Isogeometric free vibration of an elastic block

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SIGA 2012
Splines and IsoGeometric Analysis 2012
February 8, 2012
Institute of Thermomechanics AS CR, v.v.i., Prague

Contents and Motivation

Contents:

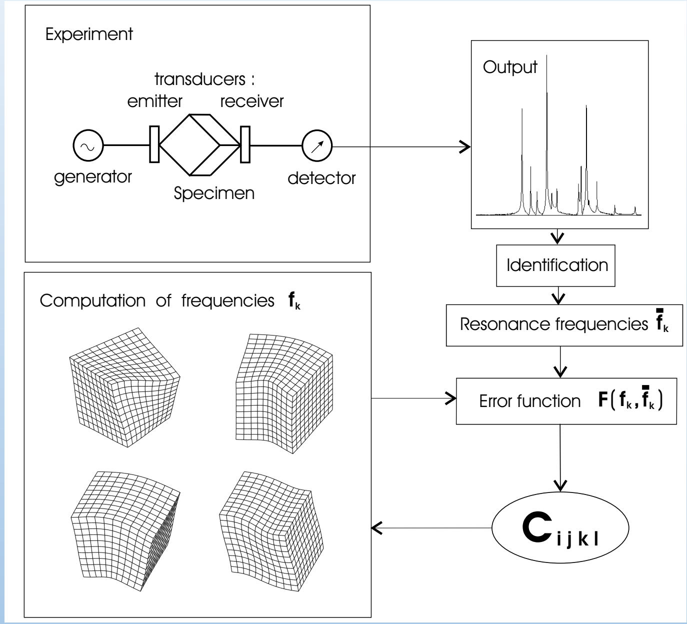
- Free vibration problem
- Resonant ultrasound spectroscopy (RUS)
- Finite element method - formulation
- Choice of shape functions satisfied stress-free boundary conditions
- Isogeometric analysis (B-spline, NURBS shape functions)
- Comparison of convergence rates for different shape functions

Motivation:

- Validation and verification of IGA in free vibration and elastodynamics problems
- RUS - free vibration of anisotropic specimens (cylinder, sphere,...)
  - utilization of eigen-vibration modes
  - determination of structure orientation

Resonant ultrasound spectroscopy

Determination of all independent elastic moduli
from knowledge of frequency spectrum.
Generally, all 21 components of the elastic tensor could be determined.



Plešek J., Kolman R., Landa M.: Using finite element method for the determination of elastic moduli by resonant ultrasound spectroscopy. *Journal of the Acoustical Society of America*, **116**, 282–287, 2004.

Formulation of elastodynamics problem - strong form

Equations of motion

$$\sigma_{ij,j} = \rho u_{i,tt} \quad \text{in} \quad \mathbf{x} \in \Omega \times (0,T)$$

Hooke's law

$$\sigma_{ij} = C_{ijkl}\epsilon_{kl}$$

Infinitesimal strain tensor

$$\epsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$

Stress-zero boundary conditions

$$\sigma_{ij}n_j = 0 \quad \text{on} \quad \Gamma \times (0,T), \Gamma = \partial\Omega$$

u_i - the ith component of the displacement vector $\mathbf{u}$, σ_{ij} - Cauchy stress tensor, ϵ_{ij} - infinitesimal strain, C_{ijkl} - elastic tensor, ρ - mass density, n_i - the ith component of outward normal

Formulation of elastodynamics problem - weak form

Find $\mathbf{u} \in S_t$ (space of trial solutions) such that for all $\mathbf{w} \in W$ (space of weighting functions)

$$(\mathbf{w}, \rho\ddot{\mathbf{u}}) + a(\mathbf{w}, \mathbf{u}) = 0$$

where

$$(\mathbf{w}, \rho\ddot{\mathbf{u}}) = \int_\Omega \rho \mathbf{w} \ddot{\mathbf{u}} \, \mathrm{d}\Omega$$

$$a(\mathbf{w}, \mathbf{u}) = \int_\Omega w_{i,j} C_{ijkl} u_{k,l} \, \mathrm{d}\Omega$$

Separation of variables - free vibration

$$u_j = u_j(\mathbf{x}) \, e^{\mathrm{i}\omega t},$$

then the eigenproblem is obtained

$$-\omega^2(\mathbf{w}, \rho\mathbf{u}) + a(\mathbf{w}, \mathbf{u}) = 0$$

i - imaginary unit, ω - angular velocity, $f = \omega/2\pi$ - frequency

Free vibration of elastic objects

Rayleigh-Ritz method: the actual motion in the form $\mathbf{u}e^{\mathrm{i}\omega t}$ is given by such a displacement function $\mathbf{u}$ that renders integral

$$\int_\Omega \frac{1}{4}\left(u_{i,j}+u_{j,i}\right) C_{ijkl}\left(u_{k,l}+u_{l,k}\right)\,\mathrm{d}\Omega$$

an extremum under the normalization condition

$$\int_\Omega \sum_i u_i^2\,\mathrm{d}\Omega = 1$$

and the extremum given $\rho\omega^2$.

In practice,

$$\mathbf{u}=\sum_p^N a_p \Phi_p$$

where functions $\Phi_p, p=1,\dots,N$ should satisfied the boundary conditions.

Eigenvalue problem

$$\left(\Gamma_{pq}-\lambda\delta_{pq}\right)a_{pq}=0, \quad p,q=1,\dots,N.$$

Free vibration of elastic objects

Galerkin continuous formulation of FEM method:

Find $\mathbf{u}^h \in S_t^h$ such that for all $\mathbf{w}^h \in W^h$

$$(\mathbf{w}^h, \rho\ddot{\mathbf{u}}^h) + a(\mathbf{w}^h, \mathbf{u}^h) = 0$$

Representations of $\mathbf{u}^h$, $\mathbf{w}^h$

$$u_i^h = \sum_A N_A(\mathbf{x}) d_{iA}$$

$$w_i^h = \sum_A N_A(\mathbf{x}) c_{iA},$$

where $N_A(\mathbf{x})$ are shape functions.

By separation of variables, eigenproblem is obtained

$$-\omega^2(\mathbf{w}^h, \rho\mathbf{u}^h) + a(\mathbf{w}^h, \mathbf{u}^h) = 0$$

Rayleigh quotient

$$\omega^2 = \frac{a(\mathbf{u}^h, \mathbf{u}^h)}{(\mathbf{u}^h, \rho\mathbf{u}^h)}$$

Finite element method

Discretized eigenvalue problem

$$-\omega^2 \mathbf{M}\mathbf{u}^h + \mathbf{K}\mathbf{u}^h = \mathbf{0}$$

Element (local) mass and stiffness matrices are given by

$$\mathbf{M}^e = \int_h \varrho \mathbf{N}^\intercal \mathbf{N} \, \mathrm{d}V^e, \qquad \mathbf{K}^e = \int_h \mathbf{B}^\intercal \mathbf{C} \mathbf{B} \, \mathrm{d}V^e$$

Global matrices

$$\mathbf{M} = \mathbf{A}_{e=1}^{nel} \mathbf{M}^e, \qquad \mathbf{K} = \mathbf{A}_{e=1}^{nel} \mathbf{K}^e$$

Normalization of eigenvectors:

$$\mathbf{U}^\intercal \mathbf{M} \mathbf{U} = \mathbf{I} \quad \Rightarrow \quad \mathbf{U}^\intercal \mathbf{K} \mathbf{U} = \mathbf{\Lambda}$$

$$\mathbf{U} = [\mathbf{u_1}, \ldots, \mathbf{u}_{NDOF}], \quad \mathbf{\Lambda} = \mathsf{diag}(\omega_1^2, \ldots, \omega_{NDOF}^2)$$

Free vibration of elastic solids

Exact solution of free vibration of solid with the arbitrary shape is not possible to find in the close form.

Historical background:

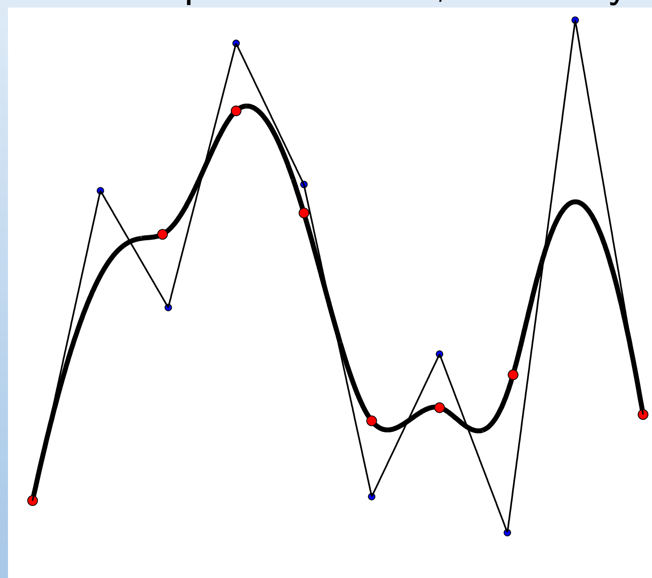
- sphere - [Love, 1927] (analytical solution by Bessel functions), [Frazer, 1964]
- block, parallelepiped rectangle - [Holland, 1968] (trigonometric functions), [Demarest, 1971] (Legendre's polynomials), [Ohno, 1976], [Visscher, 1991] (basis functions in the form $x^l y^m z^n$)
- laminated spheres and cylinders - [Yoneda, 2000]
- cylinder - [Love, 1927], [Ostrovsky, 1998] (combination of Bessel functions and FEM)
- potato, ellipsoid - [Visscher, 1991] (base functions in the form $x^l y^m z^n$)
- arbitrary shape of specimen - FEM or IGA (exact description)

B-spline curve

Piegl, L., Tiller, W. *The NURBS Book, 2nd Edition.* Springer-Verlag, 1997.

B-spline curve - a parametric described piecewise polynomial curve of degree p.

Degree $p = 3$, number of control points $n = 10$, uniformly-spaced control points.



NURBS curve - a generalization of B-spline curve by introducing weights of control points.

B-spline curve

Piegl, L., Tiller, W. *The NURBS Book, 2nd Edition.* Springer-Verlag, 1997.

B-spline curve of degree p is expressed by

$$\mathbf{C}(\xi) = \sum_{i=1}^{n} N_{i,p}(\xi)\,\mathbf{P}_i,$$

where $\mathbf{P}_i$, $i = 1, \ldots, n$ are coordinates of control points and $N_{i,p}(\xi)$ are basic functions of degree p, for example ξ is parameter, $\xi \in [0, 1]$.

Possibilities of control of the B-spline curves:

- by coordinates of control points $\mathbf{P}_i$, $i = 1, 2, \ldots, n$
- by degree p
- by knot vector $\Xi = \{\xi_1, \xi_2, ..., \xi_{n+p+1}\}$

B-spline curve - basis functions

Piegl, L., Tiller, W. *The NURBS Book, 2nd Edition.* Springer-Verlag, 1997.

For a given knot vector $\Xi = \{\xi_1, \xi_2, ..., \xi_{n+p+1}\}$, the B-spline basis functions are defined recursively starting with piecewise constants ($p = 0$)

$$N_{i,0}(\xi) = \begin{cases} 1 & \text{if} \quad \xi_i \le \xi \le \xi_{i+1}, \\ 0 & \text{otherwise.} \end{cases}$$

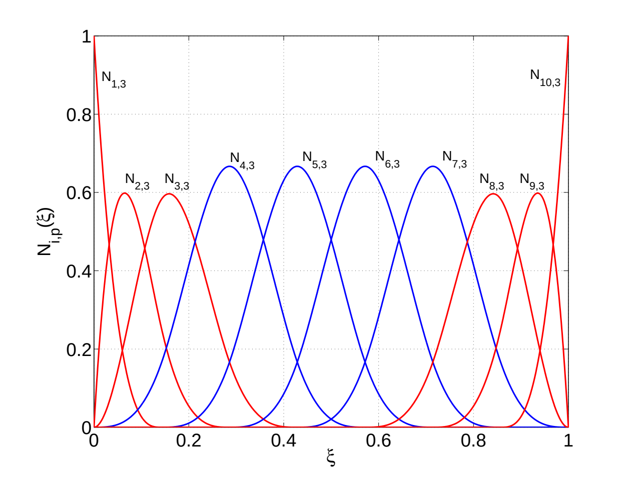
For $p = 1, 2, 3, ...,$ they are defined by the Cox-de Boor formula

$$N_{i,p}(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_{i,p-1}(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1,p-1}(\xi).$$

Remark. A efficient algorithm for the numerical evaluation of B-spline basis functions is necessary to employ.

B-spline curve - basis functions

Degree $p = 3$, $n = 10$ control points, uniform knot vector:

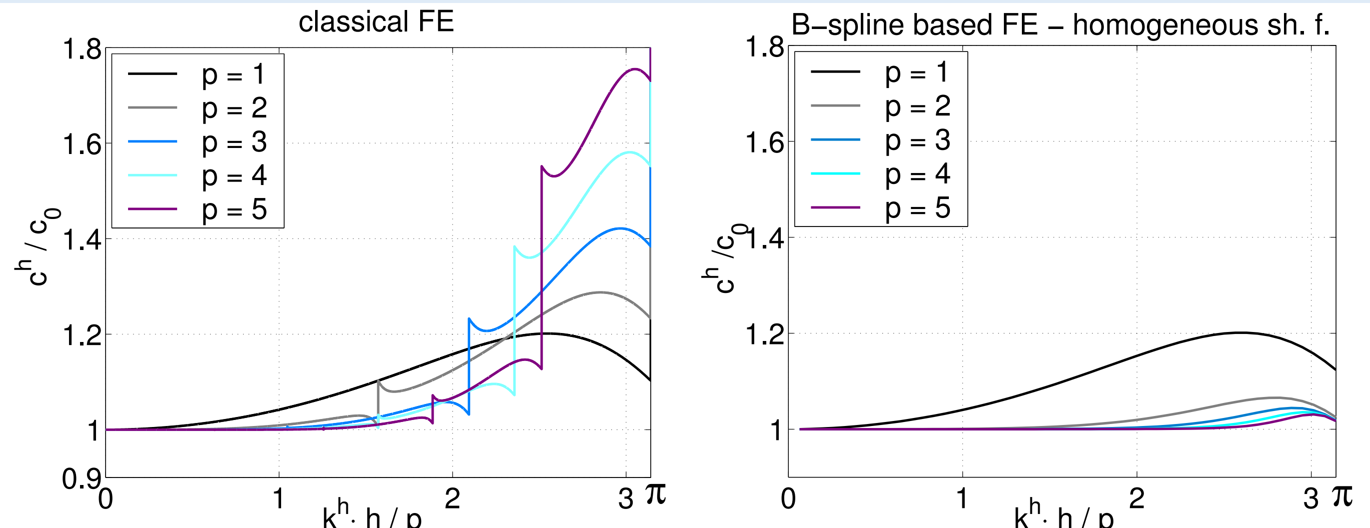


Basic properties:

- A partition of unity, that is, $\sum_{i=1}^{n} N_{i,p}(\xi) = 1$
- The support of each $N_{i,p}$ is compact and contained in the interval $[\xi_i, \xi_{i+p+1}]$
- B-spline basis functions are non-negative: $N_{i,p}(\xi) \geq 0 \, \forall \xi$
- C^{p-k} continuous piecewice polynomials, k is order of multiplicity of knot. The continuity can be controlled by the multiplicity of knot in the knot vector Ξ.

Dispersion errors in 1D, frequency errors in vibration of fixed-fixed bar

Hughes T.J.R., Reali A., Sangalli G. Duality and Unified Analysis of Discrete Approximations in Structural Dynamics and Wave Propagation: Comparison of p-method Finite Elements with k-method NURBS. *Comput. Methods Appl. Mech. Engrg.*, **197**, 4104-4124, 2008.



Existing of optical modes for classical FEs. The attenuation solutions are appearing.

h is an edge length of a classical FE. h/p is a distance of nodes.

$\bar{h}$ is a distance of control points.

B-spline surface and solid

A tensor product B-spline surface of degree p, q is defined by

$$\mathbf{S}(\xi, \eta) = \sum_{i=1}^{n} \sum_{j=1}^{m} N_{i,p}(\eta)\, M_{j,q}(\xi)\, \mathbf{P}_{i,j},$$

where $N_{i,p}(\eta), M_{j,q}(\xi)$ are univariate B-spline functions of order p and q corresponding to knot Ξ and H, respectively. $\mathbf{P}_{i,j}$, $i = 1, \ldots, n, j = 1, \ldots, m$ are coordinates of control points.

A tensor product B-spline solid of degree p, q, r is expressed by

$$\mathbf{S}(\xi, \eta, \zeta) = \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{l} N_{i,p}(\eta)\, M_{j,q}(\xi)\, L_{k,r}(\zeta)\, \mathbf{P}_{i,j,k}$$

Note: global refinement, local refinement by subdivision, hierarchically B-spline (NURBS, T-spline)

NURBS curve, surface and solid

NURBS - Non-Uniform Rational B-spline

A NURBS curve of degree p is defined by

$$\mathbf{C}(\xi) = \sum_{i=1}^{n} R_i^p(\xi)\,\mathbf{P}_i,$$

where NURBS basis is given by

$$R_i^p(\xi) = \frac{N_{i,p}(\xi)\,w_i}{\sum_{j=1}^{n} N_{\hat{i},p}(\eta)\,w_{\hat{i}}}$$

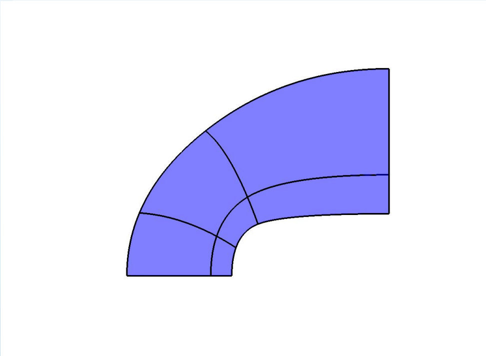
and w_i is referred to as the i-th **weight** corresponding to i-th control point.
For surfaces and solids, respectively

$$R_{i,j}^{p,q}(\xi,\eta) = \frac{N_{i,p}(\xi)\,M_{j,q}(\eta)\,w_{i,j}}{\sum_{\hat{i}=1}^{n}\sum_{\hat{j}=1}^{m} N_{\hat{i},p}(\xi)\,M_{\hat{j},q}(\eta)\,w_{\hat{i},\hat{j}}}$$

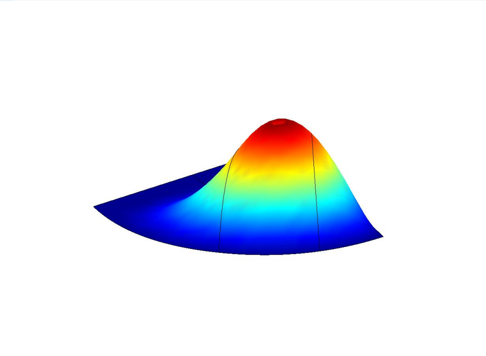
$$R_{i,j,k}^{p,q,r}(\xi,\eta,\zeta) = \frac{N_{i,p}(\xi)\,M_{j,q}(\eta)\,L_{k,r}(\zeta)\,w_{i,j,k}}{\sum_{\hat{i}=1}^{n}\sum_{\hat{j}=1}^{m}\sum_{\hat{k}=1}^{l} N_{\hat{i},p}(\xi)\,M_{\hat{j},q}(\eta)\,L_{\hat{k},r}(\zeta)\,w_{\hat{i},\hat{j},\hat{k}}}$$

B-spline and NURBS surface and shape function

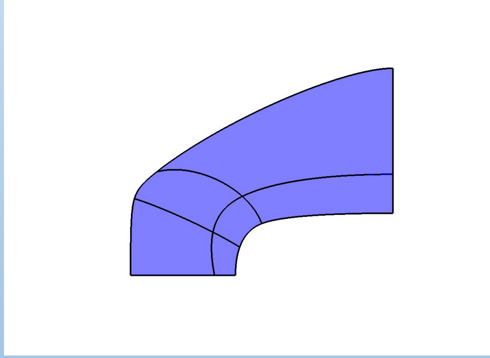
B-spline surface



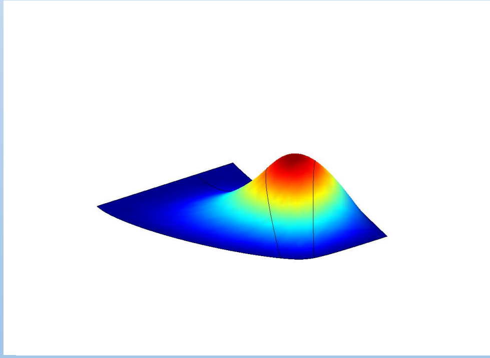
B-spline shape function



NURBS surface



NURBS shape function



Evaluation of shape functions and their derivatives

1. Direct evaluation B-spline basic functions $\mathbf{B}$ by the Cox-de Boor formula [Piegl, Tiller, 1997] – inefficient process, check of division by zero

2. Evaluation by B-spline basic functions $\mathbf{B}$ by [Piegl, Tiller, 1997] – very efficient process, CAD

3. Expression of B-spline basic functions $\mathbf{B}$ by Bernstein polynomials $\mathbf{N}$ [Borden, 2010] – evaluation of extraction operator $\mathbf{C}$ defined by a knot vector, $\mathbf{B} = \mathbf{C}\,\mathbf{N}$

4. Extraction NURBS description to the Bezier's description [Borden, 2010] – the same structure as standard FEM

Numerical test - an elastic block

- Dimensions

$$a = 2.333 \text{ [mm]}$$

$$b = 2.889 \text{ [mm]}$$

$$c = 3.914 \text{ [mm]}$$

- Mass density

$$\rho = 2459.9 \; [kg/m^3]$$

- Elastic moduli - isotropic case

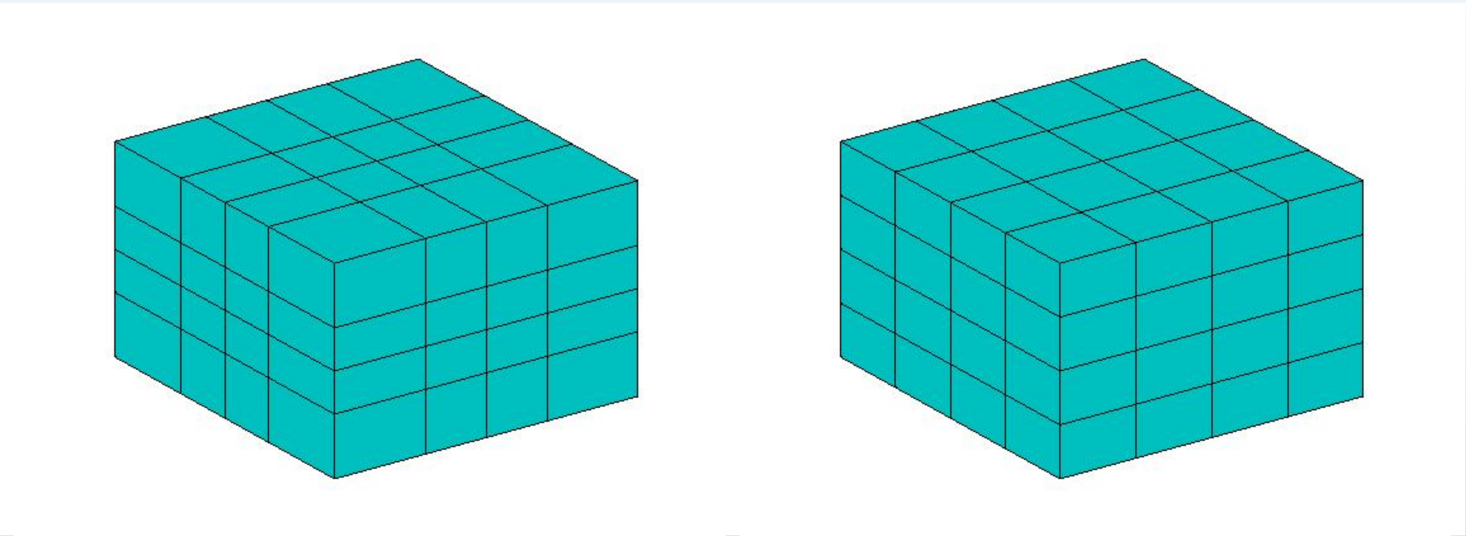
$$C_{11} = 82.0407 \text{ [GPa]}$$

$$C_{12} = 23.5666 \text{ [GPa]}$$

$$C_{44} = 29.2371 \text{ [GPa]}$$

Plešek J., Kolman R., Landa M.: Using finite element method for the determination of elastic moduli by resonant ultrasound spectroscopy. *Journal of the Acoustical Society of America*, **116**, 282–287, 2004.

IGA - Linear versus non-linear parameterization



Non-linear parameterization - uniform knot vector, uniformly spaced control points

Linear parameterization - uniform knot vector, positions of control points given by Greville abscissa; $x_i = \frac{\xi_{i+1}+\ldots+\xi_{i+p}}{p}$ [Greville, 1967]

Numerical test - an elastic block

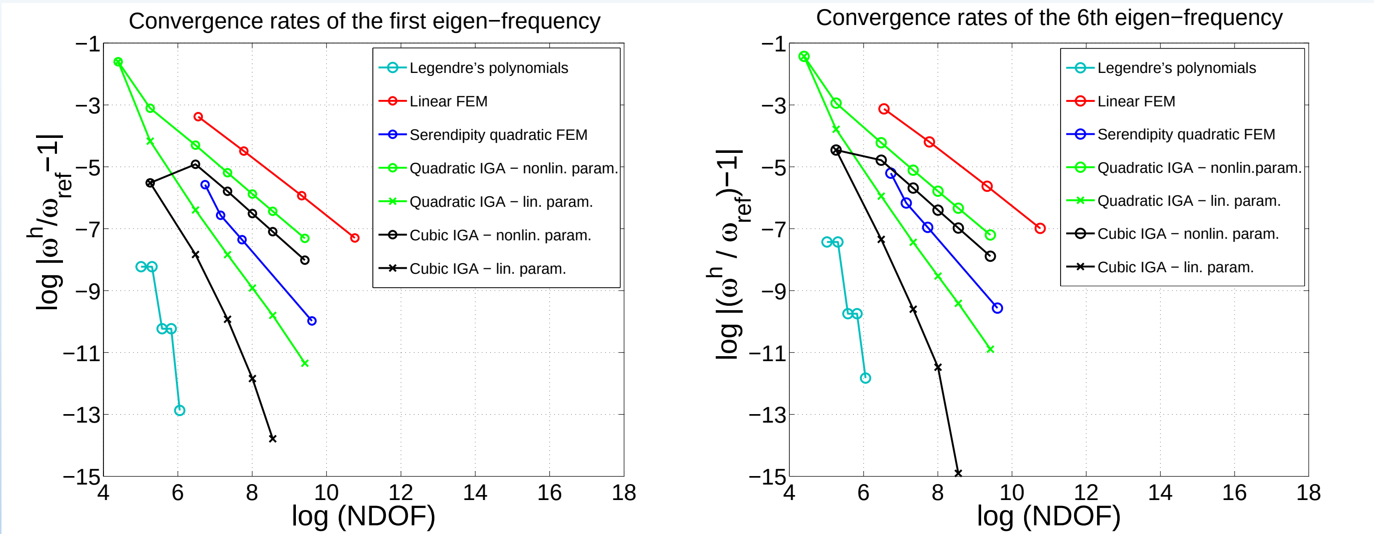
Freq. number	Ritz method order 10, $x^l y^m z^n$	Linear FEM 1000	Serend. quadr. FEM 1000	Quadr. NP IGA 1000	Quadr. LP IGA 1000	Cubic NP IGA 1000	Cubic LP IGA 1000	Experim. data
1	389154	390170	389158	390229	389192	389719	389143	390195
2	483641	485224	483680	484405	483756	484094	483641	482385
3	523541	525046	523580	524444	523654	524065	523541	521235
4	643221	644500	643253	644381	643328	643770	643218	640560
5	669073	669420	669079	669711	669088	669346	669073	664640
6	684101	686523	684109	686157	684196	685197	684068	684450
7	714654	717079	714716	715894	714873	715267	714648	712135
8	723969	727324	723784	726522	723928	725292	723718	723825
9	742704	744788	742760	744294	742881	743613	742697	741780
10	805803	808032	805840	807347	805930	806649	805791	803200
11	813858	816768	813875	815119	813931	814471	813844	809640
12	829406	831475	829346	831343	829466	830332	829292	825390
13	831333	833803	831275	833022	831372	832224	831226	831760
14	856984	860732	856580	860164	856798	858498	856489	854790
15	912615	913178	912624	913542	912640	913011	912615	906720

LP - linear parameterization

LP - non-linear parameterization by Greville abscissa

$x_i = \frac{\xi_{i+1} + \ldots + \xi_{i+p}}{p}$ [Greville, 1967]

Numerical test - convergence rates



Reference state - serendipity quadratic FEM, mesh 20x20x20 elements (PMD)

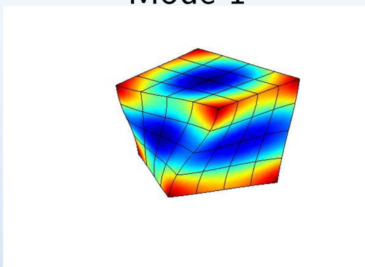
<u>Error estimation</u> [Strang, Fix, 2008, 2nd edition]:

eigen-values: $\lambda_{(l)} \leq \lambda^h_{(l)} \leq \lambda_{(l)} + Ch^{2(k+1-m)}\lambda^{(k+1)/m}$

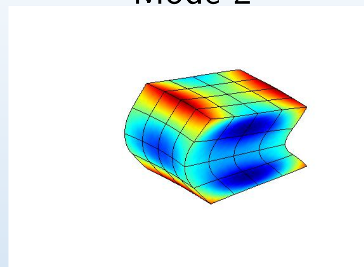
eigen-vectors: $\|\mathbf{u}^h_{(l)} - \mathbf{u}_{(l)}\|_m \leq Ch^{(k+1-m)}\lambda^{(k+1)/2m}_{(l)}$

Vibration modes of an elastic block

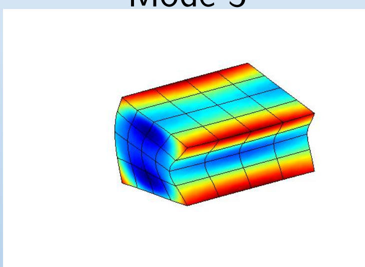
Mode 1



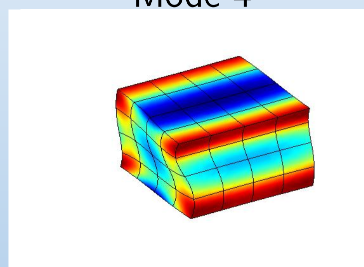
Mode 2



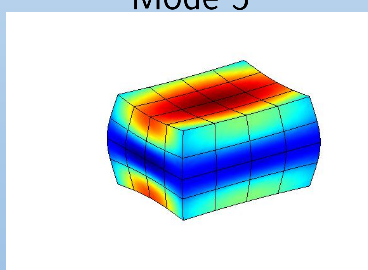
Mode 3



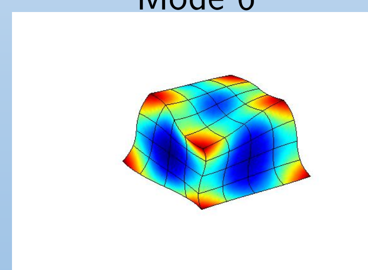
Mode 4



Mode 5



Mode 6



Conclusions and summary

- The classical C^0 continuity FE produces the optical modes and spurious oscillations, considerable band gaps and cut-off frequency ranges.
- The high mode behaviour of B-spline based FE is convergent with order of approximation. B-spline based FEM appears a smoothing effect for the dispersion curves.
- Good convergence properties of higher order IGA
- Future work: NURBS, T-splines for vibration, wave propagation and impact problems in solids and shells, mass matrix lumping, temporal-spatial dispersion analysis for direct time integration methods, stability analysis of explicit methods. Numerical solution of Radial Dirac equation.

Thank you for your attention!