REDUCTION THEOREMS FOR OPERATORS ON THE CONES OF MONOTONE FUNCTIONS

AMIRAN GOGATISHVILI AND VLADIMIR D. STEPANOV

Abstract. For a quasilinear operator on the semiaxis a reduction theorem is proved on the cones of monotone functions in $L^p - L^q$ setting for $0 < q < \infty, 1 \le p < \infty$. The case $0 < p < 1$ is also studied for operators with additional properties. In particular, we obtain critera for three-weight inequalities for the Hardy-type operators with Oinarov' kernel on monotone functions in the case $0 < q < p \le 1$.

1. Introduction

Let $\mathbb{R}_+ := [0, \infty)$. Denote $\mathfrak{M}^+$ the set of all non-negative measurable functions on $\mathbb{R}_+$ and $\mathfrak{M}^\downarrow \subset \mathfrak{M}^+$ ($\mathfrak{M}^\uparrow \subset \mathfrak{M}^+$) the subset of all non-increasing (non-decreasing) functions. For the last two decades the weighted norm $L^p - L^q$ inequalities have extensively been studied. In particular, much attention was paid to the inequalities restricted to the cones of monotone functions, see for instance [1], [21], [25], [26], [6], [12], [22], survey [5], the monographs [15], [16] and references given there. At the initial stage the main tool was the Sawyer duality principle [21] (see also [23], [24]), which allowed to reduce an $L^p - L^q$ inequality for monotone functions with $1 \le q \le \infty, 0 < p < \infty$ to a more menageable inequality for arbitrary non-negative functions. The case $p \le q, 0 < p \le 1$ was alternatively characterized in [25], [26], [6], [3]. Later on some direct reduction theorems were found [9], [10] [4] involving the supremum operators which work for the case $0 < q < p \le 1$.

Let $T : \mathfrak{M}^+ \to \mathfrak{M}^+$ be a positive quasilinear operator such that

(i) $T(\lambda f) = \lambda T f$ for all $\lambda \ge 0$ and $f \in \mathfrak{M}^+$,
(ii) $T(f + g) \le c(Tf + Tg)$ for all $f, g \in \mathfrak{M}^+$ with a constant $c > 0$ independent on f and g,
(iii) $Tf(x) \le cTg(x)$ for almost every $x \in \mathbb{R}_+$, if $f(x) \le g(x)$ for almost every $x \in \mathbb{R}_+$ with a constant $c > 0$ independent on f and g.

Let v and w be weights, that is non-negative locally integrable functions on $\mathbb{R}_+$. The first our result is a reduction of the inequality

$$\left(\int_0^\infty (Tf(t))^q w(t) dt \right)^{\frac{1}{q}} \le C \left(\int_0^\infty (f(t))^p v(t) dt \right)^{\frac{1}{p}}, \ f \in \mathfrak{M}^\downarrow \tag{1.1}$$

Date: November 1, 2011 and, in revised form, — 00, 2011.

1991 *Mathematics Subject Classification.* Primary 26D10; Secondary 26D15, 26D07.

Key words and phrases. Quasilinear operator, integral inequality, Lebesgue space, weight, Hardy operator, monotone functions.

This work was done as part of the research program Approximation Theory and Fourier Analysis at the Centre de Recerca Matematica (CRM), Bellaterra in the Fall semester of 2011. The authors are grateful to CRM for its support and hospitality. The work of the first author was also partially supported by the grant 201/08/0383 of the Grant Agency of the Czech Republic and RVO: 67985840. The work of the second author was also partially supported by the Russian Fund for Basic Research (Projects 09-01-00093, 09-01-00586 and 09-II-CO-01-003).

to a similar one on $\mathfrak{M}^+$ in the case $0 < q < \infty, 1 \le p < \infty$ (see Teorems 2.1-2.4). When $0 < p \le q < \infty, 0 < p < 1$ we supplement these results in Section 3 by an extension of [3] and [26].

It is well known that the case $0 < q < p \le 1$ is the most difficult for a characterization of inequalities like (1.1) (see [2], [4], [5], [9], [7] [11], [13], [22]). We study this case in Section 4 including the three-weight inequality of the form

$$(1.2) \qquad \left(\int_0^\infty \left(\int_x^\infty f(t)u(t)dt\right)^q w(x)dx\right)^{\frac{1}{q}} \le C\left(\int_0^\infty (f(t))^p v(t)dt\right)^{\frac{1}{p}}$$

for all $f \in \mathfrak{M}^\downarrow$ and give three alternative reductions and a criterion (see Theorem 4.1) and section 5 contains a characterization of (1.2) for $0 < p, q \le \infty$ (see Theorems 5.1 and 5.3)

Also we study the inequality

$$(1.3) \quad \left(\int_0^\infty \left(\int_0^x k(x,t)f(t)u(t)dt\right)^q w(x)dx\right)^{\frac{1}{q}} \le C\left(\int_0^\infty (f(t))^p v(t)dt\right)^{\frac{1}{p}}, \ f \in \mathfrak{M}^\downarrow,$$

where $k(x,t) \ge 0$ is Oinarov's kernel and give a full description for $0 < p, q < \infty$ (see Theorems 4.5 and 5.7).

We use signs := and =: for determining new quantities and $\mathbb{Z}$ for the set of all integers. For positive functionals F and G we write $F \ll G$, if $F \le cG$ with some positive constant c, which depends only on irrelevant parameters. $F \approx G$ means $F \ll G \ll F$ or $F = cG$. χ_E denotes the characteristic function (indicator) of a set E. Uncertainties of the form $0 \cdot \infty, \frac{\infty}{\infty}$ and $\frac{0}{0}$ are taken to be zero. We use notations C or C with lower indices for the constants (possibly different in different occasions) in the inequalities like (1.1). $\Box$ stands for the end of proof.

2. Quasilinear operators

Put $V(t) := \int_0^t v$ and denote $\mathbf{1}$ the function on $\mathbb{R}_+$ identically equal to 1.

Theorem 2.1. *Let $0 < q \le \infty, 1 < p < \infty$ and let $T : \mathfrak{M}^+ \to \mathfrak{M}^+$ be a positive quasilinear operator, satisfying* (i)-(iii). *Then the inequality* (1.1) *holds iff the following two inequalities are valid:*

$$(2.1) \qquad \left(\int_0^\infty \left(T\left(\int_x^\infty h\right)\right)^q w\right)^{\frac{1}{q}} \le C\left(\int_0^\infty h^p V^p v^{1-p}\right)^{\frac{1}{p}}, \ h \in \mathfrak{M}^+$$

and

$$(2.2) \qquad \left(\int_0^\infty (T\mathbf{1})^q w\right)^{\frac{1}{q}} \le C\left(\int_0^\infty v\right)^{\frac{1}{p}}.$$

Proof. Let $0 < q < \infty$. *Necessity.* Let $h \in \mathfrak{M}^+$ be integrable on $[x, \infty)$ for all $x > 0$. Then $f(x) = \int_x^\infty h \in \mathfrak{M}^\downarrow$ and by (1.1) and Hardy's inequality we have

$$\begin{aligned}\left(\int_0^\infty \left(T\left(\int_x^\infty h\right)\right)^q w\right)^{\frac{1}{q}} &\le C\left(\int_0^\infty \left(\int_x^\infty h\right)^p v(x)dx\right)^{\frac{1}{p}} \\ &\ll C\left(\int_0^\infty h^p V^p v^{1-p}\right)^{\frac{1}{p}}.\end{aligned}$$

(2.2) follows from (1.1) with $f = \mathbf{1}$.

Sufficiency. Suppose that $V(\infty)=\infty$ and $f\in\mathfrak{M}^{\downarrow}$. Then

$$\begin{aligned} f(x)=\frac{f(x)V(x)}{V(x)} &= \left(\int_x^\infty \frac{v}{V^2}\right) f(x)V(x) \\ &\le \left(\int_x^\infty \frac{v}{V^2}\right)\int_0^x fv \le \int_x^\infty \left(\int_0^t fv\right)\frac{v(t)dt}{V^2(t)}. \end{aligned}$$

Applying (iii) and (2.1) with

$$h(t)=\left(\int_0^t fv\right)\frac{v(t)}{V^2(t)}$$

and applying Hardy's inequality, we find

$$\begin{aligned} \left(\int_0^\infty (Tf)^q w\right)^{\frac{1}{q}} &\le C\left(\int_0^\infty\left(\int_0^t fv\right)^p \frac{v(t)dt}{V^p(t)}\right)^{\frac{1}{p}} \\ &\ll C\left(\int_0^\infty f^p v\right)^{\frac{1}{p}}. \end{aligned}$$

If $V(\infty)<\infty$, then by Hölder's inequality

$$\begin{aligned} f(x) &= \left[\frac{1}{V(x)}-\frac{1}{V(\infty)}\right] f(x)V(x)+\frac{V(x)}{V(\infty)}f(x) \\ &\le \left(\int_x^\infty \frac{v}{V^2}\right)\int_0^x fv+\frac{V^{1/p'}(x)V^{1/p}(x)}{V(\infty)}f(x) \\ &\le \left(\int_x^\infty\left(\int_0^t fv\right)\frac{v(t)dt}{V^2(t)}\right)+\frac{\left(\int_0^\infty f^p v\right)^{1/p}}{V^{1/p}(\infty)} =: \int_x^\infty h+\lambda\mathbf{1}. \end{aligned}$$

Applying (i), (ii), (2.1), (2.2) and Hardy's inequality, we obtain

$$\begin{aligned} \left(\int_0^\infty (Tf)^q w\right)^{\frac{1}{q}} &\ll \left(\int_0^\infty\left(\int_x^\infty h\right)^q w(x)dx\right)^{\frac{1}{q}}+\lambda\left(\int_0^\infty (T\mathbf{1})^q w(x)dx\right)^{\frac{1}{q}} \\ &\ll C\left(\left(\int_0^\infty\left(\int_0^t fv\right)^p\frac{v(t)dt}{V^p(t)}\right)^{\frac{1}{p}}+\left(\int_0^\infty f^p v\right)^{\frac{1}{p}}\right) \\ &\ll C\left(\int_0^\infty f^p v\right)^{\frac{1}{p}}. \end{aligned}$$

The case $q=\infty$ is treated similarly. □

To study the case $p=1$ we suppose that an operator $T:\mathfrak{M}^+\to\mathfrak{M}^+$ satisfies the following axiom:

(iv) If $\{f_n\}\subset\mathfrak{M}^{\downarrow}$ and $f_n(x)\uparrow f(x)\in\mathfrak{M}^{\downarrow}$ for almost every $x\in\mathbb{R}_+$, then $Tf_n(x)\uparrow Tf(x)$ for almost every $x\in\mathbb{R}_+$.

We also need the following simple case of ([23], Lemma 1.2).

Lemma 2.2. *Let $f\in\mathfrak{M}^{\downarrow}$. Then there exist the sequence of non-negative finitely supported integrable functions $\{h_n\}\subset\mathfrak{M}^+$ such, that the functions $f_n(x):=\int_x^\infty h_n(s)ds$ are increasing with respect to n for any $x>0$ and $f(x)=\lim_{n\to\infty}\int_x^\infty h_n(y)dy$ for almost all $x>0$.*

Theorem 2.3. *Let $0 < q < \infty, p = 1$ and let $T : \mathfrak{M}^+ \to \mathfrak{M}^+$ be a positive quasilinear operator, satisfying* (i)-(iv). *Then the inequality* (1.1) *holds iff the inequality* (2.1) *is valid.*

Proof. The necessity is obvious. For sufficiency we suppose that $f \in \mathfrak{M}^\downarrow$ and by Lemma 2.2 there exists $\{h_n\} \subset L^1(\mathbb{R}_+)$ such that

$$f_n(x) := \int_x^\infty h_n(y)dy \uparrow f(x).$$

Then by (i)-(iv) and Fatou's lemma

$$\begin{aligned}
\left(\int_0^\infty (Tf)^q w\right)^{\frac{1}{q}} &\ll \left(\int_0^\infty \left(\lim_{n\to\infty} Tf_n\right)^q w\right)^{\frac{1}{q}} \\
&\le \lim_{n\to\infty} \left(\int_0^\infty (Tf_n)^q w\right)^{\frac{1}{q}} \\
&= \lim_{n\to\infty} \left(\int_0^\infty \left(T\left(\int_x^\infty h_n\right)\right)^q w\right)^{\frac{1}{q}} \\
&\le C \lim_{n\to\infty} \left(\int_0^\infty h_n V\right) \\
&= C \lim_{n\to\infty} \left(\int_0^\infty f_n v\right) = C\int_0^\infty fv.
\end{aligned}$$

□

Analogously we reduce the inequality for non-decresing functions of the form

$$\left(\int_0^\infty (Tf(t))^q w(t)dt\right)^{\frac{1}{q}} \le C\left(\int_0^\infty (f(t))^p v(t)dt\right)^{\frac{1}{p}}, \ f \in \mathfrak{M}^\uparrow \tag{2.3}$$

provided the axiom (iv) is replaced by

(iv') If $\{f_n\} \subset \mathfrak{M}^\uparrow$ and $f_n(x) \uparrow f(x) \in \mathfrak{M}^\uparrow$ for almost every $x \in \mathbb{R}_+$, then $Tf_n(x) \uparrow Tf(x)$ for almost every $x \in \mathbb{R}_+$.

Put $V_*(t) := \int_t^\infty v$.

Theorem 2.4. *Let $0 < q < \infty, 1 < p < \infty$ and let $T : \mathfrak{M}^+ \to \mathfrak{M}^+$ be a positive quasilinear operator, satisfying* (i)-(iii). *Then the inequality* (2.3) *holds iff* (2.2) *and the inequalitiy*

$$\left(\int_0^\infty \left(T\left(\int_0^x h\right)\right)^q w\right)^{\frac{1}{q}} \le C\left(\int_0^\infty h^p V_*^p v^{1-p}\right)^{\frac{1}{p}}, \ h \in \mathfrak{M}^+ \tag{2.4}$$

are valid.

Theorem 2.5. *Let $0 < q < \infty, p = 1$ and let $T : \mathfrak{M}^+ \to \mathfrak{M}^+$ be a positive quasilinear operator, satisfying* (i)-(iii) *and* (iv'). *Then the inequality* (2.3) *holds iff the inequalities* (2.4) *and* (2.2) *are valid.*

3. The case $0 < p \le q < \infty$

Let $f \in \mathfrak{M}^{\downarrow}$. Then there exist $\{x_n\} \subset \mathbb{R}_+$ such that

$$\begin{aligned} f(x) &\approx \sum_n 2^{-n}\chi_{[0,x_n]}(x) \\ &= \sum_{n: x_n \ge x} 2^{-n}\chi_{[0,x_n]}(x) \\ &= \int_{[x,\infty)} \left(\sum_n 2^{-n}\delta_{x_n}(s)\right) ds \\ &=: \int_{[x,\infty)} h(s)ds, \end{aligned} \tag{3.1}$$

where $\delta_t(s)$ is the Dirac delta-function at a point t. Observe that

$$[f(x)]^r \approx \left(\sum_n 2^{-n}\chi_{[0,x_n]}(x)\right)^r \approx \sum_n 2^{-nr}\chi_{[0,x_n]}(x), \quad r > 0. \tag{3.2}$$

Theorem 3.1. *Let $0 < p \le q < \infty$ and let $T : \mathfrak{M}^+ \to \mathfrak{M}^+$ be a positive quasilinear operator, satisfying* (i)-(iii)*, such that*

$$T\left(\sum_n f_n\right) \ll \left(\sum_n [Tf_n]^p\right)^{\frac{1}{p}} \tag{3.3}$$

for any $f_n \ge 0$. Then the inequality (1.1) *is equivalent to the validity one of the following condtions:*

$$\left(\int_0^\infty \left(\int_0^\infty \left[T\chi_{[0,s]}(x)\right]^p h(s)ds\right)^{\frac{q}{p}} w(x)dx\right)^{\frac{p}{q}} \le C_2^p \int_0^\infty hV, \ h \in \mathfrak{M}^+, \tag{3.4}$$

$$\left(\int_0^\infty \left[\sup_{s>0} T\chi_{[0,s]}(x)f(s)\right]^q w(x)dx\right)^{\frac{1}{q}} \le C_3 \left(\int_0^\infty f^p v\right)^{\frac{1}{p}}, \ f \in \mathfrak{M}^{\downarrow}, \tag{3.5}$$

$$\left(\int_0^\infty \left[\sup_{s>0} \left(T\chi_{[0,s]}(x)\right)^p \int_s^\infty h\right]^{\frac{q}{p}} w(x)dx\right)^{\frac{p}{q}} \le C_4^p \int_0^\infty hV, \ h \in \mathfrak{M}^+, \tag{3.6}$$

or

$$\mathbf{D} := \sup_{t>0} \left(\int_0^\infty \left[T\chi_{[0,t]}(x)\right]^q w(x)dx\right)^{\frac{1}{q}} V^{-\frac{1}{p}}(t) < \infty. \tag{3.7}$$

Moreover,

$$C \approx C_2 = \mathbf{D} \approx C_3 = C_4. \tag{3.8}$$

Proof. (3.5)$\Leftrightarrow$(3.6) follows by Lemma 2.2 with equality $C_3 = C_4$. (3.5)$\Rightarrow$(3.7) follows by applying (3.5) to a test function $f_t(s) := \chi_{[0,t]}(s)$, $t > 0$. Similarly, we obtain (1.1)$\Rightarrow$(3.7). From the properties (i)-(iii) we find, that for all $s > 0$

$$Tf(x) \ge T(\chi_{[0,s]}f)(x) \ge T\chi_{[0,s]}(x)f(s)$$

and (1.1)⇒(3.5) follows. Let

$$k(x,s) := \left[T\chi_{[0,s]}(x)\right]^p$$

and

$$\mathbf{K}h(x) := \int_0^\infty k(x,s)h(s)ds.$$

Then (3.4) is equivalent to the boundedness $\mathbf{K} : L^1_V \to L^{\frac{q}{p}}_w$ and

$$C_2^p = \|\mathbf{K}\|_{L^1_V \to L^{\frac{q}{p}}_w} = \mathbf{D}^p.$$

Let us show that (3.4)⇒ (1.1). It follows from (3.2) and (3.3), that

$$\begin{aligned}\left(Tf^{\frac{1}{p}}\right)(x) &\approx \left(T\left(\sum_n 2^{-n}\chi_{[0,x_n]}(x)\right)^{\frac{1}{p}}\right)(x)\\ &\approx T\left(\sum_n 2^{-\frac{n}{p}}\chi_{[0,x_n]}(x)\right)(x)\\ &\ll \left(\sum_n 2^{-n}\left[T\chi_{[0,x_n]}(x)\right]^p\right)^{\frac{1}{p}}.\end{aligned}$$

Observe that (1.1) is equivalent to

$$\left(\int_0^\infty \left(Tf^{\frac{1}{p}}\right)^q w\right)^{\frac{p}{q}} \le C^p \int_0^\infty fv, \;\; f \in \mathfrak{M}^\downarrow.$$

Now, using (3.2), we find

$$\begin{aligned}\left(\int_0^\infty \left(Tf^{\frac{1}{p}}\right)^q w\right)^{\frac{p}{q}} &\ll \left(\int_0^\infty \left(\sum_n 2^{-n}\left[T\chi_{[0,x_n]}(x)\right]^p\right)^{\frac{q}{p}} w(x)dx\right)^{\frac{p}{q}}\\ &= \left(\int_0^\infty \left(\int_0^\infty \left[T\chi_{[0,s]}(x)\right]^p h(s)ds\right)^{\frac{q}{p}} w(x)dx\right)^{\frac{p}{q}}\\ &\le C_2^p \int_0^\infty hV = C_2^p \sum_n 2^{-n}V(x_n)\\ &\approx C_2^p \sum_n 2^{-n}\int_{[x_n,x_{n+1})} v \approx C_2^p \int_0^\infty fv.\end{aligned}$$

Consequently, $C \ll C_2$ and (3.8) follows.

□

Similarly, we characterize the case of non-decreasing functions.

Theorem 3.2. *Let* $0 < p \le q < \infty$ *and let* $T : \mathfrak{M}^+ \to \mathfrak{M}^+$ *be a positive quasilinear operator, satisfying* (i)-(iii) *and* (3.3). *Then the inequality* (2.3) *is equivalent to one of the following* *conditions:*

$$\left(\int_0^\infty\left(\int_0^\infty\left[T\chi_{[s,\infty)}(x)\right]^p h(s)ds\right)^{\frac{q}{p}} w(x)dx\right)^{\frac{p}{q}} \le C_2^p\int_0^\infty hV_*,\ h\in\mathfrak{M}^+, \tag{3.9}$$

$$\left(\int_0^\infty\left[\sup_{s>0} T\chi_{[s,\infty)}(x)f(s)\right]^q w(x)dx\right)^{\frac{1}{q}} \le C_3\left(\int_0^\infty f^p v\right)^{\frac{1}{p}},\ f\in\mathfrak{M}^\downarrow, \tag{3.10}$$

$$\left(\int_0^\infty\left[\sup_{s>0}\left(T\chi_{[s,\infty)}(x)\right)^p\int_0^s h\right]^{\frac{q}{p}} w(x)dx\right)^{\frac{p}{q}} \le C_4^p\int_0^\infty hV,\ h\in\mathfrak{M}^+, \tag{3.11}$$

$$\mathbf{D}_* := \sup_{t>0}\left(\int_0^\infty\left[T\chi_{[t,\infty)}(x)\right]^q w(x)dx\right)^{\frac{1}{q}} V_*^{-\frac{1}{p}}(t) < \infty. \tag{3.12}$$

Moreover,

$$C \approx C_2 = \mathbf{D}_* \approx C_3 = C_4. \tag{3.13}$$

Now we study the converse inequality

$$\left(\int_0^\infty f^q w\right)^{\frac{1}{q}} \le C\left(\int_0^\infty (Tf)^p v\right)^{\frac{1}{p}},\ f\in\mathfrak{M}^\downarrow \tag{3.14}$$

Put $W(t) := \int_0^t w,\ W_*(t) := \int_t^\infty w$.

Theorem 3.3. *Let* $0<p\le q<\infty$ *and let* $T:\mathfrak{M}^+\to\mathfrak{M}^+$ *be a positive quasilinear operator, satisfying* (i)-(iii), *such that*

$$\left(\sum_n\left[Tf_n\right]^q\right)^{\frac{1}{q}} \ll T\left(\sum_n f_n\right) \tag{3.15}$$

for any $f_n\ge 0$. *Then the inequality* (3.14) *is equivalent to the validity of the inequality*

$$\int_0^\infty hW \le C_2^q\left(\int_0^\infty\left(\int_0^\infty\left[T\chi_{[0,s]}(x)\right]^p h(s)ds\right)^{\frac{p}{q}} v(x)dx\right)^{\frac{q}{p}},\ h\in\mathfrak{M}^+, \tag{3.16}$$

or

$$\mathfrak{D} := \sup_{t>0} W^{\frac{1}{q}}(t)\left(\int_0^\infty\left[T\chi_{[0,t]}(x)\right]^p v(x)dx\right)^{-\frac{1}{p}} < \infty. \tag{3.17}$$

Moreover,

$$C \approx C_2 = \mathfrak{D}. \tag{3.18}$$

Proof. The implication (3.14)⇒(3.17) is clear. Let us show (3.17)⇒(3.16). By Minkowskii's inequality we have

$$\begin{aligned}\int_0^\infty hW &\le \mathfrak{D}^q\int_0^\infty\left(\int_0^\infty\left[T\chi_{[0,t]}(x)\right]^p v(x)dx\right)^{\frac{q}{p}} h(t)dt\\ &\le \mathfrak{D}^q\left(\int_0^\infty v(x)\left(\int_0^\infty\left[T\chi_{[0,t]}(x)\right]^q h(t)dt\right)^{\frac{p}{q}} dx\right)^{\frac{q}{p}}.\end{aligned}$$

Now, (3.14) is equivalent to

$$\int_0^\infty fw \le C\left(\int_0^\infty \left(Tf^{\frac{1}{q}}\right)^p v\right)^{\frac{q}{p}}, \ f \in \mathfrak{M}^\downarrow \tag{3.19}$$

Using (3.16), (3.1), (3.2) and (3.15), we write

$$\begin{aligned}
\int_0^\infty fw &= \int_0^\infty hW \\
&\le C_2^q \left(\int_0^\infty \left(\int_0^\infty \left[T\chi_{[0,s]}(x)\right]^q h(s)ds\right)^{\frac{p}{q}} v(x)dx\right)^{\frac{q}{p}} \\
&= C_2^q \left(\int_0^\infty \left(\sum_n 2^{-n}\left[T\chi_{[0,x_n]}(x)\right]^q\right)^{\frac{p}{q}} v(x)dx\right)^{\frac{q}{p}} \\
&= C_2^q \left(\int_0^\infty \left(\sum_n \left[T\left(2^{-\frac{n}{q}}\chi_{[0,x_n]}\right)(x)\right]^q\right)^{\frac{p}{q}} v(x)dx\right)^{\frac{q}{p}} \\
&\le C_2^q \left(\int_0^\infty \left[T\left(\sum_n 2^{-\frac{n}{q}}\chi_{[0,x_n]}\right)(x)\right]^p v(x)dx\right)^{\frac{q}{p}} \\
&\approx C_2^q \left(\int_0^\infty \left[T\left(\sum_n 2^{-n}\chi_{[0,x_n]}\right)^{\frac{1}{q}}(x)\right]^p v(x)dx\right)^{\frac{q}{p}} \\
&\approx C_2^q \left(\int_0^\infty \left(Tf^{\frac{1}{q}}\right)^p v\right)^{\frac{q}{p}}
\end{aligned}$$

and (3.18) follows. □

Similarly we characterize the inequality

$$\left(\int_0^\infty f^q w\right)^{\frac{1}{q}} \le C\left(\int_0^\infty (Tf)^p v\right)^{\frac{1}{p}}, \ f \in \mathfrak{M}^\uparrow \tag{3.20}$$

Theorem 3.4. *Let $0 < p \le q < \infty$ and let $T : \mathfrak{M}^+ \to \mathfrak{M}^+$ be a positive quasilinear operator, satisfying* (i)-(iii) *and* (3.15). *Then the inequality* (3.20) *is valid iff*

$$\int_0^\infty hW_* \le C_2^q \left(\int_0^\infty \left(\left[T\chi_{[s,\infty)}(x)\right]^p h(s)ds\right)^{\frac{p}{q}} w(x)dx\right)^{\frac{q}{p}}, \ h \in \mathfrak{M}^+, \tag{3.21}$$

or

$$\mathfrak{D}_* := \sup_{t>0} W_*^{\frac{1}{q}}(t)\left(\int_0^\infty \left[T\chi_{[t,\infty)}(x)\right]^p v(x)dx\right)^{-\frac{1}{p}} < \infty. \tag{3.22}$$

Moreover,

$$C \approx C_2 = \mathfrak{D}_*. \tag{3.23}$$

Remark 3.5. Let T be an integral operator

$$Tf(x) := \int_0^\infty k(x,y)f(y)dy \tag{3.24}$$

with a non-negative kernel. Then the condition (3.3) is valid for all $p \in (0,1]$ and by Theorems 3.1 and 3.2 we obtain ([26], Theorem 4.1), ([18], Theorem 2.1 (a)) and ([3], Theorem 1). Analogously, the condition (3.15) holds for all $q \geq 1$ and by Theorems 3.3 and 3.4 we obtain an extension of ([26], Theorem 4.2) ([18], Theorem 2.1 (b)) for a larger interval.

4. The case $0 < q < p \leq 1$

Let u, v and w be weights. Denote $V(t) := \int_0^t v, W(t) := \int_0^t w, U(y,x) := \int_x^y u$. For simplicity we suppose that $0 < V(t) < \infty, 0 < W(t) < \infty$ for all $t > 0$ and $V(\infty) = \infty, W(\infty) = \infty$.

Theorem 4.1. *Let $0 < q < p \leq 1, 1/r := 1/q - 1/p$. The following are equivalent:*

$$\left(\int_0^\infty \left(\int_x^\infty fu\right)^q w(x)dx\right)^{\frac{1}{q}} \leq C_1 \left(\int_0^\infty f^p v\right)^{\frac{1}{p}}, \ f \in \mathfrak{M}^\downarrow, \tag{4.1}$$

$$\left(\int_0^\infty \left(\int_x^\infty U^p(y,x)h(y)dy\right)^{\frac{q}{p}} w(x)dx\right)^{\frac{p}{q}} \leq C_2^p \int_0^\infty hV, \ h \in \mathfrak{M}^+, \tag{4.2}$$

$$\left(\int_0^\infty \left(\sup_{y\geq x} U^p(y,x)\int_x^\infty h\right)^{\frac{q}{p}} w(x)dx\right)^{\frac{p}{q}} \leq C_3^p \int_0^\infty hV, \ h \in \mathfrak{M}^+, \tag{4.3}$$

$$\left(\int_0^\infty \left(\sup_{y\geq x} U(y,x)f(y)\right)^q w(x)dx\right)^{\frac{1}{q}} \leq C_4 \left(\int_0^\infty f^p v\right)^{\frac{1}{p}}, \ f \in \mathfrak{M}^\downarrow, \tag{4.4}$$

$$\mathbb{B}^r := \sup_{\{x_k\}} \sum_{k\in\mathbb{Z}} \left(\int_{x_k}^{x_{k+1}} U^q(x_{k+1},y)w(y)dy\right)^{\frac{r}{q}} V^{-\frac{r}{p}}(x_{k+1}) < \infty. \tag{4.5}$$

Moreover,

$$C_1 \approx C_2 \approx C_3 = C_4 \approx \mathbb{B}. \tag{4.6}$$

Proof. Observe first, that (4.4)⇔(4.5) follows from ([27], Theorem 4.4). (4.4)⇒(4.3) is obvious and (4.3)⇒(4.4) follows by applying Lemma 2.2 and Fatou's lemma. For any $f \in \mathfrak{M}^\downarrow$

$$\int_x^\infty fu \geq \sup_{y\geq x} \int_x^y fu \geq \sup_{y\geq x} U(y,x)f(y).$$

Hence, (4.1)⇒(4.4). The inequality (4.1) is equivalent to

$$\left(\int_0^\infty \left(\int_x^\infty f^{\frac{1}{p}}u\right)^q w\right)^{\frac{p}{q}} \leq C_1^p \left(\int_0^\infty fv\right)^{\frac{1}{p}}, \ f \in \mathfrak{M}^\downarrow, \tag{4.7}$$

Let $f(x) = \int_x^\infty h$. Then by Minkowskii's inequality

$$\int_x^\infty f^{\frac{1}{p}}u = \int_x^\infty \left(\int_z^\infty h\right)^{\frac{1}{p}} u(z)dz \leq \left(\int_x^\infty U^p(y,x)h(y)dy\right)^{\frac{1}{p}}$$

and (4.2)⇒(4.1) follows by Lemma 2.2.

Thus, the only (4.3)⇒(4.2) remains to prove. To this end let us denote the left hand sides of (4.2) and (4.3) as $A^{\frac{p}{q}}$ and $B^{\frac{p}{q}}$, respectively. Suppose, that (4.3) is true and denote $\{x_n\}$ such a sequence, that $W(x_n) = 2^n, n \in \mathbb{Z}$. Put $\Delta_n := [x_n, x_{n+1})$. Then

$$\begin{aligned}
A &:= \sum_n \int_{\Delta_n} \left(\int_x^\infty U^p(y,x)h(y)dy \right)^{\frac{q}{p}} w(x)dx \\
&\approx \sum_n 2^n \left(\int_{x_n}^\infty U^p(y,x)h(y)dy \right)^{\frac{q}{p}} \\
&= \sum_n 2^n \left(\int_{x_n}^\infty U^p(y,x_n)h(y)dy \right)^{\frac{q}{p}} \\
&= \sum_n 2^n \left(\sum_{i=n}^\infty \int_{\Delta_i} U^p(y,x_n)h(y)dy \right)^{\frac{q}{p}} \\
&\approx \sum_n 2^n \left(\sum_{i=n}^\infty \int_{\Delta_i} U^p(y,x_i)h(y)dy \right)^{\frac{q}{p}} \\
&+ \sum_n 2^n \left(\sum_{i=n+1}^\infty U^p(x_i,x_n) \int_{\Delta_i} h(y)dy \right)^{\frac{q}{p}} \\
&=: A_1 + A_2.
\end{aligned}$$

Applying well known equivalence

$$\sum_n 2^n \left(\sum_{i=n}^\infty a_i \right)^s \approx \sum_n 2^n a_n^s \tag{4.8}$$

valid for any sequence $\{a_n\}$ of non-negative numbers and $s > 0$, we obtain

$$A_1 \approx \sum_n 2^n \left(\int_{\Delta_n} U^p(y,x_n)h(y)dy \right)^{\frac{q}{p}} \tag{4.9}$$

By Jensen's inequality and (4.8)

$$\begin{aligned}
A_2 &:= \sum_n 2^n \left(\sum_{i=n+1}^\infty \left(\sum_{j=n}^{i-1} U(x_{j+1},x_j) \right)^p \int_{\Delta_i} h(y)dy \right)^{\frac{q}{p}} \\
&\le \sum_n 2^n \left(\sum_{i=n+1}^\infty \sum_{j=n}^{i-1} U^p(x_{j+1},x_j) \int_{\Delta_i} h(y)dy \right)^{\frac{q}{p}} \\
&= \sum_n 2^n \left(\sum_{j=n}^\infty U^p(x_{j+1},x_j) \int_{x_{j+1}}^\infty h(y)dy \right)^{\frac{q}{p}} \\
&\approx \sum_n 2^n \left(U^p(x_{n+1},x_n) \int_{x_{n+1}}^\infty h(y)dy \right)^{\frac{q}{p}}. \qquad (4.10)
\end{aligned}$$

Similarly, for the constant B we have

$$\begin{aligned}
B &:= \int_0^\infty \left(\sup_{y\ge x} U^p(y,x)\int_y^\infty h\right)^{\frac{q}{p}} w(x)dx \\
&= \sum_n \int_{\Delta_n} \left(\sup_{y\ge x} U^p(y,x)\int_y^\infty h\right)^{\frac{q}{p}} w(x)dx \\
&\approx \sum_n 2^n \left(\sup_{y\ge x_n} U^p(y,x_n)\int_y^\infty h\right)^{\frac{q}{p}} \\
&= \sum_n 2^n \left(\sup_{i\ge n}\sup_{y\in\Delta_i} U^p(y,x_n)\int_y^\infty h\right)^{\frac{q}{p}} \\
&\approx \sum_n 2^n \left(\sup_{i\ge n}\sup_{y\in\Delta_i} U^p(y,x_i)\int_y^\infty h\right)^{\frac{q}{p}} \\
&+ \sum_n 2^n \left(\sup_{i\ge n+1} U^p(x_i,x_n)\int_{x_i}^\infty h\right)^{\frac{q}{p}} \\
&\approx \sum_n 2^n \left(\sup_{y\in\Delta_n} U^p(y,x_n)\int_y^\infty h\right)^{\frac{q}{p}} \\
&+ \sum_n 2^n \left(\sup_{i\ge n+1} U^p(x_i,x_n)\int_{x_i}^\infty h\right)^{\frac{q}{p}} \\
&=: B_1 + B_2.
\end{aligned}$$

Here we applied $\sum_n 2^n \left(\sup_{i\ge n} a_i\right)^s \approx \sum_n 2^n a_n^s$, valid for any sequence $a_n \ge 0$ and $s > 0$. Suppose, that (4.3) holds. Then

$$B_i \ll C_3^q \left(\int_0^\infty hV\right)^{\frac{q}{p}}, \quad i = 1,2. \tag{4.11}$$

By (4.10)

$$A_2 \ll B_2 \ll C_3^q \left(\int_0^\infty hV\right)^{\frac{q}{p}}. \tag{4.12}$$

By Hölder's inequality

$$\begin{aligned}
A_1 &\approx \sum_n 2^n \left(\int_{\Delta_n} U^p(y,x_n)V^{-1}(y)h(y)V(y)dy\right)^{\frac{q}{p}} \\
&\le \sum_n 2^n \left(\sup_{y\in\Delta_n} U^p(y,x_n)V^{-1}(y)\right)^{\frac{q}{p}} \left(\int_{\Delta_n} hV\right)^{\frac{q}{p}} \\
&\le \left(\sum_n 2^{\frac{nr}{q}} \left(\sup_{y\in\Delta_n} U^p(y,x_n)V^{-1}(y)\right)^{\frac{r}{p}}\right)^{\frac{q}{r}} \left(\sum_n \int_{\Delta_n} hV\right)^{\frac{q}{p}} \\
&=: \mathbb{D}^q \left(\int_0^\infty hV\right)^{\frac{q}{p}}
\end{aligned}$$

It follows from (4.11), that

$$\sum_n 2^n \left(\sup_{y\in\Delta_n} U^p(y,x_n) \int_y^{x_{n+1}} h \right)^{\frac{q}{p}} \ll C_3^q \left(\int_0^\infty hV \right)^{\frac{q}{p}}. \tag{4.13}$$

Let $H_n : L^1_V[\Delta_n] \to L^\infty[\Delta_n]$ be operator of the form

$$H_n h(y) := U^p(y,x_n) \int_y^{x_{n+1}} h.$$

Then

$$d_n := \|H_n\|_{L^1_V[\Delta_n]\to L^\infty[\Delta_n]} = \sup_{y\in\Delta_n} U^p(y,x_n)V^{-1}(y).$$

Let $h_n \in L^1_V[\Delta_n]$ be such that

$$\sup_{y\in\Delta_n} U^p(y,x_n)V^{-1}(y) \geq \frac{d_n}{2} \int_{\Delta_n} h_n V.$$

Then by (4.13)

$$\begin{aligned} C_3^q &\gg \sup_{h\geq 0} \frac{\sum_n 2^n \left(\sup_{y\in\Delta_n} U^p(y,x_n) \int_y^{x_{n+1}} h \right)^{\frac{q}{p}}}{\left(\int_0^\infty hV \right)^{\frac{q}{p}}} \\ &\geq \sup_{h=\sum_n a_n h_n} \frac{\sum_n 2^n a_n^{\frac{q}{p}} \left(\sup_{y\in\Delta_n} U^p(y,x_n) \int_y^{x_{n+1}} h \right)^{\frac{q}{p}}}{\left(\sum_n a_n \int_{\Delta_n} hV \right)^{\frac{q}{p}}} \\ &\gg \sup_{h=\sum_n a_n h_n} \frac{\sum_n 2^n d_n^{\frac{q}{p}} \left(a_n \int_{\Delta_n} hV \right)^{\frac{q}{p}}}{\left(\sum_n a_n \int_{\Delta_n} hV \right)^{\frac{q}{p}}} = \mathbb{D}^q. \end{aligned}$$

Hence, $\mathbb{D} \ll C_3$ and

$$A_1 \ll \mathbb{D}^q \left(\int_0^\infty hV \right)^{\frac{q}{p}} \ll C_3^q \left(\int_0^\infty hV \right)^{\frac{q}{p}}.$$

This and (4.12) imply (4.3)⇒(4.2). □

Symmetric version of the previous theorem is the following.

Theorem 4.2. *Let $0 < q < p \leq 1$. The following are equivalent:*

$$\left(\int_0^\infty \left(\int_0^x fu \right)^q w(x)dx \right)^{\frac{1}{q}} \leq \bar{C}_1 \left(\int_0^\infty f^p v \right)^{\frac{1}{p}}, \ f \in \mathfrak{M}^\uparrow, \tag{4.14}$$

$$\left(\int_0^\infty \left(\int_0^x U^p(x,y)h(y)dy \right)^{\frac{q}{p}} w(x)dx \right)^{\frac{p}{q}} \leq \bar{C}_2^p \int_0^\infty hV_*, \ h \in \mathfrak{M}^+,$$

$$\left(\int_0^\infty \left(\sup_{0<y\leq x} U^p(x,y) \int_0^x h \right)^{\frac{q}{p}} w(x)dx \right)^{\frac{p}{q}} \leq \bar{C}_3^p \int_0^\infty hV_*, \ h \in \mathfrak{M}^+,$$

$$\left(\int_0^\infty \left(\sup_{0<y\le x} U(x,y)f(y)\right)^q w(x)dx\right)^{\frac{1}{q}} \le \bar{C}_4 \left(\int_0^\infty f^p v\right)^{\frac{1}{p}}, \ f \in \mathfrak{M}^\uparrow,$$

$$\bar{\mathbb{B}}^r := \sup_{\{x_k\}} \sum_{k\in\mathbb{Z}} \left(\int_{x_k}^{x_{k+1}} U^q(y,x_k)w(y)dy\right)^{\frac{r}{q}} V_*^{-\frac{r}{p}}(x_k) < \infty.$$

Moreover,

$$\bar{C}_1 \approx \bar{C}_2 \approx \bar{C}_3 = \bar{C}_4 \approx \bar{\mathbb{B}}.$$

Remark 4.3. The result of Theorem 4.2 supplements [12].

Definition 4.4. A measurable function $k(x,y) \ge 0$ on $\{(x,y) : x \ge y \ge 0\}$, we name *Oinarov kernel*, $k(x,y) \in \mathcal{O}$, if there exist a constant $D \ge 1$, independent of x, y and z such, that

$$D^{-1}\left(k(x,z) + k(z,y)\right) \le k(x,y) \le D\left(k(x,z) + k(z,y)\right)$$

for all $x \ge z \ge y \ge 0$.

Let $k(x,z) \ge 0$ be a measurable kernel. Put

$$K(x,y) = \int_0^y k(x,z)u(z)dz.$$

Theorem 4.5. *Let $0 < q < p \le 1, 1/r := 1/q - 1/p$. Let $k(x,y)$ be a continuous Oinarov kernel. The following inequalities are equivalent:*

$$\left(\int_0^\infty \left(\int_0^x k(x,y)f(y)u(y)dy\right)^q w(x)dx\right)^{\frac{1}{q}} \le C_1 \left(\int_0^\infty f^p v\right)^{\frac{1}{p}}, \ f \in \mathfrak{M}^\downarrow, \tag{4.15}$$

$$\left(\int_0^\infty \left(\int_0^x K^p(x,y)h(y)dy + K^p(x,x)\int_x^\infty h(y)dy\right)^{\frac{q}{p}} w(x)dx\right)^{\frac{p}{q}} \le C_2^p \int_0^\infty hV, \ h \in \mathfrak{M}^+, \tag{4.16}$$

$$\left(\int_0^\infty \left(\sup_{0<y\le x} K^p(x,y)\int_y^\infty h\right)^{\frac{q}{p}} w(x)dx\right)^{\frac{p}{q}} \le C_3^p \int_0^\infty hV, \ h \in \mathfrak{M}^+, \tag{4.17}$$

$$\left(\int_0^\infty \left(\sup_{0<y\le x} K(x,y)f(y)\right)^q w(x)dx\right)^{\frac{1}{q}} \le C_4 \left(\int_0^\infty f^p v\right)^{\frac{1}{p}}, \ f \in \mathfrak{M}^\downarrow, \tag{4.18}$$

$$\mathbb{B}^r := \sup_{\{x_k\}} \sum_{k\in\mathbb{Z}} \left(\int_{x_k}^{x_{k+1}} K^q(y,x_k)w(y)dy\right)^{\frac{r}{q}} V^{-\frac{r}{p}}(x_k) < \infty. \tag{4.19}$$

Moreover,

$$C_1 \approx C_2 \approx C_3 = C_4 \approx \mathbb{B}. \tag{4.20}$$

Proof. We will prove the following implications (4.16)⇒(4.15)⇒ (4.18) ⇔ (4.17)⇒ (4.19)⇒ (4.16).

The inequality (4.15) is equivalent to

$$\left(\int_0^\infty \left(\int_0^x k(x,y) f^{\frac{1}{p}}(y) u(y) dy\right)^q w(x) dx\right)^{\frac{p}{q}} \le C_1^p \left(\int_0^\infty fv\right)^{\frac{1}{p}}, \quad f \in \mathfrak{M}^{\downarrow}, \tag{4.21}$$

Let $f(x) = \int_x^\infty h$. Then by Minkowskii's inequality

$$\begin{aligned}\int_0^x k(x,z) f^{\frac{1}{p}}(z) u(z) dz &= \int_0^x \left(\int_z^\infty h\right)^{\frac{1}{p}} k(x,z) u(z) dz \\ &\le \left(\int_0^\infty \left(\int_0^x \chi_{(z,\infty)}(y) k(x,z) u(z) dz\right)^p h(y) dy\right)^{\frac{1}{p}} \\ &\approx \left(\int_0^x K^p(x,y) h(y) dy + K^p(x,x) \int_x^\infty h(y) dy\right)^{\frac{1}{p}}\end{aligned}$$

and (4.16)⇒(4.15) follows by Lemma 2.2.

For any $f \in \mathfrak{M}^{\downarrow}$

$$\int_0^x k(x,z) f(z) u(z) dz \ge \sup_{0<y\le x} \int_0^y k(x,z) u(z) dz f(y) \ge \sup_{0<y\le x} K(x,y) f(y).$$

Hence, (4.15)⇒(4.18). (4.18)⇒(4.17) is obvious and (4.17)⇒(4.18) follows by applying Lemma 2.2 and Fatou's lemma.

Suppose, that (4.17) is true and let $\{x_n\} \subset (0,\infty)$ be an increasing sequence. For any $k \in \mathbb{Z}$, let $\varepsilon_k \in (x_k, x_{k+1})$ be such that $V(\varepsilon_k) \le 2V(x_k)$ and for any sequence $\{a_k\} \subset (0,\infty)$ of positive numbers we define the function $h(x) := \sum_{k\in\mathbb{Z}} \frac{a_k}{x_k - \varepsilon_k} \chi_{(x_k,\varepsilon_k)}(x)$. If we put the function in the inequality (4.17), we get

$$\left(\sum_{k\in\mathbb{Z}} a_k^{\frac{q}{p}} \int_{x_k}^{x_{k+1}} K^q(x, x_k) w(x) dx\right)^{\frac{p}{q}} \le 2C_3^q \sum_{k\in\mathbb{Z}} a_k V(x_k) \tag{4.22}$$

and by the Landau theorem it implies $\mathbb{B} \ll C_3$.

Thus, the only (4.19)⇒(4.16) it remains to prove. Using the definition of Oinarov's kernel, we see that

$$K(x,y) \approx k(x,y) \int_0^y u(z) dz + \int_0^y k(y,z) u(z) dz = k(x,y) U(y) + K(y,y)$$

and it implies, that (4.16) is equivalent to the following three inequalities:

$$\left(\int_0^\infty \left(\int_0^x k(x,y)^p U(y)^p h(y)dy\right)^{\frac{q}{p}} w(x)dx\right)^{\frac{p}{q}} \le C_2^p \int_0^\infty hV, \;\; h \in \mathfrak{M}^+, \tag{4.23}$$

$$\left(\int_0^\infty \left(\int_0^x K(y,y)^p h(y)dy\right)^{\frac{q}{p}} w(x)dx\right)^{\frac{p}{q}} \le C_2^p \int_0^\infty hV, \;\; h \in \mathfrak{M}^+, \tag{4.24}$$

$$\left(\int_0^\infty \left(\int_x^\infty h(y)dy\right)^{\frac{q}{p}} K(x,x)^q w(x)dx\right)^{\frac{p}{q}} \le C_2^p \int_0^\infty hV, \;\; h \in \mathfrak{M}^+, \tag{4.25}$$

By [17, Theorem 5] (4.23) holds if and only if

$$B_1^r := \sup_{\{x_k\}} \sum_{k\in\mathbb{Z}} \left(\int_{x_k}^{x_{k+1}} k(y,x_k)^q w(y)dy\right)^{\frac{r}{q}} \sup_{y\in(x_{k-1},x_k)} U(y)^r V^{-\frac{r}{p}}(y) < \infty \tag{4.26}$$

$$B_2^r := \sup_{\{x_k\}} \sum_{k\in\mathbb{Z}} \left(\int_{x_k}^{x_{k+1}} w(y)dy\right)^{\frac{r}{q}} \sup_{y\in(x_{k-1},x_k)} k^r(x_k,y) U(y)^r V^{-\frac{r}{p}}(y) < \infty \tag{4.27}$$

as well as (4.24) holds if and only if

$$B_3^r := \sup_{\{x_k\}} \sum_{k\in\mathbb{Z}} \left(\int_{x_k}^{x_{k+1}} w(y)dy\right)^{\frac{r}{q}} \sup_{y\in(x_{k-1},x_k)} K(y,y)^r V^{-\frac{r}{p}}(y) < \infty$$

and by the dual form of [17, Theorem 5] (4.25) holds if and only if

$$B_4^r := \sup_{\{x_k\}} \sum_{k\in\mathbb{Z}} \left(\int_{x_{k-1}}^{x_k} K(y,y)^q w(y)dy\right)^{\frac{r}{q}} V^{-\frac{r}{p}}(x_k) < \infty$$

Let $y_k \in (x_{k-1}, x_k)$ be such that

$$\sup_{y\in(x_{k-1},x_k)} U(y)^r V^{-\frac{r}{p}}(y) = U(y_k)^r V^{-\frac{r}{p}}(y_k).$$

Then

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}} \left(\int_{x_k}^{x_{k+1}} k(y,x_k)^q w(y)dy\right)^{\frac{r}{q}} \sup_{y\in(x_{k-1},x_k)} U(y)^r V^{-\frac{r}{p}}(y) \\
&\ll \sum_{k\in\mathbb{Z}} \left(\int_{y_k}^{y_{k+2}} K(y,y_k)^q w(y)dy\right)^{\frac{r}{q}} V^{-\frac{r}{p}}(y_k) \\
&\ll \sum_{k\in\mathbb{Z}} \left(\int_{y_{2k}}^{y_{2k+2}} K(y,y_{2k})^q w(y)dy\right)^{\frac{r}{q}} V^{-\frac{r}{p}}(y_{2k}) \\
&+ \sum_{k\in\mathbb{Z}} \left(\int_{y_{2k+1}}^{y_{2k+3}} K(y,y_{2k+1})^q w(y)dy\right)^{\frac{r}{q}} V^{-\frac{r}{p}}(y_{2k+1}) \ll \mathbb{B}^r.
\end{aligned}$$

Therefore,

$$B_1 \ll \mathbb{B}.$$

Let $y_k \in (x_{k-1}, x_k)$ be such that

$$\sup_{y\in(x_{k-1},x_k)} k^r(x_k, y)U(y)^r V^{-\frac{r}{p}}(y) = k^r(x_k, y_k)U(y_k)^r V^{-\frac{r}{p}}(y_k).$$

Then

$$\sum_{k\in\mathbb{Z}} \left(\int_{x_k}^{x_{k+1}} w(y)dy\right)^{\frac{r}{q}} \sup_{y\in(x_{k-1},x_k)} k^r(x_k, y)U(y)^r V^{-\frac{r}{p}}(y)$$
$$\ll \sum_{k\in\mathbb{Z}} \left(\int_{y_k}^{y_{k+2}} K(y, y_k)^q w(y)dy\right)^{\frac{r}{q}} V^{-\frac{r}{p}}(y_k)$$
$$\ll \sum_{k\in\mathbb{Z}} \left(\int_{y_{2k}}^{y_{2k+2}} K(y, y_{2k})^q w(y)dy\right)^{\frac{r}{q}} V^{-\frac{r}{p}}(y_{2k})$$
$$+ \sum_{k\in\mathbb{Z}} \left(\int_{y_{2k+1}}^{y_{2k+3}} K(y, y_{2k+1})^q w(y)dy\right)^{\frac{r}{q}} V^{-\frac{r}{p}}(y_{2k+1}) \ll \mathbb{B}^r.$$

Hence,

$$B_2 \ll \mathbb{B}.$$

Let $y_k \in (x_{k-1}, x_k)$ be such that

$$\sup_{y\in(x_{k-1},x_k)} K^r(y, y)V^{-\frac{r}{p}}(y) = K(y_k, y_k)^r V^{-\frac{r}{p}}(y_k)$$

Then

$$\sum_{k\in\mathbb{Z}} \left(\int_{x_k}^{x_{k+1}} w(y)dy\right)^{\frac{r}{q}} \sup_{y\in(x_{k-1},x_k)} K(y, y)^r V^{-\frac{r}{p}}(y)$$
$$\ll \sum_{k\in\mathbb{Z}} \left(\int_{y_k}^{y_{k+2}} K(y, y_k)^q w(y)dy\right)^{\frac{r}{q}} V^{-\frac{r}{p}}(y_k)$$
$$\ll \sum_{k\in\mathbb{Z}} \left(\int_{y_{2k}}^{y_{2k+2}} K(y, y_{2k})^q w(y)dy\right)^{\frac{r}{q}} V^{-\frac{r}{p}}(y_{2k})$$
$$+ \sum_{k\in\mathbb{Z}} \left(\int_{y_{2k+1}}^{y_{2k+3}} K(y, y_{2k+1})^q w(y)dy\right)^{\frac{r}{q}} V^{-\frac{r}{p}}(y_{2k+1}) \ll \mathbb{B}^r.$$

Consequently,

$$B_3 \ll \mathbb{B}$$

Since $K(y, y) \ll K(y, x_{k-1})$, $y \in (x_{k-1}, x_k)$, we have that $B_4 \ll \mathbb{B}$. Combining the above upper bounds we conclude that $C_2 \ll \mathbb{B}$ and finish the proof. □

Analogously, we obtain the dual version of the previous theorem.

Theorem 4.6. *Let $0 < q < p \le 1, 1/r := 1/q - 1/p$ and $k(x, y)$ is a continuous Oinarov kernel and $K_*(y, x) = \int_y^\infty k(z, x)u(z)dz$. Then the following inequalities are equivalent:*

$$\left(\int_0^\infty \left(\int_x^\infty k(y, x)f(y)u(y)dy\right)^q w(x)dx\right)^{\frac{1}{q}} \le C_1 \left(\int_0^\infty f^p v\right)^{\frac{1}{p}}, \ f \in \mathfrak{M}^\uparrow,$$

$$\left(\int_0^\infty \left(\int_x^\infty K_*^p(y,x)h(y)dy + K_*^p(x,x)\int_0^x h(y)dy\right)^{\frac{q}{p}} w(x)dx\right)^{\frac{p}{q}}$$
$$\leq C_2^p \int_0^\infty hV_*, \ h \in \mathfrak{M}^+,$$

$$\left(\int_0^\infty \left(\sup_{y\geq x} K_*^p(y,x)\int_0^y h\right)^{\frac{q}{p}} w(x)dx\right)^{\frac{p}{q}} \leq C_3^p \int_0^\infty hV_*, \ h \in \mathfrak{M}^+,$$

$$\left(\int_0^\infty \left(\sup_{y\geq x} K_*(y,x)f(y)\right)^q w(x)dx\right)^{\frac{1}{q}} \leq C_4 \left(\int_0^\infty f^p v\right)^{\frac{1}{p}}, \ f \in \mathfrak{M}^\uparrow,$$

$$\mathbb{B}_*^r := \sup_{\{x_k\}} \sum_{k\in\mathbb{Z}} \left(\int_{x_{k-1}}^{x_k} K_*^q(x_k,y)w(y)dy\right)^{\frac{r}{q}} V_*^{-\frac{r}{p}}(x_k) < \infty.$$

Moreover,

$$C_1 \approx C_2 \approx C_3 = C_4 \approx \mathbb{B}_*.$$

5. Further results

Keeping the notations and assumptions of the previous section we obtain the complete characterization of the inequality (4.1).

Theorem 5.1. *Let $0 < q, p < \infty$ Then the inequality* (4.1) *with the best constant C_1 holds for every $f \in \mathfrak{M}^\downarrow$ if and only if:*

(i) $0 < p \leq 1$, $p \leq q < \infty$

$$C_1 = C_2 := \sup_{x\in(0,\infty)} \left(\int_0^x U^q(x,y)w(y)dy\right)^{\frac{1}{q}} V^{-\frac{1}{p}}(x) < \infty.$$

(ii) $0 < q < p \leq 1$,

$$C_1 \approx C_3 := \left(\sup_{\{x_k\}} \sum_{k\in\mathbb{Z}} \left(\int_{x_k}^{x_{k+1}} U^q(x_{k+1},y)w(y)dy\right)^{\frac{r}{q}} V^{-\frac{r}{p}}(x_{k+1})\right)^{1/r} < \infty.$$

(iii) $1 < p \leq q < \infty$, $1/p' := 1 - 1/p$. *Then $C_1 \approx C_2 + C_4$, where*

$$C_4 := \sup_{x\in(0,\infty)} W^{\frac{1}{q}}(x) \left(\int_x^\infty U^{p'}(y,x)V^{-p'}(y)v(y)dy\right)^{\frac{1}{p'}} < \infty,$$

(iv) $1 < q < p < \infty$,

$$C_5 := \left(\int_0^\infty \left(\int_0^x U^q(x,y)w(y)dy\right)^{\frac{r}{q}} V^{-\frac{r}{q}}(x)v(x)dx\right)^{\frac{1}{r}} < \infty$$

$$C_6 := \left(\int_0^\infty W^{\frac{r}{p}}(x)w(x) \left(\int_x^\infty U^{p'}(y,x)V^{-p'}(y)v(y)dy\right)^{\frac{r}{p'}} dx\right)^{\frac{1}{r}} < \infty,$$

and $C_1 \approx C_5 + C_6$.

(v) $q = 1 < p < \infty$, *then* $C_1 = C_7$, *where*

$$C_7 := \left(\int_0^\infty \left(\int_0^x W(y)u(y)dy\right)^{p'} V^{-p'}(x)v(x)dx\right)^{\frac{1}{p'}} < \infty.$$

(vi) $0 < q < 1 < p < \infty$, *then*

$$C_1 \approx C_3 + C_6 < \infty.$$

Proof. The part (i) follows by [18] and part (ii) by Theorem 4.1. Applying Theorem 2.1 we reduce (4.1) to the inequality for the integral operator with Oinarov's kernel. Then parts (iii) and (iv) follow by using the dual version of the results of [19] or [28] and assertion of (v) is a corollary of a well-nown result ([14], Chapter XI, § 1.5, Theorem 4). Thus, we need to prove only (vi). Applying Theorem 4.1 and dual version of ([17], Theorem 5), we get

$$C_1 \approx B_1 + B_2,$$

where

$$\begin{aligned} B_1^r &:= \sup_{\{x_k\}} \sum_{k\in\mathbb{Z}} \left(\int_{x_{k-1}}^{x_k} U^q(x_k, y)w(y)dy\right)^{\frac{r}{q}} \left(\int_{x_k}^{x_{k+1}} V^{-p'}(y)v(y)dy\right)^{\frac{r}{p'}}, \\ B_2^r &:= \sup_{\{x_k\}} \sum_{k\in\mathbb{Z}} \left(\int_{x_{k-1}}^{x_k} w(y)dy\right)^{\frac{r}{q}} \left(\int_{x_k}^{x_{k+1}} U^{p'}(y, x_k)V^{-p'}(y)v(y)dy\right)^{\frac{r}{p'}}. \end{aligned}$$

It is clear, that

$$B_1 \le C_3.$$

Now,

$$\begin{aligned} B_2^r &\ll \sup_{\{x_k\}} \sum_{k\in\mathbb{Z}} \left(\int_{x_{k-1}}^{x_k} W^{\frac{r}{p}}(y)w(y)dy\right) \left(\int_{x_k}^{\infty} U^{p'}(y, x_k)V^{-p'}(y)v(y)dy\right)^{\frac{r}{p'}} \\ &\ll \sup_{\{x_k\}} \sum_{k\in\mathbb{Z}} \int_{x_{k-1}}^{x_k} W^{\frac{r}{p}}(y)w(y) \left(\int_y^\infty U^{p'}(z, y)V^{-p'}(z)v(z)dz\right)^{\frac{r}{p'}} dy \\ &\ll \int_0^\infty W^{\frac{r}{p}}(y)w(y) \left(\int_y^\infty U^{p'}(z, y)V^{-p'}(z)v(z)dz\right)^{\frac{r}{p'}} dy \\ &= C_6^r. \end{aligned}$$

Therefore,

$$B_1 + B_2 \ll C_3 + C_6.$$

Now, let $\{x_k\} \subset (0,\infty)$ be a covering sequence. Denote $i_k = \sup\{i \in \mathbb{Z} : V(x_i) \le 2^k\}$ and $I_k := (i_{k-1}, i_k], k \in \mathbb{Z}$. Then, applying Jensen's inequality, we find

$$\begin{aligned}
& \sum_{k\in\mathbb{Z}} \left(\int_{x_{k-1}}^{x_k} U^q(x_k,y)w(y)dy \right)^{\frac{r}{q}} V^{-\frac{r}{p}}(x_k) \\
= & \sum_{k\in\mathbb{Z}} \sum_{i\in I_k} \left(\int_{x_{i-1}}^{x_i} U^q(x_i,y)w(y)dy \right)^{\frac{r}{q}} V^{-\frac{r}{p}}(x_i) \\
\ll & \sum_{k\in\mathbb{Z}} 2^{-\frac{kr}{p}} \sum_{i\in I_k} \left(\int_{x_{i-1}}^{x_i} U^q(x_{i_k},y)w(y)dy \right)^{\frac{r}{q}} \\
\le & \sum_{k\in\mathbb{Z}} 2^{-\frac{kr}{p}} \left(\int_{x_{i_{k-2}}}^{x_{i_k}} U^q(x_{i_k},y)w(y)dy \right)^{\frac{r}{q}} \\
\ll & \sum_{k\in\mathbb{Z}} 2^{-\frac{2kr}{p}} \left(\int_{x_{i_{2k-2}}}^{x_{i_{2k}}} U^q(x_{i_{2k}},y)w(y)dy \right)^{\frac{r}{q}} \\
+ & \sum_{k\in\mathbb{Z}} 2^{-\frac{(2k+1)r}{p}} \left(\int_{x_{i_{2k-1}}}^{x_{i_{2k+1}}} U^q(x_{i_{2k+1}},y)w(y)dy \right)^{\frac{r}{q}} \\
\ll & \sum_{k\in\mathbb{Z}} \left(\int_{x_{i_{2k-2}}}^{x_{i_{2k}}} U^q(x_{i_{2k}},y)w(y)dy \right)^{\frac{r}{q}} \left(\int_{x_{i_{2k}}}^{x_{i_{2k+2}}} V^{-p'}(y)v(y)dy \right)^{\frac{r}{p'}} \\
+ & \sum_{k\in\mathbb{Z}} \left(\int_{x_{i_{2k-1}}}^{x_{i_{2k+1}}} U^q(x_{i_{2k+1}},y)w(y)dy \right)^{\frac{r}{q}} \left(\int_{x_{i_{2k+1}}}^{x_{i_{2k+3}}} V^{-p'}(y)v(y)dy \right)^{\frac{r}{p'}} \\
\ll & B_1^r.
\end{aligned}$$

Hence,

$$C_3 \ll B_1.$$

Now, let $\{x_k\} \subset (0,\infty)$ be such a sequence that $2^k = \int_0^{x_k} w$. We have

$$\begin{aligned}
C_6^r &= \sum_{k\in\mathbb{Z}} \int_{x_k}^{x_{k+1}} W^{\frac{r}{p}}(y)w(y) \left(\int_y^\infty U^{p'}(z,y)V^{-p'}(z)v(z)dz \right)^{\frac{r}{p'}} dy \\
&\ll \sum_{k\in\mathbb{Z}} 2^{\frac{kr}{q}} \left(\int_{x_k}^\infty U^{p'}(z,x_k)V^{-p'}(z)v(z)dz \right)^{\frac{r}{p'}} \\
&= \sum_{k\in\mathbb{Z}} 2^{\frac{kr}{q}} \left(\sum_{i=k}^\infty \int_{x_i}^{x_{i+1}} U^{p'}(z,x_k)V^{-p'}(z)v(z)dz \right)^{\frac{r}{p'}} \\
&\approx \sum_{k\in\mathbb{Z}} 2^{\frac{kr}{q}} \left(\sum_{i=k}^\infty \int_{x_i}^{x_{i+1}} U^{p'}(z,x_i)V^{-p'}(z)v(z)dz \right)^{\frac{r}{p'}}
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{k\in\mathbb{Z}} 2^{\frac{kr}{q}}\left(\sum_{i=k}^{\infty} U^{p'}(x_i,x_k)\int_{x_i}^{x_{i+1}} V^{-p'}(z)v(z)dz\right)^{\frac{r}{p'}}\\
&\approx \sum_{k\in\mathbb{Z}} 2^{\frac{kr}{q}}\left(\int_{x_k}^{x_{k+1}} U^{p'}(z,x_k)V^{-p'}(z)v(z)dz\right)^{\frac{r}{p'}}\\
&+\sum_{k\in\mathbb{Z}} 2^{\frac{kr}{q}}\left(\sum_{i=k}^{\infty}\left(\sum_{j=k}^{i-1} U(x_{j+1},x_j)\right)^{p'}\int_{x_i}^{x_{i+1}} V^{-p'}(z)v(z)dz\right)^{\frac{r}{p'}}\\
&= I + II.
\end{aligned}
$$

Then

$$
I \approx \sum_{k\in\mathbb{Z}}\left(\int_{x_{k-1}}^{x_k} w\right)^{\frac{r}{q}}\left(\int_{x_k}^{x_{k+1}} U^{p'}(z,x_k)V^{-p'}(z)v(z)dz\right)^{\frac{r}{p'}} \le B_2^r.
$$

Using Minkowski inequality, we find

$$
\begin{aligned}
&\left(\sum_{i=k}^{\infty}\left(\sum_{j=k}^{i-1} U(x_{j+1},x_j)\right)^{p'}\int_{x_i}^{x_{i+1}} V^{-p'}(z)v(z)dz\right)^{\frac{1}{p'}}\\
&\le \sum_{j=k}^{\infty} U(x_{j+1},x_j)\left(\sum_{i=j+1}^{\infty}\int_{x_i}^{x_{i+1}} V^{-p'}(z)v(z)dz\right)^{\frac{1}{p'}}\\
&= \sum_{j=k}^{\infty} U(x_{j+1},x_j)\left(\int_{x_{i+1}}^{\infty} V^{-p'}(z)v(z)dz\right)^{\frac{1}{p'}}\\
&\le \sum_{j=k}^{\infty} U(x_{j+1},x_j)V^{-\frac{1}{p}}(x_{j+1}).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
II &\ll \sum_{k\in\mathbb{Z}} 2^{\frac{kr}{q}}\left(\sum_{j=k}^{\infty} U(x_{j+1},x_j)V^{-\frac{1}{p}}(x_{j+1})\right)^{r}\\
&\approx \sum_{k\in\mathbb{Z}}\left(\int_{x_{k-1}}^{x_k} w\right)^{\frac{r}{q}} U^{r}(x_{k+1},x_k)V^{-\frac{r}{p}}(x_{k+1})\\
&\ll \sum_{k\in\mathbb{Z}}\left(\int_{x_{k-1}}^{x_k} U^{q}(x_{k+1},y)w(y)dy\right)^{\frac{r}{q}} V^{-\frac{r}{p}}(x_{k+1})\\
&\ll \sum_{k\in\mathbb{Z}}\left(\int_{x_{k-1}}^{x_{k+1}} U^{q}(x_{k+1},y)w(y)dy\right)^{\frac{r}{q}} V^{-\frac{r}{p}}(x_{k+1})\\
&= \sum_{k\in\mathbb{Z}}\left(\int_{x_{2k-1}}^{x_{2k+1}} U^{q}(x_{2k+1},y)w(y)dy\right)^{\frac{r}{q}} V^{-\frac{r}{p}}(x_{2k+1})
\end{aligned}
$$

$$+\sum_{k\in\mathbb{Z}}\left(\int_{x_{2k}}^{x_{2k+2}} U^q(x_{2k+2},y)w(y)dy\right)^{\frac{r}{q}} V^{-\frac{r}{p}}(x_{2k+2})$$
$$\leq 2C_3^r \ll B_1^r.$$

Thus,

$$C_6 \ll B_1 + B_2$$

and finally

$$C_3 + C_6 \ll B_1 + B_2.$$

□

Remark 5.2. Theorem 5.1 corrects the results of M.L.Goldman ([11], Theorem 1.1.) in the cases (i), (ii) and (v).

For the case $q = \infty$ we have the following.

Theorem 5.3. *Let* $0 < p < \infty$ *Then the inequality*

$$\operatorname*{ess\,sup}_{x\in(0,\infty)}\left(\int_x^\infty fu\right) w(x) \leq C_8\left(\int_0^\infty f^p v\right)^{\frac{1}{p}}, \tag{5.1}$$

holds for every $f \in \mathfrak{M}^\downarrow$ *if and only if*

(i) $0 < p \leq 1$,

$$C_9 := \sup_{x\in(0,\infty)}\left(\operatorname*{ess\,sup}_{y\in(0,x)} U(x,y)w(y)\right) V^{-\frac{1}{p}}(x) < \infty.$$

Moreover, $C_8 \approx C_9$.

(ii) $1 < p < \infty$. *Then*

$$C_8 = \operatorname*{ess\,sup}_{x\in(0,\infty)} w(x)\left(\int_x^\infty U^{p'}(y,x)V^{-p'}(y)v(y)dy\right)^{\frac{1}{p'}} < \infty.$$

Remark 5.4. Analogously Theorems 5.1 and 5.3 the characterizations take place for the inequality (4.14) and (5.1) on the cone $\mathfrak{M}^\uparrow$. In particular, these results supplements ([12], Theorem 2.2 (ii)). We omit details.

Similarly, for the case $p = \infty$ we obtain the following.

Theorem 5.5. *Let* $\|\cdot\|_X$ *be any quasinorm defined on* $\mathfrak{M}^+$ *and let* $T : \mathfrak{M}^+ \to \mathfrak{M}^+$ *be a positive operator. Then the inequality*

$$\|T(f)\|_X \leq C_{10}\|fv\|_\infty$$

holds for every $f \in \mathfrak{M}^\downarrow$ *if and only if*

$$C_{11} := \left\|T\left(\frac{1}{\operatorname{ess\,sup}_{y\in(0,x)} v(y)}\right)\right\|_X < \infty$$

and $C_{10} = C_{11}$.

Corollary 5.6. *Let $\|\cdot\|_X$ be any quasinorm defined on $\mathfrak{M}^+$. Then the inequality*

$$\left\|\int_x^\infty f\right\|_X \le C_{12}\|fv\|_\infty$$

holds for every $f \in \mathfrak{M}^\downarrow$ if and only if

$$C_{13} := \left\|\int_x^\infty \frac{dy}{\operatorname{ess\,sup}_{z\in(0,y)} v(z)}\right\|_X < \infty$$

and $C_{12} = C_{13}$.

Now we collect the complete characterization of (4.15).

Theorem 5.7. *Let $0 < q, p < \infty$. Let $k(x,y) \ge 0$ be a measurable kernel. Then the inequality (4.15) with the best constant C_1 holds for every $f \in \mathfrak{M}^\downarrow$ if and only if:*
(i) $0 < p \le 1$, $p \le q < \infty$

$$C_1 = C_{15} := \sup_{x\in(0,\infty)} \left(\int_0^\infty K^q(x, \min(x,y))w(y)dy\right)^{\frac{1}{q}} V^{-\frac{1}{p}}(x) < \infty.$$

(ii) $q = 1 < p < \infty$, *then* $C_1 \approx C_{16}$, *where*

$$C_{16} := \left(\int_0^\infty \left(\int_x^\infty \left(\int_y^\infty k(z,y)w(z)dz\right) V^{-1}(y)u(y)dy\right)^{p'} v(x)dx\right)^{\frac{1}{p'}} < \infty.$$

If $k(x,y)$ is an Oinarov's kernel, then
(iii) $0 < q < p \le 1$ *and $k(x,y)$ is continuous,*

$$C_1 \approx C_{17} := \left(\sup_{\{x_k\}} \sum_{k\in\mathbb{Z}} \left(\int_{x_k}^{x_{k+1}} K^q(y, x_k)w(y)dy\right)^{\frac{r}{q}} V^{-\frac{r}{p}}(x_k)\right)^{1/r} < \infty.$$

(iv) $1 < p \le q < \infty$, $1/p' := 1 - 1/p$. *Then* $C_1 \approx C_{18} + C_{19} + C_{20}$, *where*

$$C_{18} := \sup_{x\in(0,\infty)} \left(\int_0^x K^q(y,y))w(y)dy\right)^{\frac{1}{q}} V^{-\frac{1}{p}}(x) < \infty,$$

$$C_{19} := \sup_{x\in(0,\infty)} W_*^{\frac{1}{q}}(x) \left(\int_0^x K^{p'}(x,y)V^{-p'}(y)v(y)dy\right)^{\frac{1}{p'}} < \infty,$$

$$C_{20} := \sup_{x\in(0,\infty)} \left(\int_x^\infty k^q(y,x)w(y)dy\right)^{\frac{1}{q}} \left(\int_0^x U^{p'}(y)V^{-p'}(y)v(y)dy\right)^{\frac{1}{p'}} < \infty.$$

(v) $1 < q < p < \infty$,

$$C_{21} := \left(\int_0^\infty \left(\int_0^x K^q(y,y) w(y) dy \right)^{\frac{r}{p}} K^q(x,x) w(x) V^{-\frac{r}{p}}(x) dx \right)^{\frac{1}{r}} < \infty,$$

$$C_{22} := \left(\int_0^\infty \left(\int_x^\infty k^q(y,x) w(y) dy \right)^{\frac{r}{q}} \times \right.$$

$$\left. \times \left(\int_0^x U^{p'}(y) V^{-p'}(y) v(y) dy \right)^{\frac{r}{p'}} U^{p'}(x) V^{-p'}(x) v(x) dx \right)^{\frac{1}{r}} < \infty,$$

$$C_{23} := \left(\int_0^\infty W_*^{\frac{r}{p}}(x) w(x) \left(\int_0^x K^{p'}(x,y) V^{-p'}(y) v(y) dy \right)^{\frac{r}{p'}} dx \right)^{\frac{1}{r}} < \infty,$$

and $C_1 \approx C_{21} + C_{22} + C_{23}$.

(vi) $0 < q < 1 < p < \infty$, *then* $C_1 \approx C_{21} + C_{24} + C_{25} < \infty$, *where*

$$C_{24} := \left(\sup_{\{x_k\}} \sum_{k \in \mathbb{Z}} \left(\int_{x_k}^{x_{k+1}} w \right)^{\frac{r}{q}} \left(\int_{x_{k-1}}^{x_k} K^{p'}(x_k, y) V^{-p'}(y) v(y) dy \right)^{\frac{r}{p'}} \right)^{1/r} < \infty,$$

$$C_{25} := \left(\sup_{\{x_k\}} \sum_{k \in \mathbb{Z}} \left(\int_{x_k}^{x_{k+1}} k^q(y, x_k)^q w(y) dy \right)^{\frac{r}{q}} \left(\int_{x_{k-1}}^{x_k} U^{p'}(y) V^{-p'}(y) v(y) dy \right)^{\frac{r}{p'}} \right)^{1/r} < \infty.$$

Proof. Part (i) and (ii) follow from [18] and ([24], Theorem 2.2), respectively, and (iii) is Theorem 4.5. Parts (iv) and (v) were proved in ([20], Theorem 7) and (vi) follows by applying Theorem 2.1 and ([17], Theorem 5). □

The border case $q = \infty$ of the previous theorem is governed by the following.

Theorem 5.8. *Let* $0 < p < \infty$. *Let* $k(x,y) \geq 0$ *be a measurable kernel. Then the inequality*

$$\operatorname*{ess\,sup}_{x \in (0,\infty)} \left(\int_0^x k(x,y) f(y) u(y) dy \right) w(x) \leq C_{26} \left(\int_0^\infty f^p v \right)^{\frac{1}{p}}, \tag{5.2}$$

holds for every $f \in \mathfrak{M}^{\downarrow}$ *if and only if*

(i) $0 < p \leq 1$,

$$C_{26} := \sup_{x \in (0,\infty)} \left(\operatorname*{ess\,sup}_{y \in (0,\infty)} K(x, \min(x,y)) w(y) \right) V^{-\frac{1}{p}}(x) < \infty.$$

(ii) $1 < p < \infty$. *Then*

$$C_{26} := \operatorname*{ess\,sup}_{s \in (0,\infty)} w(s) \left(\int_0^s \left(\int_t^s k(s,y) u(y) V^{-1}(y) dy \right)^{p'} v(t) dt \right)^{\frac{1}{p'}} < \infty.$$

Proof. It follows by applying ([24], Theorem 2.2). □

For the case $p = \infty$, from Theorem 5.5 we obtain the following.

Corollary 5.9. *Let $\|\cdot\|_X$ be any quasinorm defined on $\mathfrak{M}^+$. Let $k(x,y) \geq 0$ be a measurable function on $\{(x,y) : x \geq y \geq 0\}$, Then the inequality*

$$\left\| \int_0^x k(x,y) f(y) dy \right\|_X \leq C_{29} \|fv\|_\infty$$

holds for every $f \in \mathfrak{M}^\downarrow$ if and only if

$$C_{30} := \left\| \int_0^x \frac{k(x,y)dy}{\operatorname{ess\,sup}_{z \in (0,y)} v(z)} \right\|_X < \infty$$

and $C_{29} = C_{30}$.

References

[1] Ariño M. and Muckenhoupt B. Maximal funcrions on classical Lorentz spaces and Hardy's inequality with weights for non-increasing functions. Trans. Amer. Math. Soc. 320 (1990), 727-735.

[2] Bennett G. and Grosse-Erdmann K.-G. Weighted Hardy inequality for decreasing sequences and functions. Math. Ann. 334 (2006), 489–531.

[3] Burenkov V.I. and Goldman M. L. Calculation of the norm of a positive operator on the cone of monotone functions. Proc. Steklov Inst. Math. 210 (1995), 47–65.

[4] Carro M., Gogatishvili A., Martin J. and Pick L. Weighted inequalities involving two Hardy operators with applications to embeddings of function spaces, J. Operator Theory. 59 (2008), 309-332.

[5] Carro M., Pick L., Soria J. and Stepanov V.D. On embeddings between clasical Lorentz spaces, Math. Inequal. Appl. 4 (2001), 397-428.

[6] Carro M. and Soria J. Boundedness of some inegral operators, Canad. J. Math. 45 (1993), 1155-1166.

[7] Gogatishvili A., Johansson M., Okpoti C.A. and Persson L.E. Characterisation of embeddings in Lorentz spaces. Bull. Austral. Math. Soc. 76 (2007), 69-92.

[8] Gogatishvili A., Opic B. and Pick L., Weighted inequalities for Hardy-type operators involving suprema. Collect. Math. 57 (2006), 227–255.

[9] Gogatishvili A. and Pick L. A reduction theorem for supremum operators. J. Comp. Appl. Math. 208 (2007), 270–279.

[10] Gogatishvili A. and Pick L. Duality principles and reduction theorems. Math. Ineq. Appl. 3 (2000), 539-558.

[11] Goldman M. L. Sharp estimates for the norms of Hardy-type operators on cones of quasimonotone functions. Proc. Steklov Inst. Math. 232 (2001), 109–137.

[12] Heinig H.P. and Stepanov V. D. Weighted Hardy inequalities for increasing functions. Canad. J. Math. 45 (1993), 104–116.

[13] Johansson M., Stepanov V. D. and Ushakova E.P. Hardy inequality with three measures on monotone functions. Math. Inequal. Appl. 11 (2008), 393–413.

[14] Kantorovich L. V. and Akilov G. P. Functional Analysis, Pergamon, Oxford, 1982.

[15] Kufner A., Maligranda L. and Persson L.-E. The Hardy inequality. About its history and some related results, Vydavatelsky Servis, Pilsen, 2007.

[16] Kufner A. and Persson L.-E. Weighted inequalities of Hardy type, World Scientific, New Jersey, 2003.

[17] Lai Q. Weighted modular inequalities for Hardy-type operators. Proc. London Math. Soc. 79 (1999), 649–672.

[18] Myasnikov E. A., Persson L.-E. and Stepanov V. D. On the best constants in certain integral inequalities for monotone functions. Acta Sci. Math. (Szeged). 59 (1994), 613–624.

[19] Oinarov R. Two-sided estimates of the norm of some classes of integral operators. Proc. Steklov Inst. Math. 204 (3) (1994), 205–214.

[20] Popova O.V. Inequalities of Hardy type on the cones of monotone functions. Siberian Math. J. 53, N 1 (2012), to appear.

[21] Sawyer E. Boundedness of classical operators on classical Lorentz spaces. Studia Math. 96 (1990), 145–158.

[22] Sinnamon G. and Stepanov V. D. The weighted Hardy inequality: new proofs and the case p=1. J. London Math. Soc. 54 (1996), 89–101.

[23] Sinnamon G. Transferring monotonicity in weighted norm inequalities, Collect. Math. 54 (2003), 181–216.

[24] Sinnamon G. Hardy's inequality and monotonocity. In: Function Spaces and Nonlinear Analysis (Eds.: P. Drábec and J. Rákosnik), Mathematical Institute of the Academy of Sciences of the Czech Republic, Prague, 2005, 292–310.

[25] Stepanov V. D. The weighted Hardy's inequality for nonincreasing functions. Trans. Amer. Math. Soc. 338 (1993), 173–186.

[26] Stepanov V. D. Integral operators on the cone of monotone functions. J. London Math. Soc. 48 (1993), 465–487.

[27] Stepanov V.D. On a supremum operator. Spectral Theory, Function Spaces and Inequalities. New Technique and Recent Trends, Operator Theory: Advances and Applications. Birkhäuser. Basel. 2012, 233-242.

[28] Stepanov V. D. Weighted norm inequalities of Hardy type for a class of integral operators. J. London Math. Soc. 50 (1994), 105–120.

Insitute of Mathematics of the Academy of Sciences of the Czech Republic, Žitna 25, 11567 Praha 1, Czech Republic

E-mail address: gogatish@math.caz.cz

Peoples Friendship University, Miklucho Maklai 6, 117198 Moscow, Russia

E-mail address: vstepanov@sci.pfu.edu.ru