THE NAVIER - STOKES EQUATIONS WITH MATERIAL DIFFERENCES

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Introduction

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary $\partial\Omega$. In the present paper we construct a method for the approximate solution of the nonstationary Navier-Stokes equations for incompressible fluid flow contained in Ω for $0 < t < T$. The approach is based on a coupling of the Lagrangian and the Eulerian representation of the fluid.

The Lagrangian representation of stationary fluid flow is given by a function $t \to x(t) =: X(t,\, x_0)$ solving the autonomous system

$$\dot{x}(t) = v(x(t))\,, \qquad x(0) = x_0\,, \tag{1}$$

where $x_0 \in \Omega$ and $v : \Omega \to \mathbb{R}^3$ is a continuous velocity field. This function represents the trajectory of a particle of the fluid, which at initial time $t = 0$ is located in x_0. The initial value problem (1) has a uniquely determined global solution if we assume $v \in C_0^{lip}(\Omega)$, i.e. v is a lipschitz continuous function with compact support in Ω.

Due to the uniqueness of the solution the set of mappings $\{X(t,\cdot) : \overline{\Omega} \to \overline{\Omega}\,|\, t \in \mathbb{R}\}$ defines a commutative group of C^1- diffeomorphisms in the closure $\overline{\Omega}$ with the inverse mapping $X(t,\cdot)^{-1} = X(-t,\cdot)$. Moreover, if in addition we require $\nabla \cdot v = 0$ in Ω, then from Liouville's differential equation $\partial_t \det \nabla X(t,\, x) = \det \nabla X(t,\, x) \;\; \nabla_X \cdot v(X(t,x)) = 0$ we obtain the identity $\det \nabla X(t,x) = \det \nabla X(0,x) = \det \nabla x = 1$. This property of the mappings $X(t,\,\cdot)$ means the conservation of measure. As a consequence, for $v \in L^p(\Omega)$, $1 \le p \le \infty$, we find $||v(X(t,\cdot))||_p = ||v||_p$, where $||\cdot||_p$ denotes the norm in $L^p(\Omega)$ ([2]).

Besides the representation of steady flow by the trajectories $t \to x(t) = X(t,\, x_0)$, for nonstationary flow we use the Eulerian representation in form of the nonlinear Navier-Stokes equations concerning the unknown velocity field $(t,\, x) \to v(t,\, x) = (v_1(t,\, x), v_2(t,\, x), v_3(t,\, x))$ and an unknown pressure function $(t,\, x) \to p(t,\, x)$ satisfying

$$\begin{aligned} &\partial_t v - \nu\Delta v + v\cdot\nabla v + \nabla p = f \quad \text{in} \quad (0,\,T)\times\Omega,\\ &\nabla\cdot v = 0 \quad \text{in} \quad (0,\,T)\times\Omega,\\ &v|_{\partial\Omega} = 0\,, \qquad v|_{t=0} = v_0\,. \end{aligned} \tag{2}$$

The constant $\nu > 0$ (kinematic viscosity), the external force density f, and the initial velocity v_0 are given data.

Results

Due to the strong nonlinearity of the convective term the system (2) does not allow a global unique solution. Since the convective term $v(t,x)\cdot\nabla v(t,x)$ arises from a material derivative we use material differences for approximation and replace the convective term by

$$\frac{1}{2\varepsilon}\,\{v(t, X_s(\varepsilon, x)) - v(t, X_s(-\varepsilon, x))\}.$$

It can be shown ([1]) that this term tends to $v(t,x)\cdot\nabla v(t,x)$ as $\varepsilon \to 0$ if v is divergence free and sufficiently smooth.

Now assume $0 < T \in \mathbb{R}$, $N \in \mathbb{N}\,(N \geq 2)$, $\varepsilon := \frac{T}{N}$, $t_k = k\varepsilon$. Then for $t \in [t_k,\, t_{k+1})$ we can replace the nonlinear term $v \cdot \nabla v$ as follows:

$$v(t,x)\cdot\nabla v(t,x) \;\sim\; \frac{1}{2\varepsilon}\Big(v(t,X_k) - v(t,X_k^{-1})\Big) =: L_\varepsilon^k v(t).$$

Here $X_k := X_k(\varepsilon,\, x)$, where $X_k(t,x)$ denotes the solution of

$$\dot{x}(t) = v_k(x(t)) := v(t_k,\, x(t))\,, \qquad x(0) = x_0\,.$$

The resulting discontinuity caused by the piecewise constant interpolation above can be avoided using piecewise linear interpolation as follows: For $t \in [t_k, t_{k+1}]$ replace the nonlinear term $v \cdot \nabla v$ by

$$v(t,x)\cdot\nabla v(t,x) \;\sim\; \frac{t - t_k}{\varepsilon}\, L_\varepsilon^k v(t) \;+\; \frac{t_{k+1} - t}{\varepsilon}\, L_\varepsilon^{k-1} v(t) =: Z_\varepsilon^k v(t)\,.$$

This leads to the following regularized piecewise linear Navier-Stokes system:

$$\begin{aligned} \partial_t v - \nu\Delta v + Z_\varepsilon v + \nabla p &= f \quad \text{in} \quad \Omega_T, \\ \nabla \cdot v &= 0 \quad \text{in} \quad \Omega_T, \\ v_{|\partial\Omega} &= 0, \\ v_{|t\leq 0} &= v_0\,. \end{aligned} \tag{3}$$

Here for $(t,x) \in [t_k, t_{k+1}] \times \overline{\Omega},\; k = 0, 1, \ldots, N-1$ we use $Z_\varepsilon v(t,x) := Z_\varepsilon^k v(t,x)$.

If $H^m(\Omega)$ denotes the usual Sobolev space of functions with weak derivatives up to and including the order m in $L^2(\Omega)$, and if $\mathcal{H}^0(\Omega) := \overline{C_{0,\sigma}^\infty(\Omega)}^{||\cdot||}$, $\mathcal{H}^1(\Omega) := \overline{C_{0,\sigma}^\infty(\Omega)}^{||\nabla\cdot||}$ denote the closure of divergence-free C_0^∞-vector functions having compact support in Ω with respect to the L^2- and the H^1-norm, respectively, then our main result reads as follows:

Theorem. Let $v_0 \in H^3(\Omega)$, $f \in L^2(0,\, T, H^1(\Omega))$. Then there exists a uniquely determined solution $v \in C([0,T], H^2(\Omega) \cap \mathcal{H}^1(\Omega))$ with $\partial_t v \in C([0,T], \mathcal{H}^0(\Omega))$ and a uniquely determined function $\nabla p \in C([0,T],\, L^2(\Omega))$ of (3). The solution satisfies for all $t \in [0,\, T]$ the energy equation

$$||v(t)||^2 + 2\nu \int_0^t ||\nabla v(\tau)||^2 d\tau \;=\; ||v_0||^2 + \int_0^t (f(\tau),\, v(\tau))d\tau.$$

Reference

[1] Varnhorn, W.: The Navier-Stokes Equations with Particle Methods. Necas Center Lecture Notes, 2 (2007), 121-157.

[2] Asanalieva, N., Varnhorn W.: The Navier-Stokes Equations with Regularization. P.A.M.M (2009), to appear