boldsymbol{t} = \{0, 0, 0, 1, 3, 3, 4, 4, 4\}$$

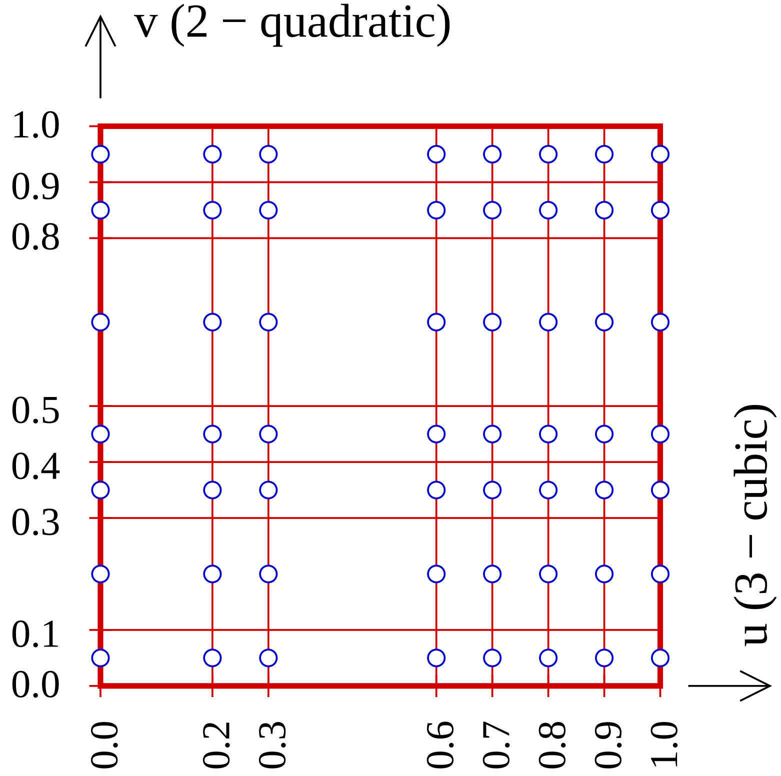


$$\boldsymbol{r}(t) = \sum_{j=1}^{6} R_i(t)\boldsymbol{P}_i$$

$$R_i(t) = \frac{N_i(t)w_i}{\sum_{j=1}^{6} N_j(t)w_j}$$

NURBS – Nonuniform Rational B-splines

- **a NURBS patch is defined by**
  - **set of control points (coordinates and weights) topologically forming regular grid**
  - **global degrees of B-spline basis functions for each parametric direction of the patch**
  - **global knot vectors for each parametric direction of the patch**



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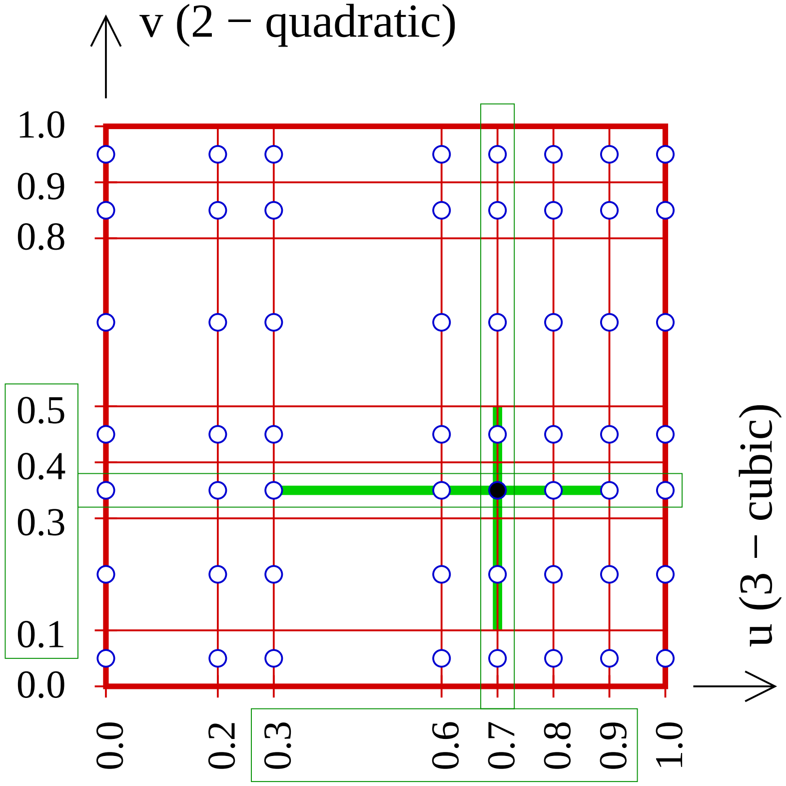
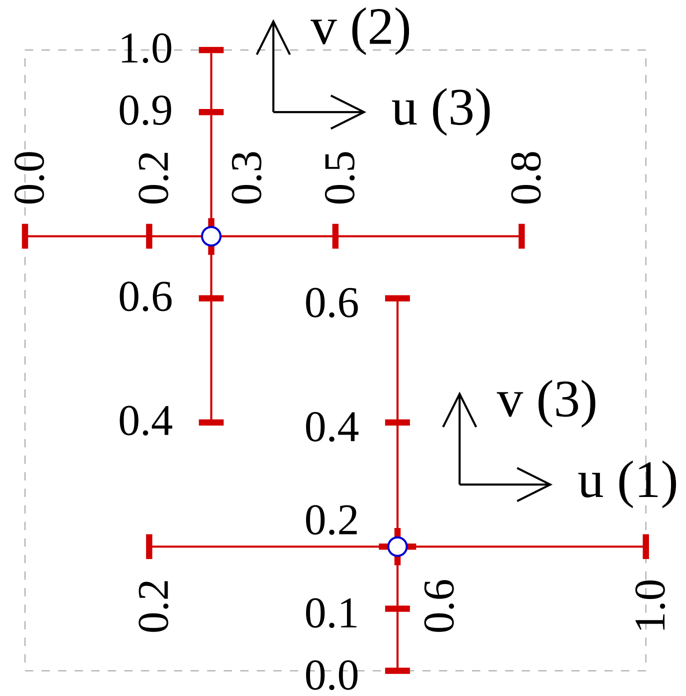
⟹ **NURBS is fully structured**



PB-splines – Point-based B-splines

- **a PB-spline patch is defined by**
  - set of control points (coordinates and weights) topologically irregular
  - local degrees of B-spline basis functions for each parametric direction of each control point
  - local knot vectors for each parametric direction of each control point



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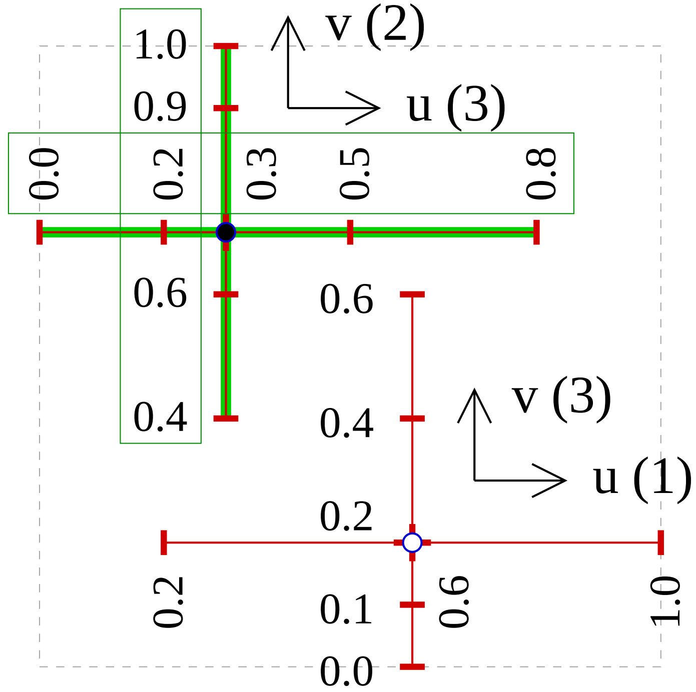
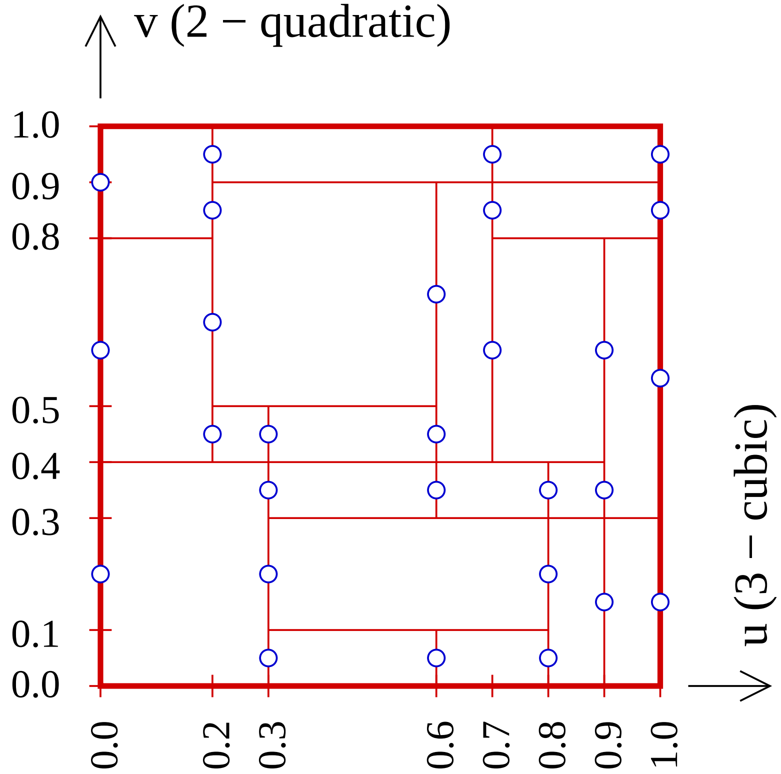
$\Longrightarrow$ **PB-spline is fully unstructured**



T-splines

- designed as compromise between NURBS and PB-splines
- a T-spline patch is defined by
  - set of control points (coordinates and weights) topologically consistent with a T-mesh
  - global degrees of B-spline basis functions for each parametric direction of the patch
  - global knot vectors for each parametric direction of the patch



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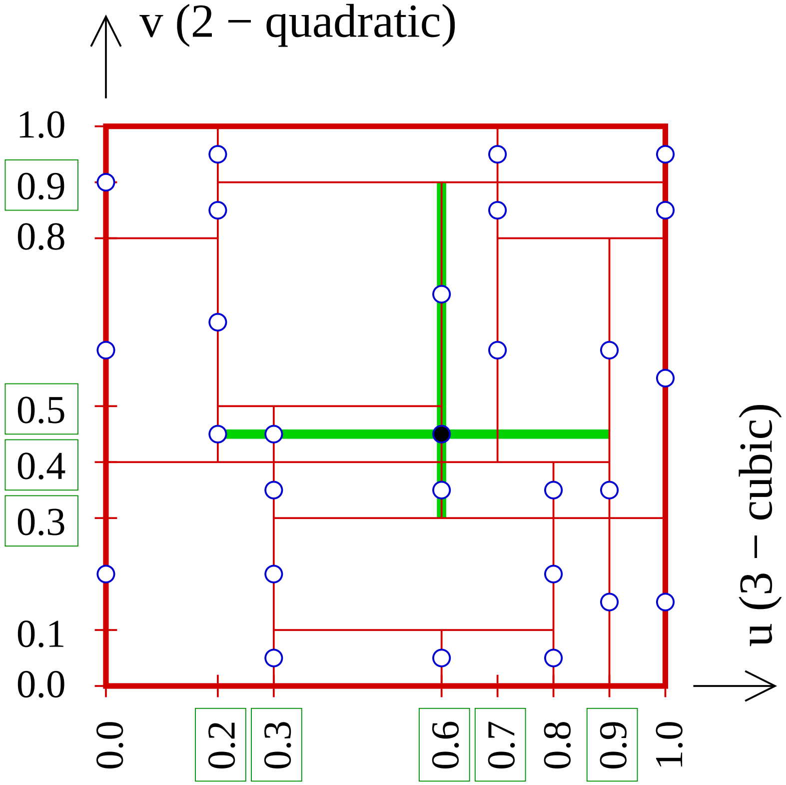
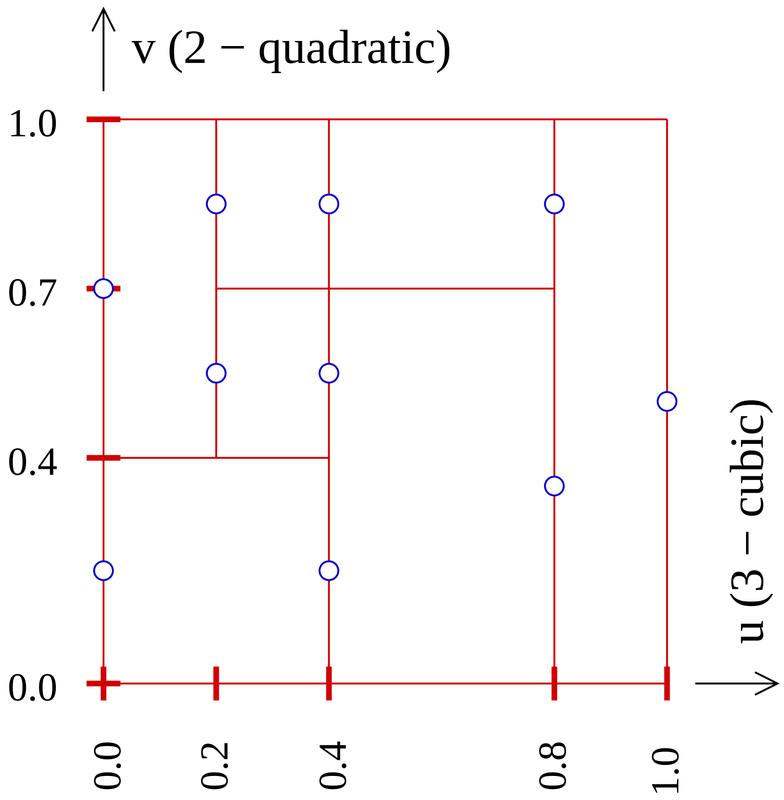
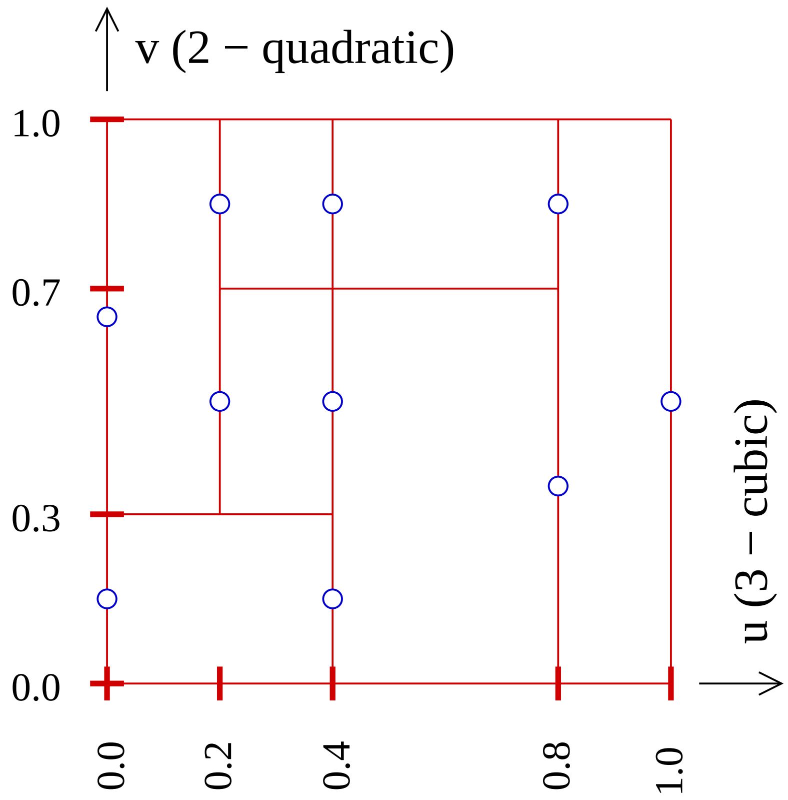


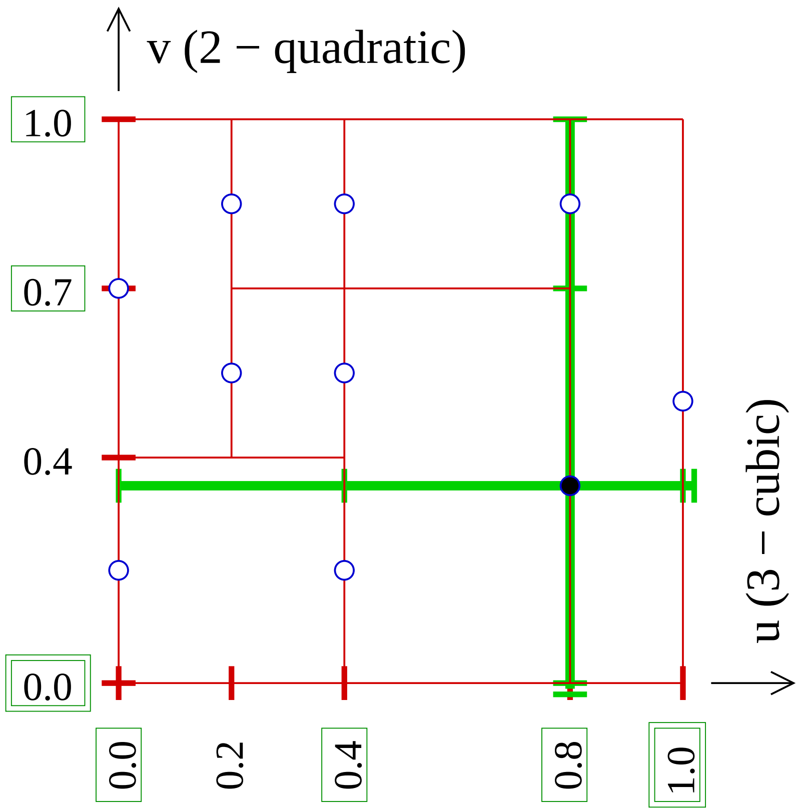
$\Longrightarrow$ T-spline is quasi-structured

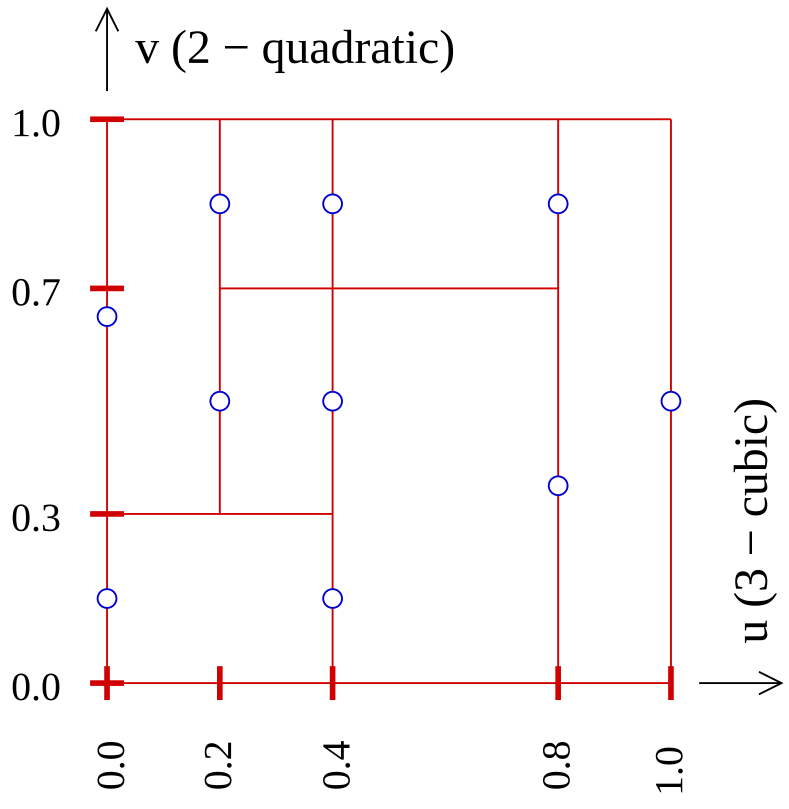
T-splines – local knot vector in parametric space



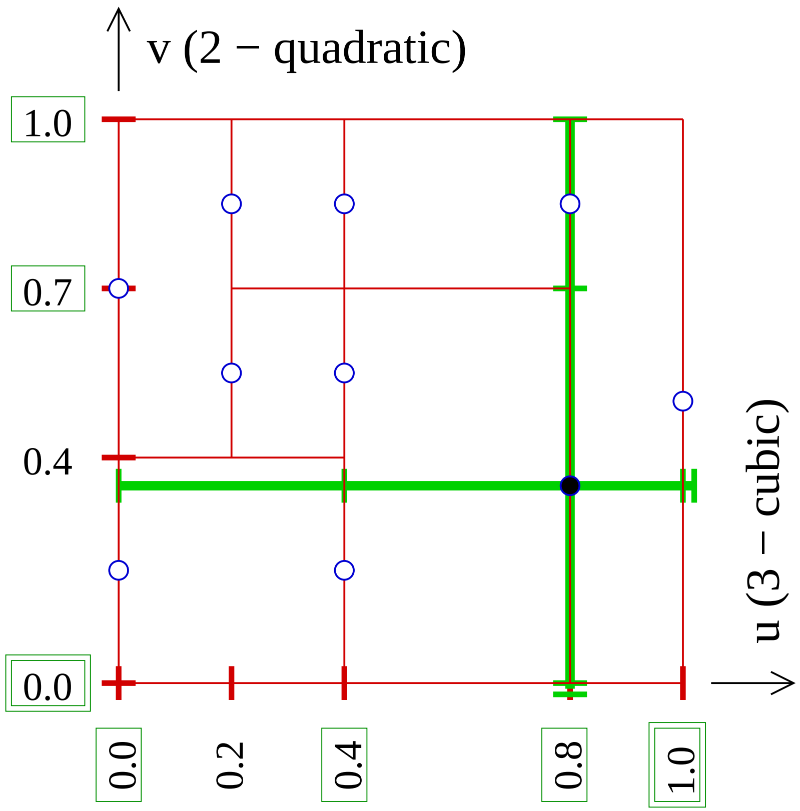


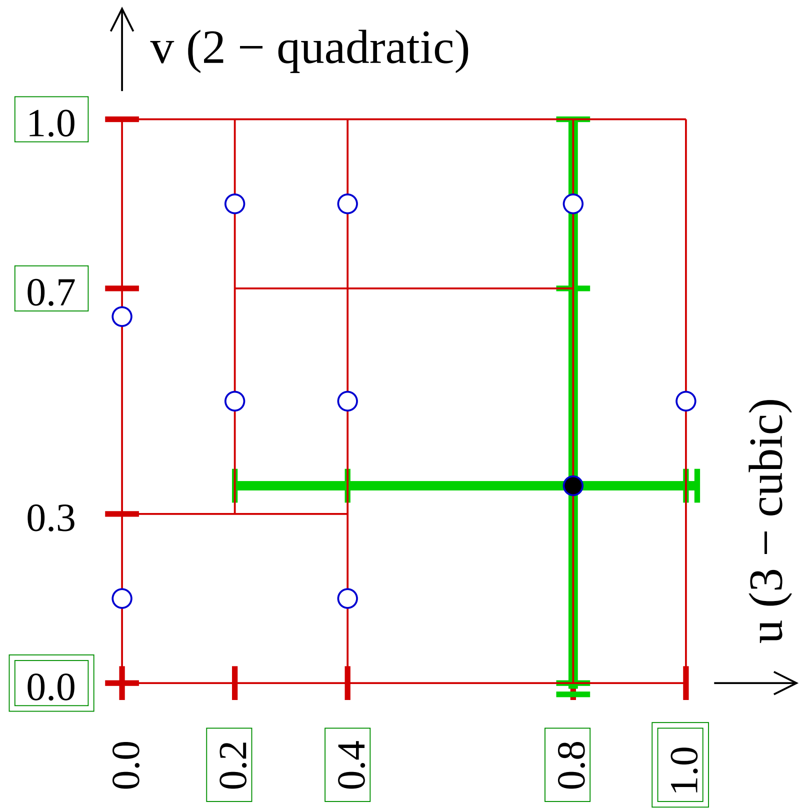
T-splines – local knot vector in parametric space



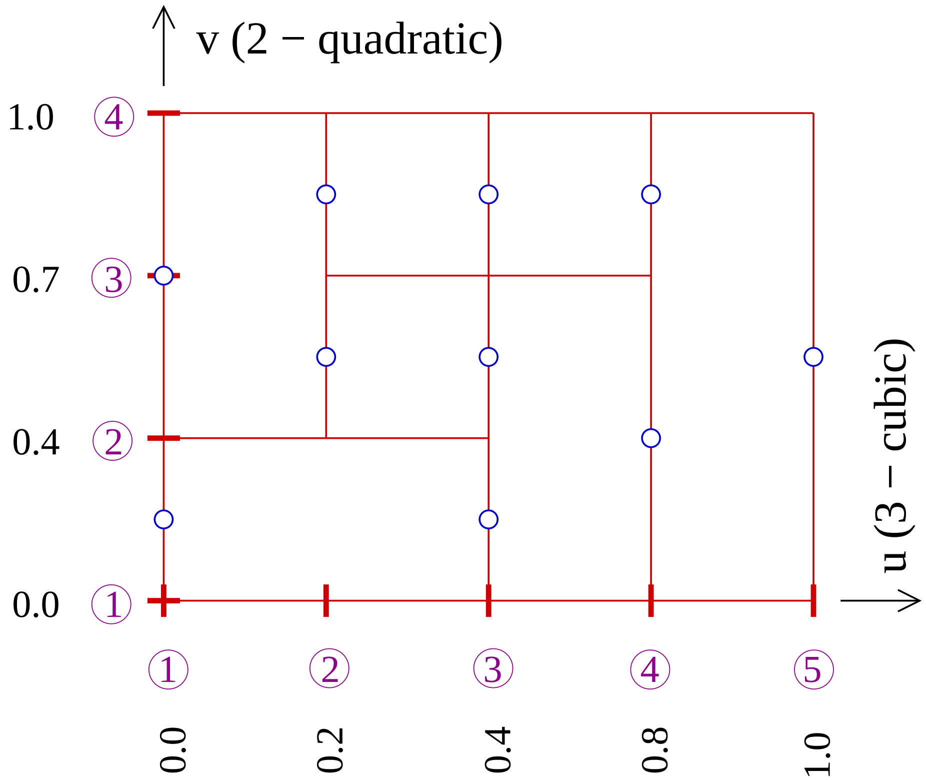


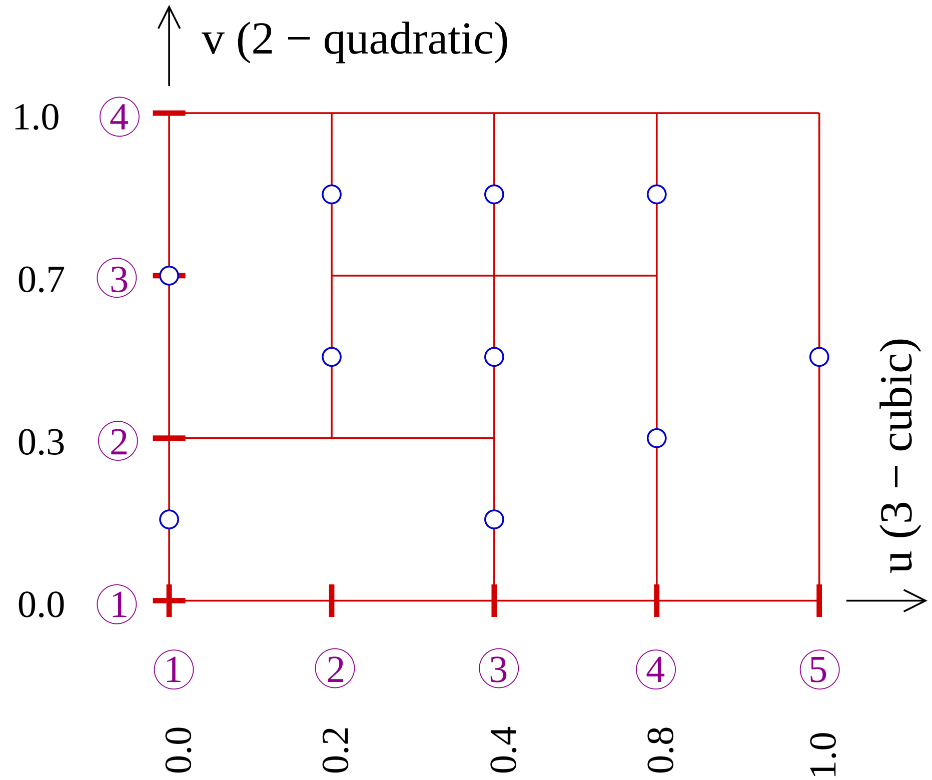
T-splines – local knot vector in parametric space



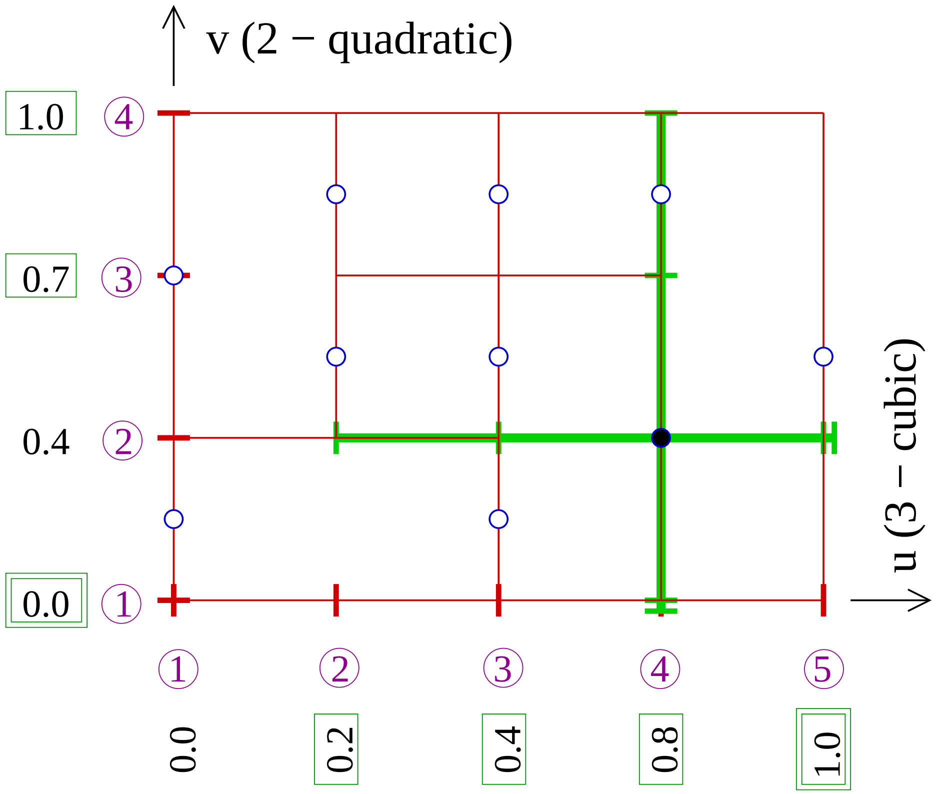


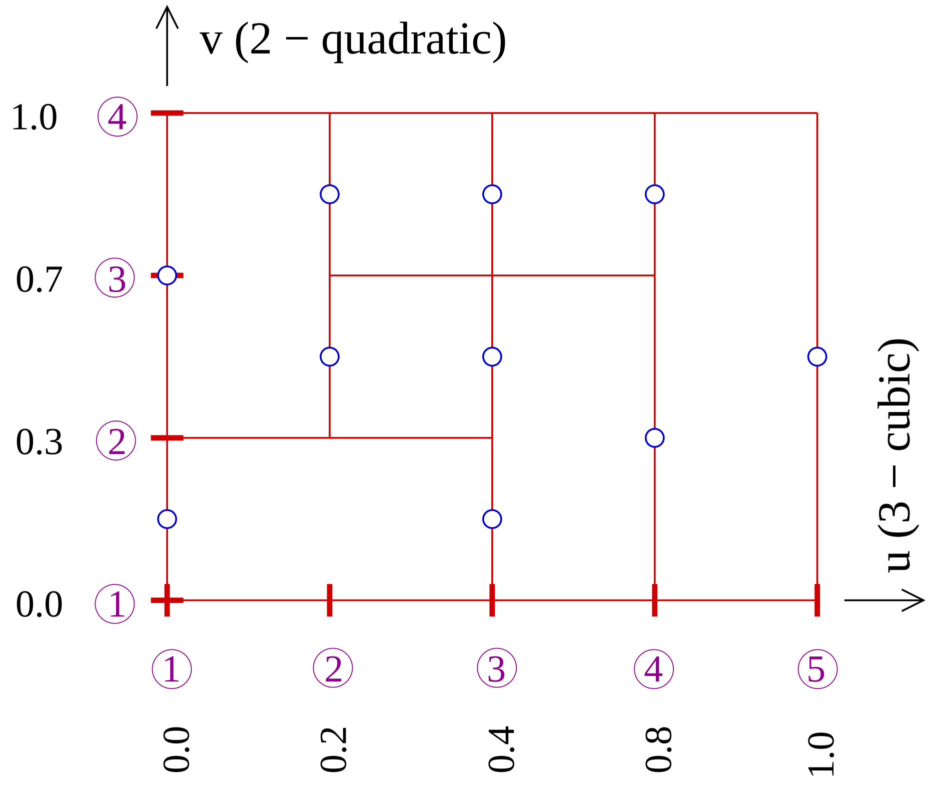
T-splines – local knot vector in index space



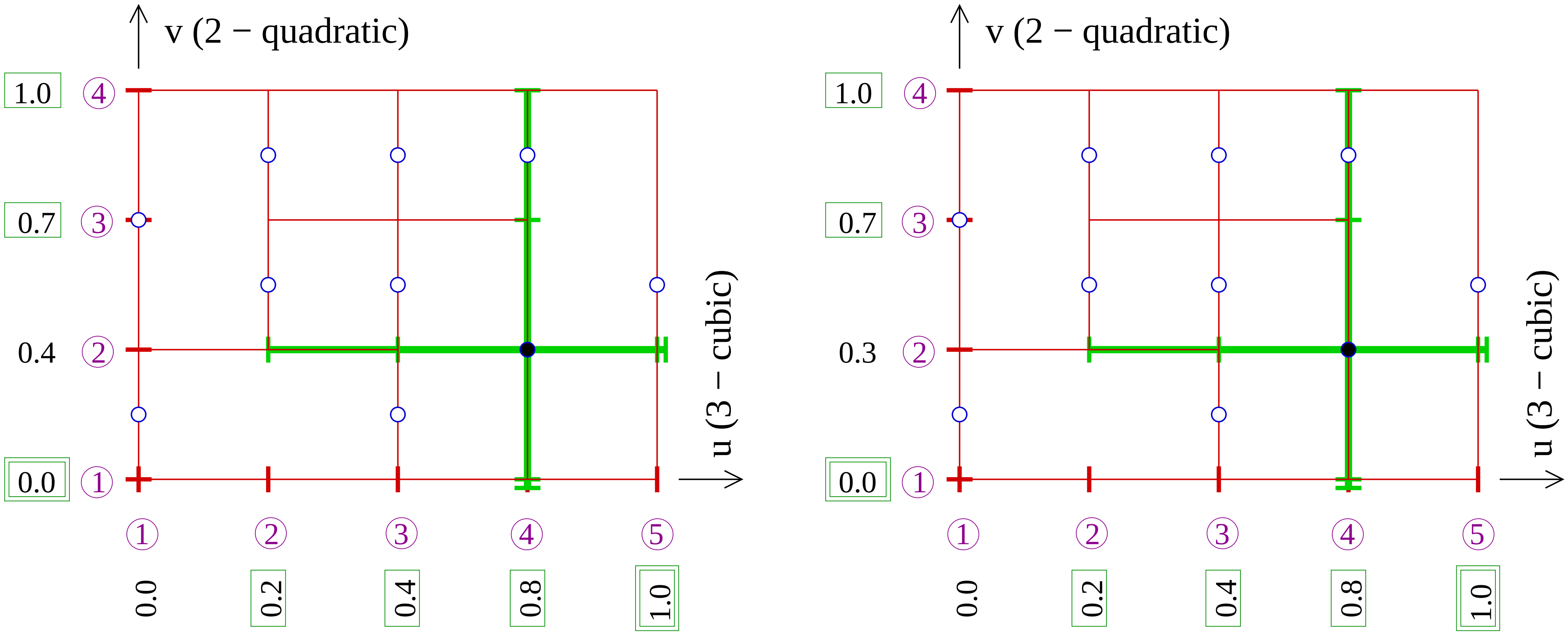


T-splines – local knot vector in index space





T-splines – local knot vector in index space



Object Oriented Design – Fundamental principles

- **encapsulation**
  (clustering together data and functionality)
- **inheritance**
  (reuse of existing code by derived classes)
- **abstraction / polymorphism**
  (transparent use of derived classes)
- **communication using messages**
  (general interface, safe data handling)

A good design is a trade-off between the level of implementation of object oriented principles and efficiency !

OOFEM

- **Object Oriented Finite Element Method computing environment**
- **open source distributed under the GNU Public License**
- **being continuously developed since 1997**
- **inspired by FEM_Object code** (EPFL Lausanne, 1993)
- **written in C++** (≈ 185.000 lines of code, ≈ 550 classes)
- **Ohloh analytics - 48 PersonYears**
- **modules for**
  - **structural mechanics**
  - **heat and mass transfer**
  - **fluid dynamics**

OOFEM – Features

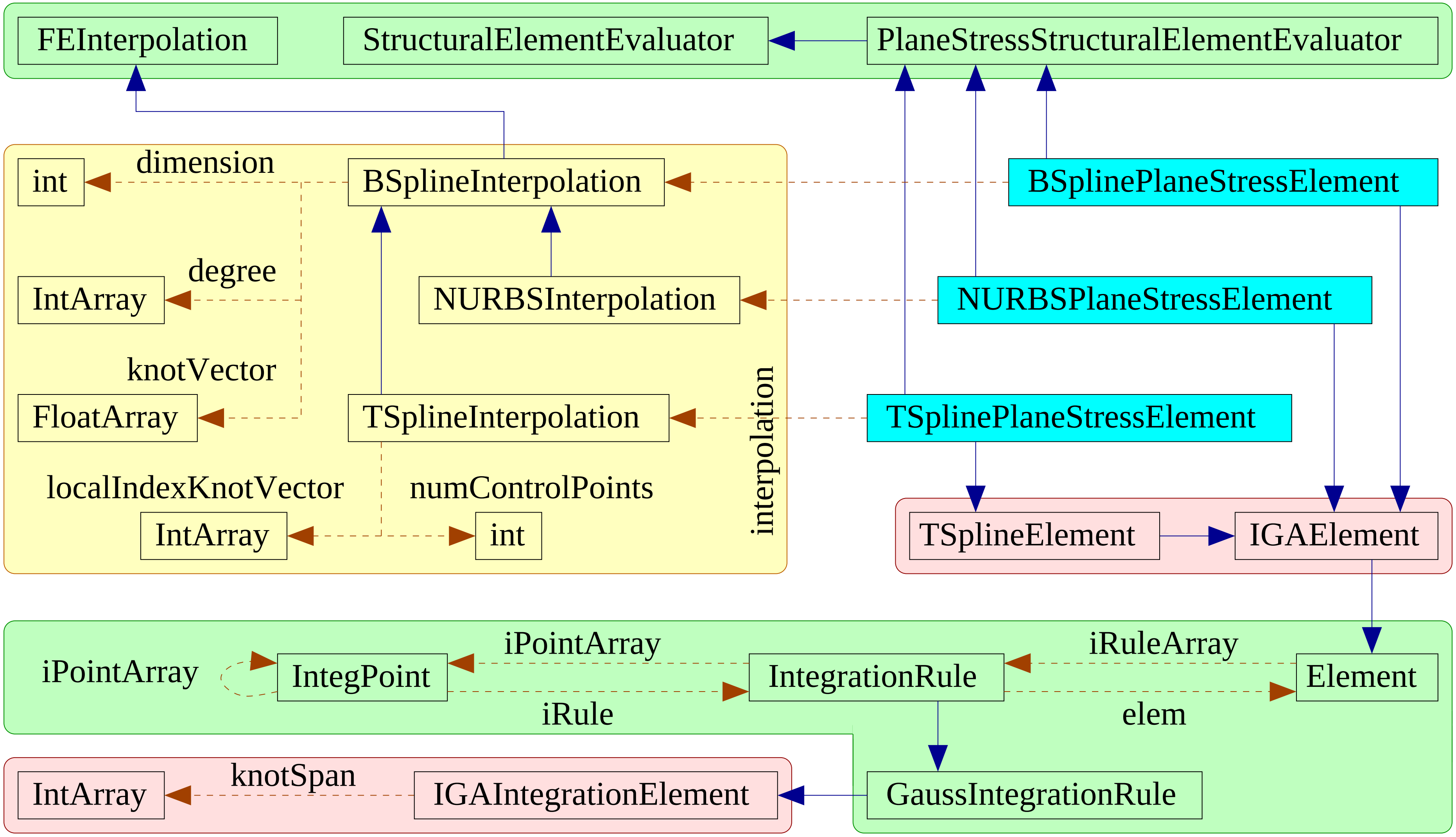
- **fully extensible** - a new element type, material model (with any internal history), BC, numerical algorithm, analysis module, ...
- **independent problem formulation, numerical solution and data storage**
- **full restart support**
- **staggered analysis support**
- **parallel processing support** - based on domain decomposition, message passing paradigms and dynamic load balancing
- **adaptive analysis support**
- **eXtended FEM support**
- **efficient sparse solvers** - interface to third party packages available

OO Design of IGA Module

- **strict separation of**
  - **interpolation**
  - **integration**
  - **analysis-specific functionality**

- **implementation of general IGA element**
- **implementation of integration on IGA element**
- **implementation of interpolation on IGA element**
- **implementation of analysis-specific IGA element**

OO Design of IGA Module



OO Design of IGA Module

```
StructuralElementEvaluator::computeStiffnessMatrix(FloatMatrix answer) {
   element = this->giveElement();
   ndofs = element->giveNumberOfDofs();

   answer.resize(ndofs, ndofs);
   answer.zero();

   loop over all integration rules (iRule) on the element {
      loop over all Gauss points (gp) of the iRule {
         B = this->computeStrainDisplacementMatrix(gp);
         D = this->computeConstitutiveMatrix(gp);
         dV = this->computeVolumeAround(gp);
         answer->add(product of B^T_D_B_dV);
      }
   }
}
```

OO Design of IGA Module

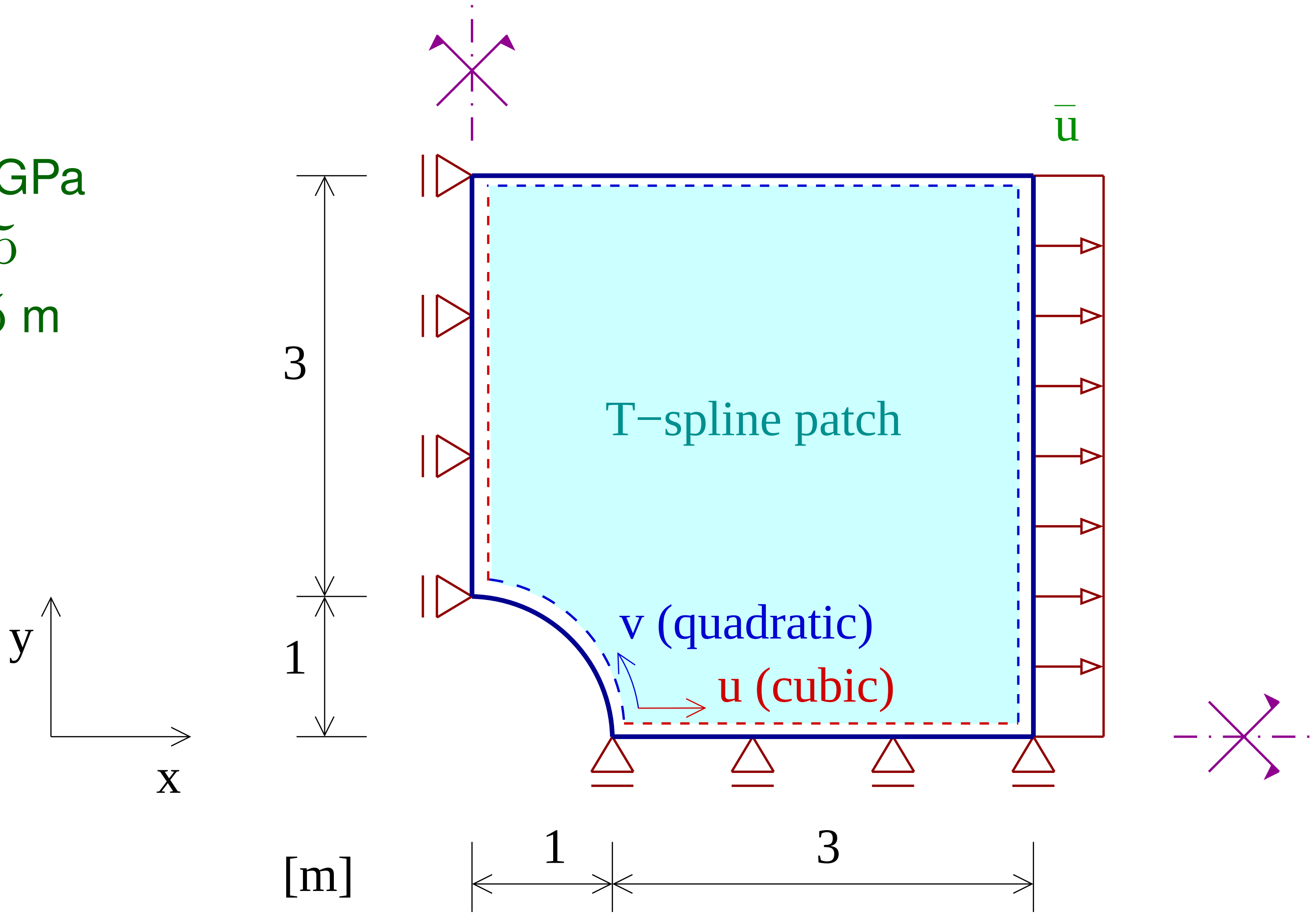
```
PlaneStressStructuralElementEvaluator::
      computeStrainDisplacementMatrix(FloatMatrix answer, IntegPoint gp) {
   FEInterpolation interp = gp->giveElement()->giveInterpolation();
   interp->evalShapeFunctDerivatives(der, gp);
   nnodes = gp->giveElement()->giveNumberOfNodes();

   answer.resize(3, 2*nnodes);                    // 2 DOFs per each node
   answer.zero();

   for i=1:nnodes{
      answer.at(1, i*2-1) = der.at(i, 1);   // dN(i)/dx
      answer.at(2, i*2)   = der.at(i, 2);   // dN(i)/dy
      answer.at(3, i*2-1) = der.at(i, 2);   // dN(i)/dy
      answer.at(3, i*2)   = der.at(i, 1);   // dN(i)/dx
   }
}
```

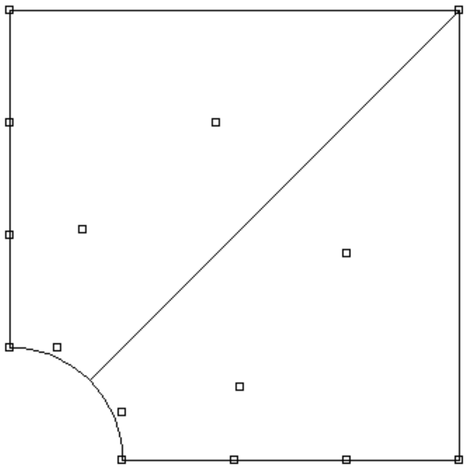
Numerical Example

$E = 15$ GPa
$\nu = 0.25$
$t = 0.15$ m
$\bar{u} = 1$ m

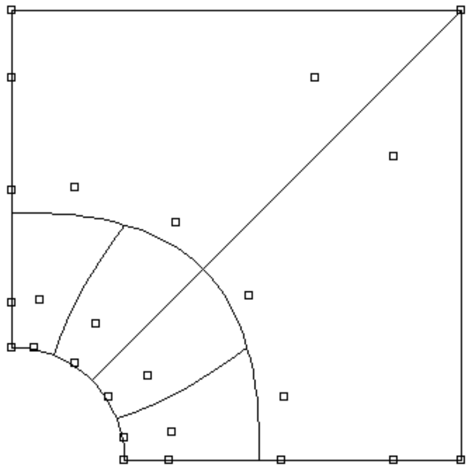


Numerical Example – IGA

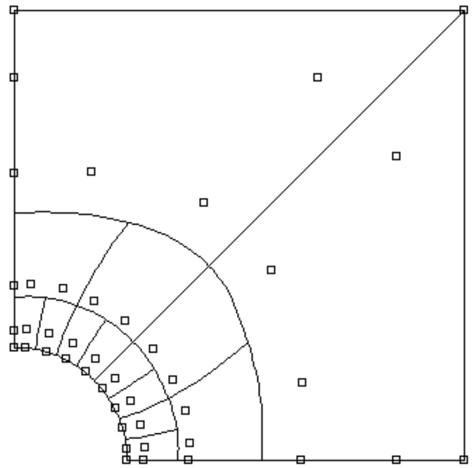
profile ε_{xx}



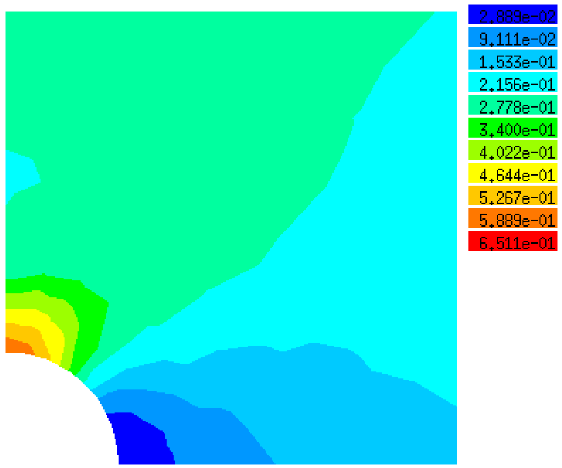
16 control points

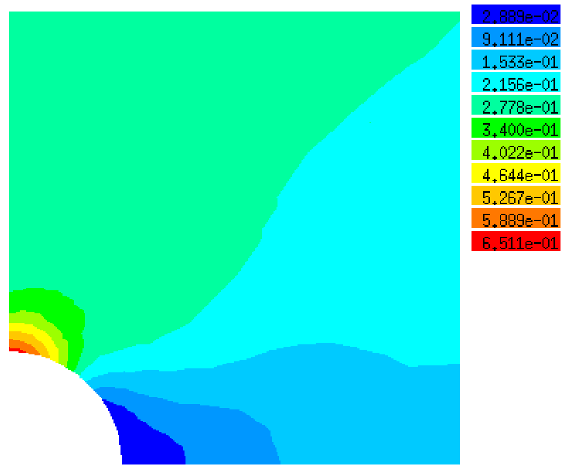


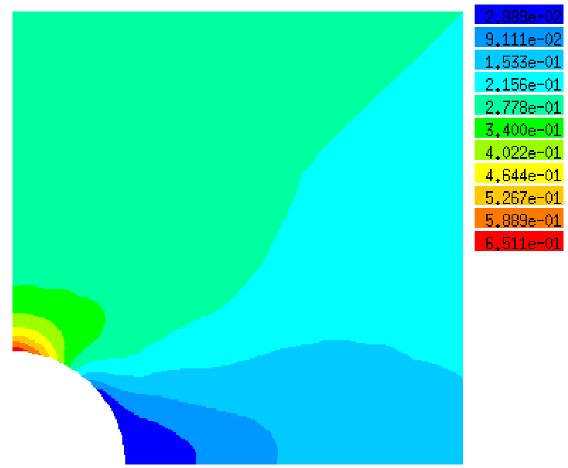
26 control points



44 control points

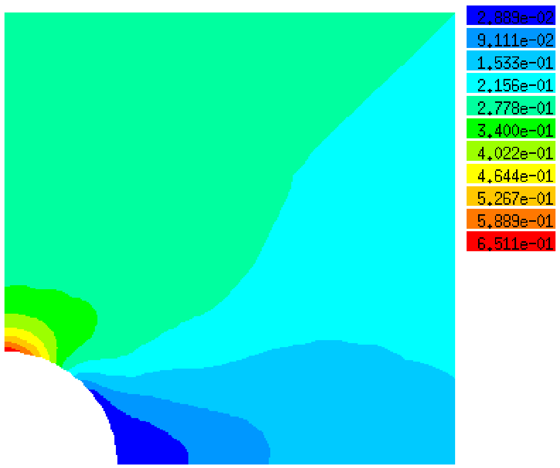


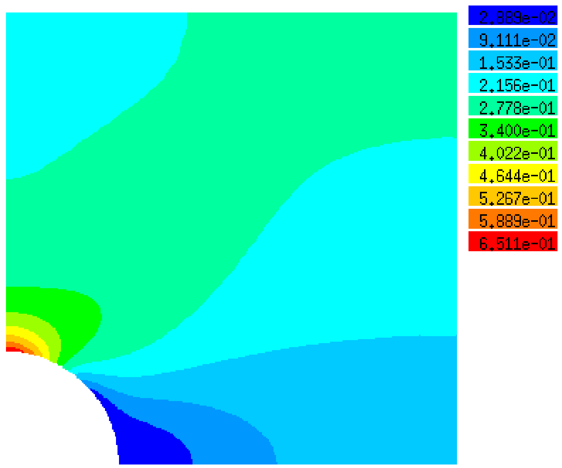


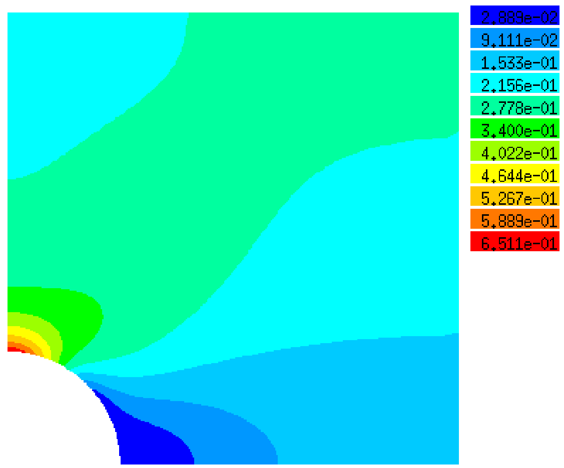


Numerical Example – IGA $\times$ FEA

profile ε_{xx}







IGA

3x2 T-spline

44 control points

IGA

5x5 NURBS

294 control points

FEA

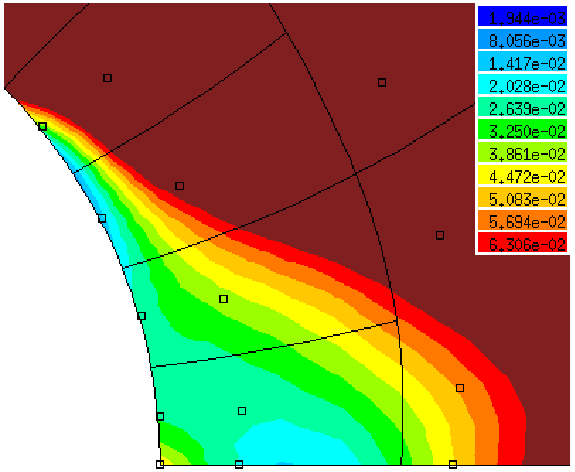
bilinear quads

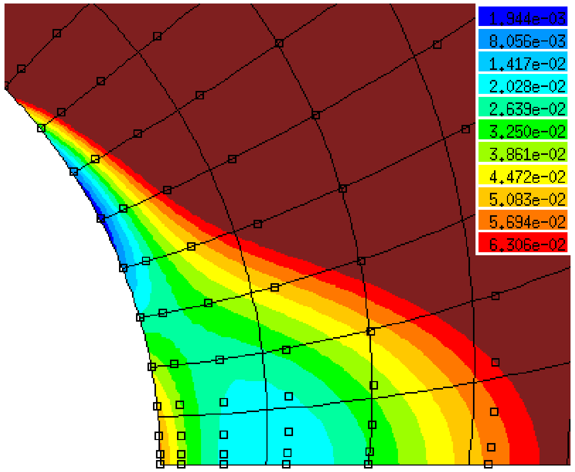
7345 nodes

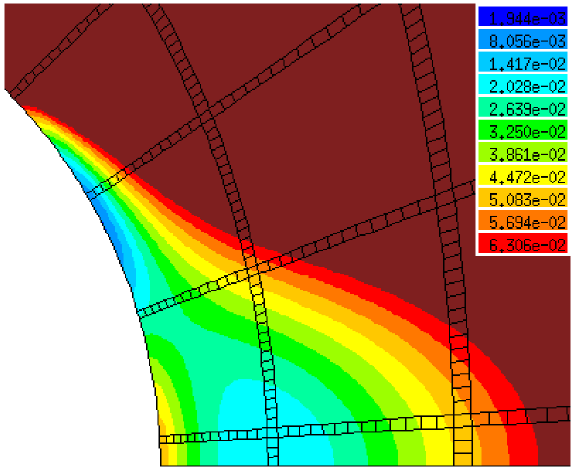
7168 elements

Numerical Example – IGA × FEA – detail

profile ε_{xx}







IGA	IGA	FEA
3x2 T-spline	5x5 NURBS	bilinear quads
44 control points	294 control points	7345 nodes
		7168 elements

Summary

- **implementation of an IGA module into an existing object oriented finite element code was presented**
- **emphasis was given on proper OO design**
  - **most of the functionality of the existing code reused**
  - **modularity and extensibility of the code preserved**
- **amount of modified and/or added code is rather limited mostly related to handling basis functions**
- **functionality of implementation was verified on numerical example**
- **T-spline based IGA proved to be a promising technology**

Acknowledgments

- **implemented into open source FEM package OOFEM**
- **the support by the Grant Agency of the Czech Republic (Project No. 103/09/2009) is gratefully acknowledged**