Mathematics of complete fluid systems

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Everything should be made as simple as possible, but not one bit simpler.

A. Einstein

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1 Introduction

This lecture series is a brief introduction to the mathematics of complete fluid systems. By *complete* we mean the systems that are energetically closed; the total energy of the fluid is a constant of motion. We stay on the platform of *classical continuum mechanics* - a fluid in motion will be described by

means of phenomenological quantities - *fields* - that are continous in space and time and obey the basic principles of *classical physics*.

We introduce a mathematical theory of complete fluid systems in the framework of *weak solutions*. The main advantage of this approach is the fact that the associated mathematical problems are solvable without any artificial (smallness and smoothness) restrictions imposed on the data and on arbitrary lapses of time, including infinite time intervals.

1.1 Balance laws

The time evolution of systems in continuum mechanics is determined by the physical principles expressed mathematically as *balance laws*. A balance law includes three basic quantities:

- a *density* d of a balanced quantity;

- a *flux vector field* $\mathbf{F}$;

- a *source* s;

where all quantities depend on the time t and the spatial variable $x \in \Omega \subset R^3$. Here, the symbol Ω denotes the *physical space* - a domain in the Euclidean space occupied by the fluid.

A general form of a balance law reads

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BALANCE LAW FOR d

[total amount of d in B at the time t_2] $-$ [total amount of d in B at the time t_1] $=$ (1.1)

$$\int_B d(t_2, x) \, \mathrm{d}x - \int_B d(t_1, x) \, \mathrm{d}x = - \int_{t_1}^{t_2} \int_{\partial B} \mathbf{F}(t, x) \cdot \mathbf{n} \, \mathrm{dS}_x \, \mathrm{d}t + \int_{t_1}^{t_2} \int_B s(t, x) \mathrm{d}x \, \mathrm{d}t$$

$=$ [total flux through the boundary ∂B during the time interval $[t_1, t_2]$] + [source]

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1.1.1 Balance law - differential form

The following argument is classical. Assuming that all quantities in (1.1) are smooth (differentiable) we may apply the Gauss-Green theorem to deduce that

$$\int_B d(t_2, x) \, \mathrm{d}x - \int_B d(t_1, x) \, \mathrm{d}x = - \int_{t_1}^{t_2} \int_B \mathrm{div}_x \mathbf{F}(t, x) \, \mathrm{d}x \, \mathrm{d}t + \int_{t_1}^{t_2} \int_B s(t, x) \mathrm{d}x \, \mathrm{d}t,$$

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furthermore, letting $t_2 \to t_1 \equiv t$,

$$\int_B \partial_t d(t,x) \, \mathrm{d}x = -\int_B \mathrm{div}_x \mathbf{F}(t,x) \, \mathrm{d}x + \int_B s(t,x)\mathrm{d}x,$$

yielding, finally,

BALANCE LAW FOR d - DIFFERENTIAL FORM

$$\partial_t d(t,x) + \mathrm{div}_x \mathbf{F}(t,x) = s(t,x) \tag{1.2}$$

1.1.2 Balance law - weak form

There are two ways how to view the so-called *weak formulation* of the balance law (1.1). First, we multiply equation (1.2) by a smooth (differentiable) *test function* $\varphi = \varphi(t,x)$, with a compact support. Integrating by parts, we obtain

BALANCE LAW FOR d - WEAK FORM

$$\int_0^T \int_\Omega \left[d(t,x)\partial_t\varphi(t,x) + \mathbf{F}(t,x) \cdot \nabla_x\varphi(t,x) + s(t,x)\varphi(t,x) \right] \mathrm{d}x \, \mathrm{d}t = 0 \tag{1.3}$$

for any $\varphi \in C_c^\infty((0,T) \times \Omega)$.

It seems a bit strange that the "weak" form was obtained from the "strong" differentiable form. Another possibility how to see (1.3) is to go back to (1.1) that can be viewed as

$$-\lim_{\varepsilon \to 0} \int_{t_1}^{t_2} \int_B d(t,x)\partial_t\varphi_\varepsilon(t,x) \, \mathrm{d}x \, \mathrm{d}t$$

$$\int_B d(t_2,x) \, \mathrm{d}x - \int_B d(t_1,x) \, \mathrm{d}x = -\int_{t_1}^{t_2} \int_{\partial B} \mathbf{F}(t,x) \cdot \mathbf{n} \, \mathrm{dS}_x \, \mathrm{d}t + \int_{t_1}^{t_2} \int_B s(t,x)\mathrm{d}x \, \mathrm{d}t$$

$$\lim_{\varepsilon \to 0} \int_0^T \int_\Omega \mathbf{F}(t,x) \cdot \nabla_x\varphi_\varepsilon(t,x) \, \mathrm{d}x \, \mathrm{d}t + \lim_{\varepsilon \to 0} \int_0^T \int_\Omega s(t,x)\varphi_\varepsilon(t,x) \, \mathrm{d}x \, \mathrm{d}t,$$

where $\varphi_\varepsilon \in C_c^\infty((t_1,t_2) \times B)$ is a suitable family of smooth functions,

$$0 \le \varphi_\varepsilon \le 1, \ \varphi_\varepsilon \nearrow 1 \text{ as } \varepsilon \to 0.$$

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1.2   Systems of balance laws - constitutive relations

Mathematical formulation of problems in continuum fluid mechanics consists of a system of balance laws for quantities $d_1, \ldots, d_N$, fluxes $\mathbf{F}_1, \ldots, \mathbf{F}_N$, and sources $s_1, \ldots, s_N$. Clearly, as we would always have more unknowns than equations, the system must be *closed* by determining the fluxes and sources in terms of the unknowns $d_1, \ldots, d_N$. These relations characterize *material properties* of a specific fluid and are termed constitutive relations.

2   Field equations of continuum fluid mechanics

The basic equations of continuum fluid mechanics correspond to the physical principles of balance of *mass*, *momentum*, and *energy*.

2.1   Fluid description - thermostatics

The problem of time in physics and chemistry is closely related to the formulation of the second law of thermodynamics.

I. Prigogine, Nobel Lecture, 8 December 1977

Consider, for a moment, a fluid at rest. We suppose that the *state* of the fluid is completely determined by two fields: the *mass density* ϱ and the temperature ϑ. The (internal) *energy* density $e = e(\varrho, \vartheta)$ and the *pressure* $p = p(\varrho, \vartheta)$ are uniquely determined by ϱ and ϑ. In accordance with *Second law of thrmodynamics*, there is another thermodynamic varible $s = s(\varrho, \vartheta)$ called (specific) *entropy*. The internal energy e, the pressure p, and the entropy s are interrelated through

GIBBS' EQUATION

$$\vartheta Ds(\varrho, \vartheta) = De(\vartheta, \vartheta) + p(\varrho, \vartheta)D\left(\frac{1}{\varrho}\right). \tag{2.1}$$

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Relation (2.1) should be viewed as the stipulation that

$$\frac{1}{\vartheta}\left[De(\vartheta,\vartheta) + p(\varrho,\vartheta)D\left(\frac{1}{\varrho}\right)\right] \text{ is a perfect differential.}$$

Note that the entropy s is determined modulo an additive constant. The latter can be fixed by means of Third law of thermodynamic that requires the entropy to approach zero for $\vartheta \to 0$.

2.2 Transport - bulk velocity

The observable (macroscopic) motion of the fluid is described by means of the *velocity field* $\mathbf{u} = \mathbf{u}(t,x)$. The streamlines are obtained by solving a system of ordinary differential equations:

$$\frac{d\mathbf{X}}{dt} = \mathbf{u}(t,\mathbf{X}(t,x)), \ \mathbf{X}(0) = \mathbf{x}_0.$$

The velocity describes *mass transport*, the quantity $\varrho\mathbf{u}$ is the mass flux, and the physical principle of mass conservation reads

EQUATION OF CONTINUITY:

$$\partial_t\varrho + \mathrm{div}_x(\varrho\mathbf{u}) = 0, \tag{2.2}$$

or, in the weak form,

$$\int_0^T \int_\Omega \left[\varrho\partial_t\varphi + \varrho\mathbf{u}\cdot\nabla_x\varphi\right]\,dx\,dt = 0 \tag{2.3}$$

for any test function $\varphi \in C_c^\infty((0,T)\times\Omega)$.

2.3 Balance of linear momentum

Similarly to the preceding part, we derive the balance of linear momentum in the (differential) form

$$\partial_t(\varrho\mathbf{u}) + \mathrm{div}_x(\varrho\mathbf{u}\otimes\mathbf{u}) = \mathrm{div}_x\mathbb{T} + \varrho\mathbf{f},$$

where $\mathbb{T}$ is the so-called *Cauchy stress*, and $\mathbf{f}$ is a (given) external force acting on the fluid.

Fluids are characterized among other materials by

Accordingly, the balance of momentum in an isotropic fluid can be written as

We have noticed that, unlike in the mass conservation, the momentum flux consists of two components: the *convective* one representes by the tensor $\varrho\mathbf{u} \otimes \mathbf{u}$ that corresponds, similarly to (2.2) to the mass transport, and the diffusive one $\mathbb{S} - p\mathbb{I}$. As we shall se below, the energy is transported in a similar way.

2.4 Balance of energy

Determining a suitable form of the balance of energy is one of the cornerstones of the theory. Our approach leans on a formulation based on *Second law of thermodynamics*, specifically, on the entropy production equation formulated below.

2.4.1 Kinetic energy balance

Taking the scalar product of the momentum balance equation (2.5) with $\mathbf{u}$ we obtain

$$\partial_t \left(\frac{1}{2}\varrho|\mathbf{u}|^2 \right) + \operatorname{div}_x \left(\frac{1}{2}\varrho|\mathbf{u}|^2\mathbf{u} \right) + \operatorname{div}_x\Big(p(\varrho, \vartheta)\mathbf{u}\Big) - \operatorname{div}_x\Big(\mathbb{S}\mathbf{u}\Big) = -\mathbb{S} : \nabla_x\mathbf{u} + p(\varrho, \vartheta)\operatorname{div}_x\mathbf{u} + \varrho\mathbf{f} \cdot \mathbf{u}. \tag{2.7}$$

Note that, in accordance with the equation of continuity (2.2), we have

$$\partial_t(\varrho\mathbf{u}) + \mathrm{div}_x(\varrho\mathbf{u}\otimes\mathbf{u}) = \big(\partial_t\varrho + \mathrm{div}_x(\varrho\mathbf{u})\big)\mathbf{u} + \varrho\partial_t\mathbf{u} + \varrho\mathbf{u}\cdot\nabla_x\mathbf{u}.$$

The quantity

$$\frac{1}{2}\varrho|\mathbf{u}|^2 \text{ is called } \textit{kinetic energy,}$$

whereas equation (2.7) represent the kinetic energy balance. Obviously, equation (2.7) contains a source term

$$-\mathbb{S} : \nabla_x\mathbf{u} + p(\varrho,\vartheta)\mathrm{div}_x\mathbf{u} + \varrho\mathbf{f}\cdot\mathbf{u}.$$

In many cases, the external force $\mathbf{f}$ is a gradient of a scalar potential, specifically,

$$\mathbf{f} = \nabla_x F, \ F = F(x).$$

Accordingly, making use of the continuity equation (2.2), we get

$$\varrho\mathbf{f}\cdot\mathbf{u} = \varrho\nabla_x F\cdot\mathbf{u} = \mathrm{div}_x\big(\varrho F\mathbf{u}\big) - F\mathrm{div}_x(\varrho\mathbf{u}) == \partial_t\big(\varrho F\big) + \mathrm{div}_x\big(\varrho F\mathbf{u}\big);$$

whence (2.7) reads

$$\partial_t\left(\frac{1}{2}\varrho|\mathbf{u}|^2 - \varrho F\right) + \mathrm{div}_x\left(\left(\frac{1}{2}\varrho|\mathbf{u}|^2 - \varrho F\right)\mathbf{u}\right) + \mathrm{div}_x\big(p(\varrho,\vartheta)\mathbf{u}\big) - \mathrm{div}_x\big(\mathbb{S}\mathbf{u}\big) \quad (2.8)$$
$$= -\mathbb{S} : \nabla_x\mathbf{u} + p(\varrho,\vartheta)\mathrm{div}_x\mathbf{u}.$$

The presence of the source term on the right-hand side of (2.8) indicates that the kinetic energy is not conserved, in particular, the *total energy balance* should also take into account the changes of the internal energy. Note, however, that if the fluid is inviscid (idealization), we would have

$$\mathbb{S} \equiv 0.$$

If, in addition, the fluid is barotropic, meaning the pressure $p = p(\varrho)$ depends solely on the density, we have, making once more use of the continuity equation,

$$p(\varrho)\mathrm{div}_x\mathbf{u} = -\partial_t\big(\varrho P(\varrho)\big) - \mathrm{div}_x\big(\varrho P(\varrho)\big),$$

where

$$P(\varrho) = \int_1^\varrho \frac{p(z)}{z^2} \ \mathrm{d}z.$$

In such a case, equation (2.8) would reduce to a *conservation law*

$$\partial_t\left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho P(\varrho) - \varrho F\right) + \mathrm{div}_x\left(\left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho P(\varrho) - \varrho F\right)\mathbf{u}\right) + \mathrm{div}_x\big(p(\varrho,\vartheta)\mathbf{u}\big) = 0 \quad (2.9)$$

for the energy

$$\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho P(\varrho) - \varrho F$$

of an inviscid barotropic fluid.

2.4.2 Internal energy balance

In accordance with *First law of thermodynamics*, the *total energy* of the system is conserved. Thus the internal energy balance must contain the source term appearing in (2.8) as a "sink" on the right-hand side. Specifically, the internal energy balance equation reads

$$\partial_t \left(\varrho e(\varrho, \vartheta) \right) + \mathrm{div}_x \left(\varrho e(\varrho, \vartheta) \mathbf{u} \right) + \mathrm{div}_x \mathbf{q} = \mathbb{S} : \nabla_x \mathbf{u} - p(\varrho, \vartheta) \mathrm{div}_x \mathbf{u}. \tag{2.10}$$

In addition to the convective transport term $\varrho e \mathbf{u}$, the internal energy can be transported without mass transfer by diffusion, represented by the extra internal energy flux $\mathbf{q}$.

Summing up (2.8), (2.10) we obtain the *total energy balance* equation

$$\partial_t \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) - \varrho F \right) + \mathrm{div}_x \left(\left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) - \varrho F \right) \mathbf{u} \right) \tag{2.11}$$

$$+ \mathrm{div}_x \left(p(\varrho, \vartheta) \mathbf{u} \right) - \mathrm{div}_x \left(\mathbb{S} \mathbf{u} \right) + \mathrm{div}_x \mathbf{q} = 0.$$

2.4.3 Entropy production

Using again the continuity equation (2.2), we can rewrite the internal energy equation (2.11) in the form

$$\varrho De(\varrho, \vartheta) \cdot \begin{bmatrix} \partial_t \varrho \\ \partial_t \vartheta \end{bmatrix} + \varrho \mathbf{u} \cdot De(\varrho, \vartheta) \cdot \begin{bmatrix} \nabla_x \varrho \\ \nabla_x \vartheta \end{bmatrix} + \mathrm{div}_x \mathbf{q} = \mathbb{S} : \nabla_x \mathbf{u} - p(\varrho, \vartheta) \mathrm{div}_x \mathbf{u}. \tag{2.12}$$

Since e, p are related to the entropy s by Gibbs'equation (2.1), we get

$$\varrho De(\varrho, \vartheta) \cdot \begin{bmatrix} \partial_t \varrho \\ \partial_t \vartheta \end{bmatrix} + \varrho \mathbf{u} \cdot De(\varrho, \vartheta) \cdot \begin{bmatrix} \nabla_x \varrho \\ \nabla_x \vartheta \end{bmatrix}$$

$$= \varrho \vartheta Ds(\varrho, \vartheta) \cdot \begin{bmatrix} \partial_t \varrho \\ \partial_t \vartheta \end{bmatrix} + \varrho \vartheta \mathbf{u} \cdot Ds(\varrho, \vartheta) \cdot \begin{bmatrix} \nabla_x \varrho \\ \nabla_x \vartheta \end{bmatrix} + \frac{p(\varrho, \vartheta)}{\varrho} \left(\partial_t \varrho + \mathbf{u} \cdot \nabla_x \varrho \right)$$

$$= \varrho \vartheta Ds(\varrho, \vartheta) \cdot \begin{bmatrix} \partial_t \varrho \\ \partial_t \vartheta \end{bmatrix} + \varrho \vartheta \mathbf{u} \cdot Ds(\varrho, \vartheta) \cdot \begin{bmatrix} \nabla_x \varrho \\ \nabla_x \vartheta \end{bmatrix} - p(\varrho, \vartheta) \mathrm{div}_x \mathbf{u};$$

whence (2.12) reads as

$$\varrho \vartheta Ds(\varrho, \vartheta) \cdot \begin{bmatrix} \partial_t \varrho \\ \partial_t \vartheta \end{bmatrix} + \varrho \vartheta \mathbf{u} \cdot Ds(\varrho, \vartheta) \cdot \begin{bmatrix} \nabla_x \varrho \\ \nabla_x \vartheta \end{bmatrix} + \mathrm{div}_x \mathbf{q} = \mathbb{S} : \nabla_x \mathbf{u}.$$

In other words, we have deduced

$$\partial_t(\varrho s(\varrho, \vartheta)) + \mathrm{div}_x(\varrho s(\varrho, \vartheta)\mathbf{u}) + \mathrm{div}_x\left(\frac{\mathbf{q}}{\vartheta}\right) = \sigma \tag{2.13}$$

or, in the weak form,

$$\int_0^T \int_\Omega \left[\varrho s(\varrho, \vartheta)\partial_t\varphi + \varrho s(\varrho, \vartheta)\mathbf{u} \cdot \nabla_x\varphi + \frac{\mathbf{q}}{\vartheta} \cdot \nabla_x\varphi\right] \, \mathrm{d}x \, \mathrm{d}t = -\int_0^T \int_\Omega \sigma\varphi \, \mathrm{d}x \, \mathrm{d}t \tag{2.14}$$

for any $\varphi \in C_c^\infty((0, T) \times \Omega)$, with the *entropy production rate*

$$\sigma = \frac{1}{\vartheta}\left(\mathbb{S} : \nabla_x\mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x\vartheta}{\vartheta}\right). \tag{2.15}$$

Although the internal energy balance (2.10), the total energy conservation (2.11), and the entropy production equation (2.13) are, in fact, *equivalent* in the framework of regular solutions, it may not be the case for the weak solutions. As we shall see, it is the entropy that provides the most relevant piece of information in the weak formulation.

2.5 Constitutive equations for the viscous stress and the internal energy flux

In accordance with *Second law of thermodynamics*, the entropy production rate σ in (2.15) must be a non-negative quantity for any admissible physical process. This fact imposes certain restrictions on the specific form of $\mathbb{S}$ and $\mathbf{q}$. Here, we suppose that $\mathbb{S}$ is a linear function of the velocity gradient $\nabla_x\mathbf{u}$ while ॥ is a linear function of $\nabla_x\vartheta$. Specifically, we impose

$$\mathbb{S} = \mu\left(\nabla_x\mathbf{u} + \nabla_x^t\mathbf{u} - \frac{2}{3}\mathrm{div}_x\mathbf{u}\mathbb{I}\right) + \eta\mathrm{div}_x\mathbf{u}\mathbb{I}, \tag{2.16}$$

together with

Note that the entropy production rate σ takes the form

$$\sigma = \frac{1}{\vartheta} \left(\frac{\mu}{2} \left| \nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \right|^2 + \eta |\mathrm{div}_x \mathbf{u}|^2 + \frac{\kappa(\vartheta)|\nabla_x \vartheta|^2}{\vartheta} \right); \tag{2.18}$$

whence, in particular, the *shear viscosity coefficient* μ, the *bulk viscosity coefficient* η as well as the *heat conductivity coefficient* κ must be non-negative. In this lecture, we always assume *strict* positivity of μ and κ.

2.6   Boundary conditions

In this series of lectures, we always assume that the physical space containing the fluid is a *bounded* domain $\Omega \subset R^3$. Accordingly, some kind of boundary behavior of certain fields must be specified. Our principal assumption is that the total energy of the fluid is conserved. A short inspection of equation (2.11) reveals that a suitable condition could be *impermeability of the boundary*:

$$\mathbf{u} \cdot \mathbf{n}|_{\partial \Omega} = 0, \tag{2.19}$$

together with

THERMAL INSULATION:

$$\mathbf{q} \cdot \mathbf{n}|_{\partial \Omega} = 0. \tag{2.20}$$

In the presence of *viscosity*, the impermeability hypothesis (2.19) is not sufficient to determine uniquely the behavior of the fluid. Alternatively, we may take the most common

NO-SLIP BOUNDARY CONDITION:

$$\mathbf{u}|_{\partial\Omega} = 0, \tag{2.21}$$

or

COMPLETE SLIP BOUNDARY CONDITION:

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ [\mathbb{S}\mathbf{n}] \times \mathbf{n}|_{\partial\Omega} = 0. \tag{2.22}$$

3 Weak formulation

The system of equations (2.2), (2.5), (2.13), with the constitutive relations for $\mathbb{S}$ and $\mathbf{q}$ given by (2.16), (2.17), will be called *Navier-Stokes-Fourier system*. Our goal in the present section is to introduce a suitable *weak* formulation of the problem that takes into account the satisfaction of the boundary conditions (2.20 - 2.22). In order to have, at least formally, a well-posed problem, we introduce the *initial conditions* that characterize the original state of the fluid system at the time $t = 0$:

$$\varrho(0, \cdot) = \varrho_0, \ \varrho\mathbf{u}(0, \cdot) = \varrho_0\mathbf{u}_0, \ \varrho s(\varrho, \vartheta) = \varrho_0 s(\varrho_0, \vartheta_0). \tag{3.1}$$

In order to comply with the basic physical principles, we suppose that

$$\varrho_0 > 0, \ \vartheta_0 > 0 \text{ in } \Omega. \tag{3.2}$$

3.1 Weak continuity

We solutions are usually quantities that are only (locally) integrable with respect to the space and time variables, satisfying some differential equations in the sense of generalized derivatives. Consider, for instance, the weak formulation of a general conservation law (1.3). Here, it is enough that d, $\mathbf{F}$, and s be integrable functions in the space-time cylinder $(0, T) \times \Omega$,

$$d \in L^1((0, T) \times \Omega), \ \mathbf{F} \in L^1((0, T) \times \Omega; R^3), \ s \in L^1((0, T) \times \Omega).$$

Take the quantity $\varphi(t,x) = \psi(t)\phi(x)$ as a test function in (1.3) to obtain

$$\int_0^T \left(\int_\Omega d(t,\cdot)\phi \ \mathrm{d}x \right) \partial_t \psi \ \mathrm{d}t = - \int_0^T \left(\int_\Omega (\mathbf{F} \cdot \nabla_x \phi - s\phi) \ \mathrm{d}x \right) \psi \ \mathrm{d}t. \tag{3.3}$$

As relation (3.3) holds for any smooth $\psi = \psi(t)$, we deduce that the function of time:

$$t \mapsto \int_\Omega d(t,\cdot)\phi \ \mathrm{d}x \text{ is absolutely continuous}$$

for any $\phi \in C_c^\infty(\Omega)$. In other words, we are able to identify the *instantaneous values* of the averages $\int_\Omega d\phi \ \mathrm{d}x$ provided ϕ is differentiable and compactly supported.

In addition, suppose that d enjoys some extra integrability, say

$$\operatorname{ess\,sup}_{t\in(0,T)} \|d(t,\cdot)\|_{L^p(\Omega)} \leq c \text{ for a certain } p > 1,$$

in other words,

$$d \in L^\infty(0,T;L^p(\Omega)).$$

Since the smooth functions $\phi \in C_c^\infty(\Omega)$ are dense in the dual space $L^{p'}(\Omega)$, $1/p + 1/p' = 1$, we get

$$t \mapsto \int_\Omega d(t,\cdot)\phi \ \mathrm{d}x \text{ is absolutely continuous for any } \phi \in L^{p'}(\Omega) \approx [L^p(\Omega)]^*,$$

in other words, the density d, viewed as a function of the time t with values in the Banach space $L^p(\Omega)$, is continuous with respect to the weak topology in $L^p(\Omega)$:

$$d \in C_{\text{weak}}([0,T];L^p(\Omega)).$$

3.2 Equation of continuity - weak formulation

Suppose that $\varrho \geq u$ and that

$$\varrho \in L^\infty(0,T;L^\gamma(\Omega)), \quad \varrho\mathbf{u} \in L^\infty(0,T;L^\beta(\Omega;R^3)) \text{ for certain } \beta, \gamma > 1.$$

We say that ϱ, $\varrho\mathbf{u}$ is a weak solution of equation (2.2), supplemented with the boundary condition (2.19) and the initial condition (3.1), if the integral identity

$$\int_\Omega \left[\varrho(\tau,\cdot)\varphi(\tau,\cdot) - \varrho_0\varphi(0,\cdot) \right] \mathrm{d}x = \int_0^\tau \int_\Omega \left(\varrho\partial_t\varphi + \varrho\mathbf{u} \cdot \nabla_x\varphi \right) \mathrm{d}x \ \mathrm{d}t \tag{3.4}$$

is satisfied for any $\tau \in [0,T]$ and any test function $\varrho \in C_c^\infty([0,T] \times \overline{\Omega})$.

It is worth-noting that (3.4) holds in the whole physical space $(0,T) \times R^3$ provided ϱ was extended to be zero outside Ω.

3.3 Momentum equation - weak formulation

In order to formulate a weak analogue of the momentum equation, we need the stress tensor to be al least integrable. As the latter is given by Newton's law (2.16), we need the spatial gradient of $\mathbf{u}$ to be at least integrable. Accordingly, we suppose that

$$\mathbf{u} \in L^q(0, T; W^{1,q}(\Omega; R^3)) \text{ for a certain } q > 1.$$

As the Sobolev functions in $W^{1,q}$ possess well-defined *traces*, the no-slip boundary condition (2.21) may be incorporated in the function space, requiring

$$\mathbf{u} \in L^q(0, T; W_0^{1,q}(\Omega; R^3)) \text{ for a certain } q > 1. \tag{3.5}$$

Similarly to the preceding section, we say that ϱ, $\mathbf{u}$, ϑ is a weak solution of the momentum equation (2.5), (2.16), with the boundary conditions (2.21) and the initial condition (3.1) if the velocity field belongs to the class specified in (3.5), $\mathbb{S}$ is given by (2.16), and if the integral identity

$$\int_\Omega \left[\varrho \mathbf{u}(\tau, \cdot) \cdot \varphi(\tau, \cdot) - \varrho_0 \mathbf{u}_0 \varphi(0, \cdot) \right] \mathrm{d}x = \int_0^\tau \int_\Omega \left(\varrho \mathbf{u} \cdot \partial_t \varphi + \varrho \mathbf{u} \otimes \mathbf{u} : \nabla_x \varphi + p(\varrho, \vartheta) \mathrm{div}_x \varphi \right) \mathrm{d}x \, \mathrm{d}t \tag{3.6}$$

$$- \int_0^\tau \int_\Omega \left(\mathbb{S} : \nabla_x \mathbf{u} - \varrho \nabla_x F \cdot \nabla_x \varphi \right) \mathrm{d}x \, \mathrm{d}t$$

holds for any $\tau \in [0, T]$ and any test function $\varphi \in C_c^\infty([0, T] \times \Omega; R^3)$.

In (3.6), we tacitly assume that all quantities are at least integrable in the set $(0, T) \times \Omega$. If the no-slip condition (2.21) is replaced by the complete slip (2.22), we have to replace (3.5) by

$$\mathbf{u} \in L^q(0, T; W^{1,2}(\Omega; R^3)), \ \mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \tag{3.7}$$

and to extend the class of admissible test functions to

$$\varphi \in C_c^\infty([0, T] \times \overline{\Omega}; R^3), \ \varphi \cdot \mathbf{n}|_{\partial\Omega} = 0.$$

3.4 Entropy production

We have reached the crucial moment of the mathematical theory. In the *weak formulation* of the entropy production equation (2.14), we replace *equality* (2.15) by *inequality*

$$\sigma \geq \frac{1}{\vartheta} \left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right). \tag{3.8}$$

To this end, we must have $\vartheta > 0$ a.a. in $(0, T) \times \Omega$.

Accordingly, the weak formulation of the entropy balance reads

$$\int_0^\tau \int_\Omega \left[\varrho s(\varrho, \vartheta)(\tau, \cdot)\varphi(\tau, \cdot) - \varrho_0 s(\varrho_0, \vartheta_0)\varphi(0, \cdot)\right] \mathrm{d}x \, \mathrm{d}t - \int_0^\tau \int_\Omega \frac{1}{\vartheta}\left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta}\right)\varphi \, \mathrm{d}x \, \mathrm{d}t$$

$$\geq \int_0^\tau \int_\Omega \left(\varrho s(\varrho, \vartheta)\partial_t \varphi + \varrho s(\varrho, \vartheta)\mathbf{u} \cdot \nabla_x \varphi + \frac{\mathbf{q}}{\vartheta} \cdot \nabla_x \varphi\right) \mathrm{d}x \, \mathrm{d}t \tag{3.9}$$

for a.a. $\tau \in (0, T)$ and any $\varphi \in C_c^\infty([0, T] \times \overline{\Omega})$, $\varphi \geq 0$.

Of course, replacing (2.15) by (3.8) means a crude simplifaction and considerable extension of the class of possible solutions. In order to compensate for this lack of information, the resulting system must be supplemented by an extra condition formulated in the following section.

3.5 Total energy balance

Thanks to our choice of the boundary conditions, we may integrate the total energy balance (2.11) to obtain

$$\int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) - \varrho F\right](\tau, \cdot) \, \mathrm{d}x = \int_\Omega \left[\frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + \varrho_0 e(\varrho_0, \vartheta_0) - \varrho_0 F\right] \mathrm{d}x \tag{3.10}$$

for a.a. $\tau \in (0, T)$.

To conclude, we recall that the *weak formulation* of the *Navier-Stokes-Fourier system* consists of

- MASS CONSERVATION. Equation of continuity in the sense of generalized derivatives.

- NEWTON'S SECOND LAW. Momentum equation + boundary conditions satisfied in the weak sense.

- ENTROPY PRODUCTION. Entropy *inequality* with a non-negative production rate.

- TOTAL ENERGY BALANCE. Conservation of the total energy in time.

4 Weak vs. strong solutions

However beautiful the strategy, you should occasionally look at the results.

Sir Winston Churchill, 1874-1965

Our concept of *weak solutions* to the Navier-Stokes-Fourier system apparently extends that of the strong solutions, in the sense that, evidently, strong solutions satisfying all equations in the classical sense, including the entropy *equation* (2.13), (2.15) are also weak solutions in the sense specified in Section 3.

4.1 Vacuum problem

One of the major open problems of the theory is the possibility for ϱ to become zero in a finite time. Although weak solutions can be constructed with $\varrho \geq 0$, we are not able to show that, in fact,

$$\varrho(t, \cdot) > 0 \text{ as long as } \varrho_0 > 0.$$

Suppose that solutions are smooth. We can rewrite the equation of continuity (2.2) in the form

$$\partial_t \varrho + \mathbf{u} \cdot \nabla_x \varrho = -\varrho \mathrm{div}_x \mathbf{u},$$

or, equivalently,

$$\partial_t \log(\varrho) + \mathbf{u} \cdot \nabla_x \log(\varrho) = -\mathrm{div}_x \mathbf{u}.$$

Thus, defining the characteristic field

$$\frac{\mathrm{d}\mathbf{X}}{\mathrm{d}t} = \mathbf{u}(t, \mathbf{X}(t)),$$

we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \log(\varrho(t, \mathbf{X}(t)) = -\mathrm{div}_x \mathbf{u}(t, \mathbf{X}(t));$$

whence

$$\exp(-Lt) \inf_{x \in \Omega} \varrho_0(x) \leq \varrho(t, x) \leq \exp(Lt) \sup_{x \in \Omega} \varrho_0(x) \text{ for all } t > 0, \tag{4.1}$$

where

$$L = \sup_{(t,x) \in (0,T) \times \Omega} |\mathrm{div}_x \mathbf{u}(t, x)|.$$

Unfortunately, *uniform* bounds on $\mathrm{div}_x \mathbf{u}$ are beyond reach of the present theory so (4.1) remains valid only in the class of strong solutions.

4.2 Weak-strong compatibility

A more delicate task is to show that a weak solution that is sufficiently regular, represents a strong solution. In particular, the entropy inequality (3.9) can be replaced by (2.13), (2.15). In order to see this, we first observe that the kinetic energy balance (2.8) holds as soon as the weak solution is smooth, in particular,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u}|^2 - \varrho F \right) \, \mathrm{d}x = \int_\Omega \left(-\mathbb{S} : \nabla_x \mathbf{u} + p(\varrho, \vartheta) \mathrm{div}_x \mathbf{u} \right) \, \mathrm{d}x. \tag{4.2}$$

The next step is to take $\vartheta \varphi$, $\varphi \geq 0$, $\varphi \in C_c^\infty((0, T) \times \overline{\Omega})$, as a test function in (3.9) to obtain:

$$\int_0^\tau \int_\Omega \mathbb{S} : \nabla_x \mathbf{u} \varphi \, \mathrm{d}x \, \mathrm{d}t \leq \int_0^\tau \int_\Omega \left(\vartheta \left[\partial_t(\varrho s(\varrho, \vartheta)) + \mathrm{div}_x(\varrho s(\varrho, \vartheta) \mathbf{u}) \right] \varphi - \mathbf{q} \cdot \nabla_x \varphi \right) \, \mathrm{d}x \, \mathrm{d}t. \tag{4.3}$$

On the other hand, Gibbs' equation (2.1) yields

$$\vartheta \left[\partial_t(\varrho s(\varrho, \vartheta)) + \mathrm{div}_x(\varrho s(\varrho, \vartheta) \mathbf{u}) \right] = \varrho \left[\vartheta \partial_t s(\varrho, \vartheta) + \vartheta \mathbf{u} \cdot \nabla_x s(\varrho, \vartheta) \right]$$

$$= \varrho \left[\partial_t e(\varrho, \vartheta) + \mathbf{u} \nabla_x e(\varrho, \vartheta) \right] + p(\varrho, \vartheta) \mathrm{div}_x \mathbf{u},$$

where we have also used the equation of continuity (2.2), together with the fact that the (regular) density remains bounded below away from zero provided we have chosen $\varrho_0 > 0$, see (4.1).

We infer that (4.3) gives rise to

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \varrho e(\varrho, \vartheta) \, \mathrm{d}x \geq \int_\Omega \left(\mathbb{S} : \nabla_x \mathbf{u} - p(\varrho, \vartheta) \mathrm{div}_x \mathbf{u} \right) \, \mathrm{d}x. \tag{4.4}$$

Thus the total energy balance imposed through (3.10) is compatible with (4.2), (4.4) only if the sign $\geq$ is replaced by $=$ in (4.4), meaning relation (3.9) holds with equality sign.

> Weak solutions of the Navier-Stokes-Fourier system introduced in Section 4 coincide with the strong solutions as soon as they are regular, specifically, if $\varrho > 0$, $\vartheta > 0$, and ϱ, ϑ, $\mathbf{u}$ are continuously differentiable in t and twice continuously differentiable in x.

4.3 Exercises

4.3.1 Maxwell's equation

Suppose that $p = p(\varrho, \vartheta)$, $e = e(\varrho, \vartheta)$, and $s = s(\varrho, \vartheta)$ satisfy Gibbs' relation (2.1). Show Maxwell's equation

$$\frac{\partial e(\varrho, \vartheta)}{\partial \varrho} = \frac{1}{\varrho^2} \left(p(\varrho, \vartheta) - \vartheta \frac{\partial p(\varrho, \vartheta)}{\partial \vartheta} \right). \tag{4.5}$$

4.3.2 Entropy vs. pressure

Using Gibbs' equation (2.1) and Maxwell's equation (4.5) show that

$$\frac{\partial s(\varrho, \vartheta)}{\partial \varrho} = -\frac{1}{\varrho^2}\frac{\partial p(\varrho, \vartheta)}{\partial \vartheta}.$$

4.3.3 Monoatomic gas state equation

A general *monoatomic gas* is characterized by the relation

$$p(\varrho, \vartheta) = \frac{2}{3}\varrho e(\varrho, \vartheta). \tag{4.6}$$

Suppose, in addition to (4.6), that p and e satisfy Gibbs' equation (2.1). Show that p takes the form

$$p(\varrho, \vartheta) = \vartheta^{5/2}P\left(\frac{\varrho}{\vartheta^{3/2}}\right) \text{ for a certain function } P.$$

5 A priori bounds

A priori bounds are constraints imposed on the solution set by the data and the differential identities satisfied. As is well known in the standard theiry of *elliptic* equations, *a priori* bounds determine the function spaces framework in which the solutions are looked for. Evolutionary equations arising in continuum fluid mechanics are known to possess only very poor *a rpriori* bounds, which makes the rigorous analysis rather delicate. To begin, we point out that *a priori* bounds are purely formal, derived under the hypothesis of smoothness of all quantities in question.

5.1 Total mass conservation

The simplest bound follows directly by integrating the equation of continuity (2.2) over the spatial domain Ω. Supposing that the velocity field $\mathbf{u}$ obeys the impermeability condition

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

we immediately obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \varrho(t, \cdot)\,\mathrm{d}x = 0,$$

in other words

$$\int_\Omega \varrho(t, \cdot)\,\mathrm{d}x = \int_\Omega \varrho_0\,\mathrm{d}x, \text{ for any } t \in [0, T]. \tag{5.1}$$

The physical meaning of (5.1) is obvious - the *total mass* of the fluid contained in the physical domain Ω is a constant of motion. Note that the same conclusion holds even in the class of the weak solutions satisfying (3.4), where it is enough to take $\varphi \equiv 1$.

Since the density is *non-negative* we get

$$\sup_{t\in[0,T]} \|\varrho(t,\cdot)\|_{L^1(\Omega)} = \|\varrho_0\|_{L^1(\Omega)} \text{ for any } t \in [0,T]. \tag{5.2}$$

5.2 Energy estimates

The total energy balance (3.10) yields immediately

$$\sup_{t\in[0,T]} \|\sqrt{\varrho}\mathbf{u}(t,\cdot)\|_{L^2(\Omega;R^3)} \le c(\text{initial data}), \tag{5.3}$$

and

$$\sup_{t\in[0,T]} \|\varrho e(\varrho,\vartheta)(t,\cdot)\|_{L^1(\Omega)} \le c(\text{initial data}) \tag{5.4}$$

provided, say,

$$F \in L^\infty(\Omega).$$

Of course, we have tacitly assumed that $e \ge 0$. Similarly to the preceding part, the energy estimates remain valid even in the class of weak solutions in the sense specified in Section 4.

5.3 Dissipation and entropy estimates

Our next step is to exploit the entropy balance equation (2.13). Integrating by parts and making use of the boundary conditions we deduce that

$$\int_\Omega \varrho s(\varrho,\vartheta)(\tau,\cdot) \, dx = (\ge) \int_\Omega \varrho_0 s(\varrho_0,\vartheta_0) \, dx + \int_0^\tau \int_\Omega \frac{1}{\vartheta} \left[\mathbb{S} : \nabla_x\mathbf{u} - \frac{\mathbf{q}_\cdot \cdot \nabla_x\vartheta}{\vartheta} \right] \, dx \, dt. \tag{5.5}$$

5.3.1 Thermodynamic stability

In accordance with *Second law of thermodynamics*, we can control the total entropy production rate provided we have an upper bound on the total entropy of the system. Since we already know that

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the total internal energy is bounded, we may attempt to obtain upper bound on $\varrho s(\varrho, \vartheta)$ in terms of $\varrho e(\varrho, \vartheta)$. To this end, we introduce

<div style="background-color: #d9d9d9; padding: 1em;">

HYPOTHESIS OF THERMODYNAMIC STABILITY

$$\frac{\partial p(\varrho, \vartheta)}{\partial \varrho} > 0, \tag{5.6}$$

$$\frac{\partial e(\varrho, \vartheta)}{\partial \vartheta} > 0. \tag{5.7}$$

</div>

Condition (5.6) asserts that *compressibility* of the fluid is always positive while $\partial_\vartheta e$ is the *specific heat at constant volume*.

5.3.2 Ballistic free energy

Let us introduce a thermodynamic potential

$$H_\Theta(\varrho, \vartheta) = \varrho e(\varrho, \vartheta) - \Theta \varrho s(\varrho, \vartheta), \text{ with } \Theta > 0 \text{ a constant,} \tag{5.8}$$

termed *ballistic free energy*.

It follows from the hypotheses (5.6), (5.7) that

-
$$\varrho \mapsto H_\Theta(\varrho, \Theta) \text{ is strictly convex;} \tag{5.9}$$

-
$$\vartheta \mapsto H_\Theta(\varrho, \vartheta) \text{ is decreasing for } \vartheta < \Theta \text{ and increasing for } \vartheta > \Theta. \tag{5.10}$$

5.3.3 Total dissipation balance

Combining the total energy conservation (3.10) with (5.5) we deduce

$$\int_\Omega \left[H_\Theta(\varrho, \vartheta) - (\varrho - \overline{\varrho}) \frac{\partial H_\Theta(\overline{\varrho}, \Theta)}{\partial \varrho} - H_\Theta(\overline{\varrho}, \Theta) - \varrho F \right] (\tau, \cdot) \; \mathrm{d}x \qquad (5.11)$$

$$+ \int_0^\tau \int_\Omega \frac{\Theta}{\vartheta} \left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right) \; \mathrm{d}x \; \mathrm{d}t$$

$$= (\leq) \int_\Omega \left[H_\Theta(\varrho_0, \vartheta_0) - (\varrho_0 - \overline{\varrho}) \frac{\partial H_\Theta(\overline{\varrho}, \Theta)}{\partial \varrho} - H_\Theta(\overline{\varrho}, \Theta) - \varrho_0 F \right] \; \mathrm{d}x,$$

where the constant $\overline{\varrho} > 0$ has been chosen so that

$$\int_\Omega (\varrho(t, \cdot) - \overline{\varrho}) \; \mathrm{d}x = 0.$$

Consequently, making use of the coercivity properties of the ballistic free energy H_Θ established in Section 5.3.2, we deduce from (5.11) the following *a priori* bounds:

$$-c(\text{data}) \leq \int_\Omega \varrho s(\varrho, \vartheta)(\tau, \cdot) \; \mathrm{d}x \leq c(\text{data}) \text{ for all } \tau \in [0, T], \qquad (5.12)$$

and

$$\int_0^T \int_\Omega \frac{1}{\vartheta} \left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right) \; \mathrm{d}x \; \mathrm{d}t \leq c(\text{data}). \qquad (5.13)$$

5.3.4 Korn inequality

Our next goal is to use (5.13) in order to derive bounds on $\nabla_x \mathbf{u}$ and $\nabla_x \vartheta$, respectively. We start with $\nabla_x \vartheta$ seeing that, in accordance with Fourier's law (2.17),

$$\int_0^T \int_\Omega \frac{\kappa(\vartheta) |\nabla_x \vartheta|^2}{\vartheta^2} \; \mathrm{d}x \; \mathrm{d}t \leq c(\text{data}).$$

Consequently, assuming

$$\underline{\kappa}(1 + \vartheta^2) \le \kappa(\vartheta) \le \overline{\kappa}(1 + \vartheta^2) \tag{5.14}$$

for certain positive constants $\underline{\kappa}$, $\overline{\kappa}$ we deduce that

$$\int_0^T \int_\Omega |\nabla_x \vartheta|^2 \, \mathrm{d}x \, \mathrm{d}t + \int_0^T \int_\Omega |\nabla_x \log(\vartheta)|^2 \, \mathrm{d}x \, \mathrm{d}t \le c(\text{data}) \tag{5.15}$$

Similar estimates on the velocity gradient are more delicate. By virtue of Newton's law (2.16), we get

$$\mathbb{S} : \nabla_x \mathbf{u} = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) : \nabla_x \mathbf{u}$$

$$= \frac{\mu(\vartheta)}{2} \left| \nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right|^2 ;$$

whence, assuming, say,

$$\underline{\mu}(1 + \vartheta) \le \mu(\vartheta) \le \overline{\mu}(1 + \vartheta), \tag{5.16}$$

we deduce from (5.13) that

$$\int_0^T \int_\Omega \left| \nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right|^2 \, \mathrm{d}x \, \mathrm{d}t \le c \tag{5.17}$$

Suppose, for a moment, that $\mathbf{u}$ vanishes on $\partial\Omega$, meaning $\mathbf{u}$ satisfies the no-slip boundary condition (2.21). By simple computation, we get

$$\left| \nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right|^2 = |\nabla_x \mathbf{u}|^2 + |\nabla_x^t \mathbf{u}|^2 + 2\nabla_x \mathbf{u} \cdot \nabla_x^t \mathbf{u} - \frac{4}{3} |\mathrm{div}_x \mathbf{u}|^2,$$

where, by means of a simple by parts integration,

$$\int_\Omega \nabla_x \mathbf{u} \cdot \nabla_x^t \mathbf{u} \, \mathrm{d}x = \int_\Omega |\mathrm{div}_x \mathbf{u}|^2 \, \mathrm{d}x.$$

Thus the bound (5.13) yields

$$\int_0^T \int_\Omega |\nabla_x \mathbf{u}|^2 \, \mathrm{d}x \le c(\text{data}). \tag{5.18}$$

Finally, since we have assumed that $\mathbf{u}|_{\partial\Omega} = 0$, we may combine (5.18) with Poincaré's inequality to conclude that

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$$\int_0^T \int_\Omega \left(|\mathbf{u}|^2 + |\nabla_x \mathbf{u}|^2 \right) \, \mathrm{d}x \le c(\text{data}). \tag{5.19}$$

We note that

$$\int_\Omega \left(|\mathbf{u}|^2 + |\nabla_x \mathbf{u}|^2 \right) \, \mathrm{d}x \equiv \|\mathbf{u}\|^2_{W^{1,2}(\Omega;R^3)},$$

where $W^{1,2}$ stands for the standard *Sobolev space* of functions having first (spatial) partial derivatives square integrable.

5.4   More hypotheses imposed on constitutive equations

The bounds derived in the preceding part are the best (known) *a priori* bounds available under the general conditions imposed on the initial data and the existence time $T > 0$. Still, they are not strong enough, in general, to render *all* terms appearing in the weak formulation at least *equi-integrable* in order to ensure their stability. Besides the general hypothesis of thermodynamic stability (5.6), (5.7), more restrictions will be imposed on the constitutive relations in order to obtain better estimates to make the problem mathematically tractable.

5.4.1   General pressure law

A general pressure law of a *monoatomic gas* reads

$$p(\varrho, \vartheta) = \frac{2}{3} \varrho e(\varrho, \vartheta), \tag{5.20}$$

see Eliezer, Ghatak, Hora [1].

It is straightforward to check that (5.20) is compatible with the general Gibbs' relation (2.1) if the pressure takes the form

$$p(\varrho, \vartheta) = \vartheta^{5/2} P \left(\frac{\varrho}{\vartheta^{3/2}} \right) \tag{5.21}$$

for a certain function $P : [0, \infty) \to R$. Now, the hypothesis (5.6) can be interpreted as

$$P = C^1[0, \infty), \ P'(Z) > 0 \text{ for any } Z \ge 0. \tag{5.22}$$

Going back to (5.20), the specific internal energy reads

$$e(\varrho, \vartheta) = \frac{3}{2} \varrho \left(\frac{\vartheta^{3/2}}{\varrho} \right) P \left(\frac{\varrho}{\vartheta^{3/2}} \right). \tag{5.23}$$

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Now, the second condition of thermodynamic stability (5.7) translates to

$$\frac{\frac{5}{3}P(Z) - P'(Z)Z}{Z} > 0 \text{ for any } Z > 0, \tag{5.24}$$

in particular, the function

$$Z \mapsto \frac{P(Z)}{Z^{5/3}} \text{ is non-increasing}, \tag{5.25}$$

and we set

$$\lim_{Z \to \infty} \frac{P(Z)}{Z^{5/3}} = p_\infty > 0. \tag{5.26}$$

Under the present circumstances, the entropy is determined, up to an additive constant, by the Gibbs' equation (2.1), specifically,

$$s(\varrho, \vartheta) = S\left(\frac{\varrho}{\vartheta^{3/2}}\right), \tag{5.27}$$

where

$$S'(Z) = -\frac{3}{2}\frac{\frac{5}{3}P(Z) - P'(Z)Z}{Z^2} < 0. \tag{5.28}$$

In accordance with *Third law of thermodynamic*, we may fix S by postulating

$$\lim_{Z \to \infty} S(Z) = 0. \tag{5.29}$$

5.4.2 Effect of thermal radiation

The pressure p can be augmented by a component $p_R = p_R(\vartheta)$ resulting from *thermal radiation*, namely,

$$p_R = \frac{a}{3}\vartheta^4, \ a > 0,$$

with a the Stefan-Boltzmann constant. Consequently, we end up with the following specific form of the constitutive equations:

PRESSURE:

$$p(\varrho, \vartheta) = \vartheta^{5/2} P\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \frac{a}{3}\vartheta^4, \qquad (5.30)$$

INTERNAL ENERGY:

$$e(\varrho, \vartheta) = \frac{3}{2}\vartheta\left(\frac{\vartheta^{3/2}}{\varrho}\right) P\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \frac{a}{\varrho}\vartheta^4, \qquad (5.31)$$

ENTROPY:

$$s(\varrho, \vartheta) = S\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \frac{4a}{3\varrho}\vartheta^3. \qquad (5.32)$$

5.5 Exercises

5.5.1 Thermodynamic stability for monaotomic gases

Show the the hypothesis of thermodynamic stability (5.7) implies relations (5.24), (5.25).

5.6 Energy estimates revisited

From now on, we shall assume that the pressure p, the internal energy e, and the entropy s are given through (5.30 - 5.32). Thus we get, as a consequence of (5.26), the following bounds

$$\sup_{t\in[0,T]} \|\varrho(t,\cdot)\|_{L^{5/3}(\Omega)} \le c(\text{data}), \qquad (5.33)$$

and, because of the "radiation" component of the internal energy,

$$\sup_{t\in[0,T]} \|\vartheta(t,\cdot)\|_{L^4(\Omega)} \le c(\text{data}). \qquad (5.34)$$

5.7 Pressure estimates

The *a priori* bounds should be at least so strong for all the expressions appearing in the weak formulation to make sense (to be at least integrable in the space-time cylinder $(0, T) \times \Omega$. As a matter of fact, slightly more is needed, namely the *equi-integrability* property in order to perform the limits with respect to the weak topology of the Lebesgue space L^1. Note that the estimates (5.33), (5.34) guarantee only that

$$p(\varrho, \vartheta), \varrho e(\varrho, \vartheta) \in L^\infty(0, T; L^1(\Omega)).$$

Better estimates can be obtained by "computing" the pressure by means of the continuity equation (2.5):

$$\Delta p(\varrho, \vartheta) = \mathrm{div}_x \mathrm{div}_x \mathbb{S} - \mathrm{div}_x \mathrm{div}_x (\varrho \mathbf{u} \otimes \mathbf{u}) - \mathrm{div}_x \partial_t (\varrho \mathbf{u}),$$

where, for the sake of simplicity, we have take $\mathbf{f} \equiv 0$. Then, very formally indeed,

$$p(\varrho, \vartheta) = \mathrm{div}_x \Delta^{-1} \mathrm{div}_x \mathbb{S} - \mathrm{div}_x \Delta^{-1} \mathrm{div}_x (\varrho \mathbf{u} \otimes \mathbf{u}) - \Delta^{-1} \mathrm{div}_x \partial_t (\varrho \mathbf{u}).$$

Let us remark that both $\varrho \mathbf{u} \otimes \mathbf{u}$ and $\mathbb{S}$ are already known to be bounded in the Lebesgue space $L^q((0, T) \times \Omega$ for a certain $q > 1$. Indeed, by virtue of the Sobolev imbedding relation

$$W^{1,2}(\Omega) \hookrightarrow L^6(\Omega), \tag{5.35}$$

combined with the uniform bounds established in (5.3), (5.19), and (5.32), we have

$$\|\varrho \mathbf{u} \otimes \mathbf{u}\|_{L^q(\Omega; R^{3 \times 3})} = \|\sqrt{\varrho}\sqrt{\varrho}\mathbf{u} \otimes \mathbf{u}\|_{L^q(\Omega; R^{3 \times 3})} \leq \|\varrho\|_{L^{5/3}(\Omega)} \|\sqrt{\varrho}\mathbf{u}\|_{L^2(\Omega; R^3)} \|\mathbf{u}\|_{W^{1,2}(\Omega; R^3)},$$

with

$$\frac{1}{\gamma} = \frac{3}{10} + \frac{1}{2} + \frac{1}{6}, \text{ meaning } \gamma = \frac{30}{29} > 1;$$

whence

$$\varrho \mathbf{u} \otimes \mathbf{u} \in L^2(0, T; L^{30/29}(\Omega; R^{3 \times 3})). \tag{5.36}$$

Similarly, by virtue of (5.19) , (5.34), and hypothesis (5.16)

$$\|\mathbb{S}\|_{L^q(\Omega, R^{3 \times 3})} \leq c \|\vartheta\|_{L^4(\Omega)} \|\nabla_x \mathbf{u}\|_{L^2(\Omega; R^{3 \times 3})}, \ q = \frac{4}{3};$$

whence

$$\mathbb{S} \in L^2(0, T; L^{4/3}(\Omega; R^{3 \times 3})). \tag{5.37}$$

Seeing that the operator $\mathrm{div}_x \Delta^{-1} \mathrm{div}_x$ is of zero-th order, the estimates (5.35), (5.36) would guarantee boundedness of the pressure provided we could handle the time derivative $\Delta^{-1} \mathrm{div}_x \partial_t (\varrho \mathbf{u})$.

In order to make these arguments rigorous, we proceed in a slightly different way. Specifically, we multiply the momentum equation by the "operator"

$$\mathcal{B}\left[b(\varrho) - \frac{1}{|\Omega|}\int_\Omega b(\varrho)\ \mathrm{d}x\right],$$

where $b(\varrho)$ is a suitable function ϱ behaving like ϱ^α, $\alpha > 0$ for $\varrho \to \infty$. We *anticipate* the following two properties of the operator $\mathcal{B}$ discussed in what follows:

- $\mathcal{B}$ is a linear operator representing the "inverse" of divergence, specifically,

$$\mathrm{div}_x\mathcal{B}[v] = v.$$

- $\mathcal{B}[v]$ vanishes on the boundary,

$$\mathcal{B}[v]|_{\partial\Omega} = 0.$$

  This property clearly implies that, necessarilly,

$$\int_\Omega v\ \mathrm{d}x = 0.$$

References

[1] S. Eliezer, A. Ghatak, and H. Hora. *An introduction to equations of states, theory and applications.* Cambridge University Press, Cambridge, 1986.