Contact treatment in isogeometric analysis

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Outline

Motivation

- We would like to solve impact problems by IGA
- Problems with normal vector in FEM due to C^0 continuity
- Negative equivalent nodal pressure in FEM

Contact Kinematics

Non-penetration condition

$$g_{\mathrm{N}} = (\mathbf{x}^2 - \mathbf{x}^1) \cdot \mathbf{n}^1 \geq 0$$

Distance function

$$f(\xi^1, \xi^2) = \left\| \mathbf{x}^2 - \bar{\mathbf{x}}^1 \right\| = \min_{\mathbf{x}^1 \subseteq \gamma^1} \left\| \mathbf{x}^2 - \mathbf{x}^1(\xi^1, \xi^2) \right\|$$

Necessary condition

$$\frac{df(\xi^1, \xi^2)}{d\xi^\alpha} = \left(\mathbf{x}^2 - \mathbf{x}^1(\xi^1, \xi^2)\right) \cdot \mathbf{x}^1_{,\alpha}(\xi^1, \xi^2) = \mathbf{0}$$

Normal vector

$$\bar{\mathbf{n}}^1 = \frac{\bar{\mathbf{x}}^1_{,\xi^1} \times \bar{\mathbf{x}}^1_{,\xi^2}}{\left\| \bar{\mathbf{x}}^1_{,\xi^1} \times \bar{\mathbf{x}}^1_{,\xi^2} \right\|} \qquad \bar{\mathbf{n}}^1 = \frac{\mathbf{x}^2 - \bar{\mathbf{x}}^1}{\left\| \mathbf{x}^2 - \bar{\mathbf{x}}^1 \right\|}$$

Contact Boundary Value Problem

Frictionless Contact in Linear Elasticity

$$-\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} = \bar{\mathbf{f}} \quad \text{in} \quad \boldsymbol{\Omega}$$

$$\varepsilon = \frac{1}{2}\left(\boldsymbol{\nabla}\mathbf{u} + \boldsymbol{\nabla}^T\mathbf{u}\right)$$

$$\boldsymbol{\sigma}(\mathbf{u}) = \mathbf{D}_{\mathrm{e}} : \varepsilon(\mathbf{u})$$

$$\mathbf{u} = \mathbf{0} \quad \text{on} \quad \Gamma_u$$

$$\boldsymbol{\sigma} \cdot \mathbf{n} = \bar{\mathbf{t}} \quad \text{on} \quad \Gamma_\sigma$$

Karush-Kuhn-Tucker conditions

$$\begin{array}{rcl} g_{\mathrm{N}} & \geq & 0 \\ p_{\mathrm{N}} & \leq & 0 \quad \text{on} \quad \Gamma_c \\ g_{\mathrm{N}}\, p_{\mathrm{N}} & = & 0 \end{array}$$

Weak form

$$\Pi^\gamma = \int_{\Omega^\gamma} \frac{1}{2}\boldsymbol{\varepsilon}^\gamma : \mathbf{D}_\mathrm{e} : \boldsymbol{\varepsilon}^\gamma \, dV - \int_{\Omega^\gamma} \bar{\mathbf{f}} \cdot \mathbf{u}^\gamma \, dV - \int_{\Gamma_\sigma^\gamma} \bar{\mathbf{t}}^\gamma \cdot \mathbf{u}^\gamma \, dS, \quad \gamma = 1, 2$$

$$\Pi = \sum_{\gamma=1}^{2} \Pi^\gamma \longrightarrow MIN$$

$$\text{subjected to } g_\mathrm{N} \geq 0 \quad \text{on } \Gamma_c$$

$$\Pi = \sum_{\gamma=1}^{2} \Pi^\gamma + \Pi_c \longrightarrow MIN$$

Treatment of Contact Constraints

- Lagrange multiplier method
- Penalty method
- Direct constraint elimination
- Nitsche method
- Perturbed Larange formulation
- Barrier method
- Augmented Lagrange method
- Cross-constraint method

Treatment of Contact Constraints

Lagrange multiplier method

$$\Pi_c^{LM} = \int_{\Gamma_c} \lambda_{\mathrm{N}} g_{\mathrm{N}} \, dS$$

$$\delta\Pi_c^{LM} = \int_{\Gamma_c} \lambda_{\mathrm{N}} \delta g_{\mathrm{N}} \, dS + \int_{\Gamma_c} \delta\lambda_{\mathrm{N}} g_{\mathrm{N}} \, dS$$

Penalty method

$$\Pi_c^{P} = \frac{1}{2} \int_{\Gamma_c} \epsilon_{\mathrm{N}} \left(\bar{g}_{\mathrm{N}}\right)^2 \, dS \quad \epsilon_{\mathrm{N}} > 0$$

$$\delta\Pi_c^{P} = \int_{\Gamma_c} \epsilon_{\mathrm{N}} \bar{g}_{\mathrm{N}} \delta\bar{g}_{\mathrm{N}} \, dS$$

Contact discretization

Contact residual

$$\int_{\Gamma_c} \lambda_{\mathrm{N}} \delta g_{\mathrm{N}} \, dS \to \sum_{i=1}^{n_c} \int_{\Gamma_i^h} \lambda_{\mathrm{N}}^h \delta g_{\mathrm{N}}^h \, dS$$

Constraint equation

$$\int_{\Gamma_c} \delta \lambda_{\mathrm{N}} g_{\mathrm{N}} \, dS = 0 \to \sum_{i=1}^{n_c} \int_{\Gamma_i^h} \delta \lambda_{\mathrm{N}}^h g_{\mathrm{N}}^h \, dS = 0$$

Interpolations

$$\lambda_{\mathrm{N}}^h = \sum_K M_K(\xi) \lambda_{\mathrm{N}K} \qquad \text{and} \qquad \delta g_{\mathrm{N}}^h = \sum_I N_I(\xi) \delta g_{\mathrm{N}I}$$

$$g_{\mathrm{N}}^h = \hat{\mathbf{C}}_i^T \mathbf{u}$$

$$\mathbf{C} = [\mathbf{C}_1 | \mathbf{C}_2 | \ldots | \mathbf{C}_{n_c}]$$

Discrete form of the potential energy

$$\Pi^{LM}(\mathbf{u},\mathbf{\Lambda}) = \frac{1}{2}\mathbf{u}^T\mathbf{K}\mathbf{u} - \mathbf{u}^T\bar{\mathbf{f}} + \mathbf{\Lambda}^T\mathbf{C}^T\mathbf{u}$$

$$\delta\mathbf{u}\left[\mathbf{K}\mathbf{u} - \bar{\mathbf{f}} + \mathbf{C}\mathbf{\Lambda}\right] = \mathbf{0}$$

$$\delta\mathbf{\Lambda}^T\left[\mathbf{C}^T\mathbf{u}\right] = \mathbf{0}$$

$$\begin{bmatrix} \mathbf{K} & \mathbf{C} \\ \mathbf{C}^T & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{u} \\ \mathbf{\Lambda} \end{Bmatrix} = \begin{Bmatrix} \bar{\mathbf{f}} \\ \mathbf{0} \end{Bmatrix}$$

Knot-to-Surface contact discretization

$$\int_{\Gamma_c} \lambda_{\mathrm{N}} \delta g_{\mathrm{N}} \, dS \approx \sum_{s=1}^{n_c} P_{\mathrm{N}s} \delta g_{\mathrm{N}}$$

$$\delta g_{\mathrm{N}s} = \delta \mathbf{u}_s^T \mathbf{N}_s$$

$$\mathbf{N}_s = \left\{ \begin{array}{c} \mathbf{n}^1 \\ -\left(1 - \bar{\xi}\right) \mathbf{n}^1 \\ -\bar{\xi} \mathbf{n}^1 \end{array} \right\}_s$$

Conclusions

- Finish implementation of 2D knot-to-knot contact discretization
- Perform numerical tests (Hertz contact, patch test)
- Incorporate into dynamic code and solve impact problems