COMPUTING UPPER BOUNDS ON FRIEDRICHS' CONSTANT

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Abstract

This contribution shows how to compute upper bounds of the optimal constant in Friedrichs' and similar inequalities. The approach is based on the method of *a priori-a posteriori inequalities* [9]. However, this method requires trial and test functions with continuous second derivatives. We show how to avoid this requirement and how to compute the bounds on Friedrichs' constant using standard finite element methods. This approach is quite general and allows variable coefficients and mixed boundary conditions. We use the computed upper bound on Friedrichs' constant in a posteriori error estimation to obtain guaranteed error bounds.

1. Introduction

This paper is dedicated to the 60th birthday of my teacher, supervisor, colleague, and good friend Michal Křížek. Naturally, my first impacted publication [8] was co-authored by him. In that paper we proposed and analyzed certain a posteriori error estimates of approximate solutions of partial differential equations. These error estimates are based on complementary energy and they contain constants which come from Friedrichs' inequality and the trace theorem. Consequently, the error estimate cannot be evaluated unless the value or an upper bound of these constants is known. This motivates our interest in numerical computations of upper bounds on Friedrichs' constant.

In general, the presence of the constants from Friedrichs-like inequalities and from the trace theorems is typical for complementary error bounds (or error majorants), see [6, 13, 16, 21] and the references therein. These error bounds are not fully reliable unless upper bounds on the involved constants are computed. These constants can be obtained from extremal eigenvalues of the corresponding differential operators. In [18], Friedrichs' constant is computed by the standard Rayleigh-Ritz method for approximations of eigenvalues. Although this method is very accurate, it provides only lower bounds on Friedrichs' constant.

Computing upper bounds on Friedrichs' constant or equivalently computing lower bounds on the corresponding minimal eigenvalue is considerably more difficult. A survey of available methods can be found in [10]. We concentrate on the method of *a priori-a posteriori inequalities* [9]. This method is quite sensitive to the proper choice of test and trial functions and in addition, these functions are required to have

continuous second derivatives. Below we show, how to avoid this practically unpleasant requirement and how to compute the upper bounds on Friedrichs' constant by the standard finite element techniques.

The rest of this paper is organized as follows. Section 2 briefly recalls the complementary error estimates for a linear second-order elliptic problem with mixed boundary conditions. Section 3 shows the relation of Friedrichs' constant and the smallest eigenvalue of the corresponding eigenproblem. Section 4 describes the method of a priori-a posteriori inequalities, shows how to overcome the requirement of C^2 regularity, and introduces an algorithm for computation of the upper bound on Friedrichs' constant. Section 5 presents the results of performed numerical experiments and indicates the accuracy of the proposed approach. Finally, Section 6 draws several conclusions and ideas for future research.

2. Complementary error estimates

Let $\Omega \subset \mathbb{R}^d$ be a domain with Lipschitz boundary. Further let Γ_{D} and Γ_{N} be relatively open parts of the boundary $\partial\Omega$ such that $\overline{\Gamma}_{\mathrm{D}} \cup \overline{\Gamma}_{\mathrm{N}} = \partial\Omega$. Further we assume that Γ_{D} and Γ_{N} have Lipschitz boundary with respect to $\partial\Omega$ and that the $d-1$ dimensional measure of Γ_{D} is nonzero. We consider the following model problem:

$$-\operatorname{div} \mathcal{A}\nabla u = f \text{ in } \Omega, \quad u = g_{\mathrm{D}} \text{ on } \Gamma_{\mathrm{D}}, \quad \boldsymbol{n}^{\top}\mathcal{A}\nabla u = g_{\mathrm{N}} \text{ on } \Gamma_{\mathrm{N}}, \tag{1}$$

where $\mathcal{A} \in \mathbb{R}^{d\times d}$ is a symmetric and uniformly positive definite tensor and $\boldsymbol{n}$ stands for the unit outward normal vector to $\partial\Omega$.

We will formulate problem (1) in the weak sense. Therefore, we assume a Dirichlet lift $\overline{g}_{\mathrm{D}} \in H^1(\Omega)$ of the Dirichlet data g_{D} such that $\overline{g}_{\mathrm{D}} = g_{\mathrm{D}}$ on Γ_{D} in the sense of traces. Further, we define the space

$$V = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_{\mathrm{D}} \text{ in the sense of traces}\}.$$

Due to integrability, we consider $\mathcal{A} \in [L^{\infty}(\Omega)]^{d\times d}$, $f \in L^2(\Omega)$, and $g_{\mathrm{N}} \in L^2(\Gamma_{\mathrm{N}})$. The weak formulation of problem (1) reads: find $u \in H^1(\Omega)$ such that $u - \overline{g}_{\mathrm{D}} \in V$ and

$$a(u,v) = F(v) \quad \forall v \in V. \tag{2}$$

The bilinear and linear forms $a(\cdot,\cdot)$ and $F(\cdot)$ are given by

$$a(u,v) = \int_{\Omega} (\nabla v)^{\top}\mathcal{A}\nabla u \,\mathrm{d}x \quad \text{and} \quad F(v) = \int_{\Omega} fv \,\mathrm{d}x + \int_{\Gamma_{\mathrm{N}}} g_{\mathrm{N}} v \,\mathrm{d}x.$$

To solve problem (2) approximately, we can use for example the finite element method. However, for the purpose of this paper we do not assume any particular method and simply consider any conforming approximation $u_h \in H^1(\Omega)$ which satisfies the Dirichlet boundary conditions, i.e. $u_h - \overline{g}_{\mathrm{D}} \in V$.

The typical complementary estimate of the error $u - u_h$ has the form:

$$|||u - u_h||| \le \left\|\mathcal{A}^{-1}\boldsymbol{y} - \nabla u_h\right\|_{\mathcal{A}} + C_{\mathrm{F}} \left\|f + \operatorname{div} \boldsymbol{y}\right\|_0 + C_{\mathrm{T}} \left\|g_{\mathrm{N}} - \boldsymbol{y}\cdot\boldsymbol{n}\right\|_{0,\Gamma_{\mathrm{N}}} \tag{3}$$

for any $\boldsymbol{y} \in \mathbf{H}(\mathrm{div}, \Omega)$, see [2, 5, 6, 7, 13, 16, 21]. The symbol $|\!|\!| v |\!|\!|^2 = a(v, v)$ stands for the energy norm, $\|\cdot\|_0$ denote the $L^2(\Omega)$ norm, $\|\cdot\|_{0,\Gamma_\mathrm{N}}$ is the $L^2(\Gamma_\mathrm{N})$ norm, and $\|\boldsymbol{y}\|_{\mathcal{A}}^2 = \int_\Omega \boldsymbol{y}^\top \mathcal{A}\boldsymbol{y} \,\mathrm{d}x$. Notice that for problem (1) we have $|\!|\!| v |\!|\!| = \|\nabla v\|_{\mathcal{A}}$ for all $v \in V$. The constants C_F and C_T come from the following variants of Friedrichs' inequality and the trace theorem [12]:

$$\|v\|_0 \le C_\mathrm{F} |\!|\!| v |\!|\!| \quad \text{and} \quad \|v\|_{0,\Gamma_\mathrm{N}} \le C_\mathrm{T} |\!|\!| v |\!|\!| \quad \forall v \in V. \tag{4}$$

Let us note that besides the complementary error estimates a variety of other approaches for a posteriori error estimation exists. For example explicit and implicit residual estimates, hierarchical estimates, estimates based on postprocessing and estimates of a quantity of interest. For more information see books [1, 3, 4, 22] and the references therein.

The standard techniques scarcely yield a guaranteed upper bound on the error $u - u_h$, however, the a posteriori error estimate (3) does. For successful practical implementation of this error bound it is crucial to find a suitable vector field $\boldsymbol{y} \in \mathbf{H}(\mathrm{div}, \Omega)$. This issue is addressed for example in [2, 7, 15, 20] etc. A straightforward approach is to minimize the right-hand side of (5) over a suitable finite dimensional subspace and we will not discuss this issue any further. The second issue is the correct value for the constants C_F and C_T.

First, we point out that it is relatively simple to eliminate the trace constant C_T from (3). It suffices to choose the vector field $\boldsymbol{y} \in \mathbf{H}(\mathrm{div}, \Omega)$ such that $\boldsymbol{y} \cdot \boldsymbol{n} = g_\mathrm{N}$ on Γ_N. Estimate (3) then simplifies to

$$|\!|\!| u - u_h |\!|\!| \le \left\| \mathcal{A}^{-1}\boldsymbol{y} - \nabla u_h \right\|_{\mathcal{A}} + C_\mathrm{F} \left\| f + \mathrm{div}\, \boldsymbol{y} \right\|_0 \tag{5}$$

for all $\boldsymbol{y} \in \mathbf{H}(\mathrm{div}, \Omega)$ satisfying $\boldsymbol{y} \cdot \boldsymbol{n} = g_\mathrm{N}$ on Γ_N.

In principle, we can use the same trick and get rid of Friedrichs' constant C_F as well. Construction of a suitable vector field $\boldsymbol{y}$ satisfying both the boundary condition $\boldsymbol{y} \cdot \boldsymbol{n} = g_\mathrm{N}$ on Γ_N and the equilibration condition $f + \mathrm{div}\, \boldsymbol{y} = 0$ in Ω is described in [19, 20]. However, practical implementation of this approach is difficult in general and, moreover, it need not to be optimal for certain problems. Therefore, we concentrate on estimate (5) in what follows.

In case of the Laplacian, the value of Friedrichs' constant C_F can be found explicitly for simple domains (balls, rectangles, cuboids, etc.) and for special combinations of Dirichlet and Neumann boundary conditions. See [10] for examples on rectangles and balls and [14] for an example on equilateral triangle.

However, these situations are rare. Practically, we have to use suitable upper bounds for this constant. In certain situations the upper bounds can be found explicitly. For example, for Laplacian with homogeneous Dirichlet boundary conditions we have an estimate [11]:

$$C_\mathrm{F} \le \frac{1}{\pi} \left(\frac{1}{|a_1|^2} + \cdots + \frac{1}{|a_d|^2} \right)^{-1/2},$$

where $|a_1|, \ldots, |a_d|$ are lengths of sides of a d-dimensional box in which the domain Ω is contained.

Anyway, this explicit upper bound is exceptional. The most of practical problems requires numerical computation of an upper bound of the constant C_{F}. A method yielding this upper bound is discussed in the following section.

3. Eigenvalue problems

The optimal constants in Friedrichs' inequality from (4) is clearly given by

$$C_{\mathrm{F}} = \sup_{0 \neq v \in V} \frac{\|v\|_0}{|||v|||}.$$

If we set $\lambda_1 = 1/C_{\mathrm{F}}^2$ then this expression can be equivalently formulated as

$$\lambda_1 = \inf_{0 \neq v \in V} \frac{|||v|||^2}{\|v\|_0^2}.$$

This is the infimum of the generalized Rayleigh quotient corresponding to the following eigenvalue problems

$$a(u_i, v) = \lambda_i (u_i, v) \quad \forall v \in V, \tag{6}$$

where we use the notation $(u, v) = \int_\Omega uv \, \mathrm{d}x$ for the $L^2(\Omega)$ inner product. Thus, the smallest eigenvalue λ_1 of this eigenproblem is related to the optimal constant in Friedrichs' inequality from (4) as $C_{\mathrm{F}} = 1/\sqrt{\lambda_1}$.

A standard numerical approach for approximate solution of differential eigenproblems is the Rayleigh-Ritz method. This method minimizes the Rayleigh quotients over a finite dimensional subspace $V_h \subset V$:

$$\lambda_1^h = \inf_{0 \neq v_h \in V_h} \frac{|||v_h|||^2}{\|v_h\|_0^2}. \tag{7}$$

However, this is clearly an upper bound of the exact eigenvalue, i.e. we have $\lambda_1 \leq \lambda_1^h$. Consequently, the value $1/\sqrt{\lambda_1^h}$ is a lower bound of the constant C_{F}.

The desired computation of the *lower bound* of the smallest eigenvalue λ_1 is a much more difficult task in general. A survey of available methods is provided in [10]. In the following section we concentrate on the method of *a priori-a posteriori* inequalities [9, 17].

4. Lower bounds of the smallest eigenvalues

Kuttler and Sigillito published in [9] the following result.

Theorem 1. *Let H be a separable Hilbert space. Let $A : H \mapsto H$ be a symmetric operator with dense domain $D(A)$. Let A have pure point spectrum $\{\lambda_i\}$ with corresponding orthonormal eigenvectors $\{u_i\}$ which are complete in H. Let A_* be an extension of A, so $D(A) \subset D(A_*) \subset H$ with $A_* u = Au$ for all $u \in D(A)$.*

For any number λ_* and any $u_* \in D(A)$, suppose there exist $w \in D(A_*)$ satisfying

$$A_* w = A_* u_* - \lambda_* u_* \quad \text{and} \quad w - u_* \in D(A).$$

Then

$$\min_i \left| \frac{\lambda_i - \lambda_*}{\lambda_i} \right| \leq \frac{\|w\|_H}{\|u_*\|_H}.$$

We will use this result to compute a lower bound for the smallest eigenvalue of problem (6) in a similar way as in [9]. However, in [9] the authors rely on the strong formulation of the eigenvalue problem and therefore they need C^2 regularity of the test and trial functions. We will use the weak formulations and utilize the standard finite element basis functions which need not to be C^2 regular.

The unique solvability of problem (2) with $\overline{g}_{\mathrm{D}} = 0$ and $g_{\mathrm{N}} = 0$ guarantees the existence of a unique solution $u \in V$ such that equality (2) holds for any $f \in L^2(\Omega)$. Consequently, we have well defined operator $B : L^2(\Omega) \mapsto V$ such that $Bf = u$. The image $\mathrm{Im}(B)$ contains those functions $u \in V$ that solve (2) for some $f \in L^2(\Omega)$ with $\overline{g}_{\mathrm{D}} = 0$ and $g_{\mathrm{N}} = 0$. Furthermore, this $f \in L^2(\Omega)$ is unique due to the density of V in $L^2(\Omega)$. Thus, we define the operator $A : \mathrm{Im}(B) \mapsto L^2(\Omega)$ such that $Au = f$.

In order to apply Theorem 1 we set $H = L^2(\Omega)$, $D(A) = \mathrm{Im}(B)$, and $A_* = A$. We consider a number λ_*, any $u_* \in \mathrm{Im}(B)$, and define the function $w \in \mathrm{Im}(B)$ by

$$a(w, v) = a(u_*, v) - \lambda_*(u_*, v) \quad \forall v \in V. \tag{8}$$

The statement of Theorem 1 then gives

$$\min_i \left| \frac{\lambda_i - \lambda_*}{\lambda_i} \right| \leq \frac{\|w\|_0}{\|u_*\|_0} \leq C_{\mathrm{F}} \frac{|\!|\!| w |\!|\!|}{\|u_*\|_0}, \tag{9}$$

where we used Friedrichs' inequality from (4).

The energy norm $|\!|\!| w |\!|\!|$ can hardly be computed exactly and therefore we use the following theorem to find its upper bound.

Theorem 2. *Let $w \in \mathrm{Im}(B)$ be given by (8). Then*

$$|\!|\!| w |\!|\!| \leq \left\| \nabla u_* - \mathcal{A}^{-1} \boldsymbol{q} \right\|_{\mathcal{A}} + C_{\mathrm{F}} \left\| \lambda_* u_* + \mathrm{div}\, \boldsymbol{q} \right\|_0 \quad \forall \boldsymbol{q} \in W, \tag{10}$$

where $W = \{\boldsymbol{q} \in \mathbf{H}(\mathrm{div}, \Omega) : \boldsymbol{q} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma_{\mathrm{N}}\}$.

Proof. Let us fix any $\boldsymbol{q} \in W$, test (8) by $v = w$ and use the divergence theorem to express

$$\begin{aligned} |\!|\!| w |\!|\!|^2 &= (\mathcal{A} \nabla u_*, \nabla w) - \lambda_*(u_*, w) - (\boldsymbol{q}, \nabla w) - (\mathrm{div}\, \boldsymbol{q}, w) \\ &= \big(\mathcal{A}(\nabla u_* - \mathcal{A}^{-1} \boldsymbol{q}), \nabla w \big) - (\lambda_* u_* + \mathrm{div}\, \boldsymbol{q}, w). \end{aligned}$$

The Cauchy-Schwarz inequality and Friedrichs' inequality from (4) yield

$$|||w|||^2 \le \left\|\nabla u_* - \mathcal{A}^{-1}\boldsymbol{q}\right\|_{\mathcal{A}} \|w\|_{\mathcal{A}} + C_{\mathrm{F}} \|\lambda_* u_* + \operatorname{div} \boldsymbol{q}\|_0 \, |||w|||.$$

Recalling the equality $\|w\|_{\mathcal{A}} = |||w|||$, we finish the proof. □

The upper bound of Friedrichs' constant can be computed from the lower bound for the smallest eigenvalue λ_1 of problem (6). We proceed as follows. We compute a sufficiently accurate Rayleigh–Ritz approximation λ_h and the corresponding approximate eigenfunction $u_h \in V_h \subset V$. We assume that λ_1 is the closest eigenvalue to λ_h. We put $\lambda_* = \lambda_h$ and $u_* = u_h$. With these data we compute an approximate minimizer $\boldsymbol{q}_h \in W$ of the upper bound (10), see Section 5 for technical details. We put $\alpha = \|\nabla u_h - \mathcal{A}^{-1}\boldsymbol{q}_h\|_{\mathcal{A}} / \|u_h\|_0$, $\beta = \|\lambda_h u_h + \operatorname{div} \boldsymbol{q}_h\|_0 / \|u_h\|_0$ and use the fact that $C_{\mathrm{F}} = 1/\sqrt{\lambda_1}$. Inequalities (9) and (10) then yield the estimate

$$\frac{\lambda_h - \lambda_1}{\lambda_1} \le \frac{1}{\sqrt{\lambda_1}} \left(\alpha + \frac{1}{\sqrt{\lambda_1}} \beta \right).$$

This is equivalent to the quadratic inequality $0 \le X^2 + \alpha X + \beta - \lambda_h$, where $X = \sqrt{\lambda_1}$. Solving this inequality we obtain the lower bound

$$X_2^2 \le \lambda_1, \quad \text{where } X_2 = \left(\sqrt{\alpha^2 + 4(\lambda_h - \beta)} - \alpha \right) /2.$$

Consequently, we have the upper bound on Friedrichs' constant

$$C_{\mathrm{F}} \le 1/X_2.$$

5. Numerical experiments

In order to compute an upper bound on Friedrichs' constant, we proceed as described above. The idea is to compute an approximate minimizer $\boldsymbol{q}_h \in W$ of the right-hand side of the estimate (10). This right-hand side is not a quadratic functional in $\boldsymbol{q}$ and, moreover, it contains the unknown Friedrichs' constant C_{F}. Since it is sufficient to compute an approximate minimizer $\boldsymbol{q}_h \in W$ only, we replace in (10) the constant C_{F} by its approximation $1/\sqrt{\lambda_h}$ where λ_h is obtained by the Rayleigh-Ritz method. Further, we use the elementary inequality

$$(a+b)^2 \le (1+\varrho^{-1})a^2 + (1+\varrho)b^2 \quad \forall \varrho > 0,\ a, b \in \mathbb{R}$$

and instead of (10) we actually minimize

$$(1+\varrho^{-1}) \left\|\nabla u_* - \mathcal{A}^{-1}\boldsymbol{q}\right\|_{\mathcal{A}}^2 + \frac{1+\varrho}{\lambda_h} \|\lambda_* u_* + \operatorname{div} \boldsymbol{q}\|_0^2 \tag{11}$$

over all $\varrho > 0$ and $\boldsymbol{q} \in W_h$, where W_h is a suitable finite dimensional subspace of W. For fixed ϱ, the expression (11) is already a quadratic functional in $\boldsymbol{q}$. After the

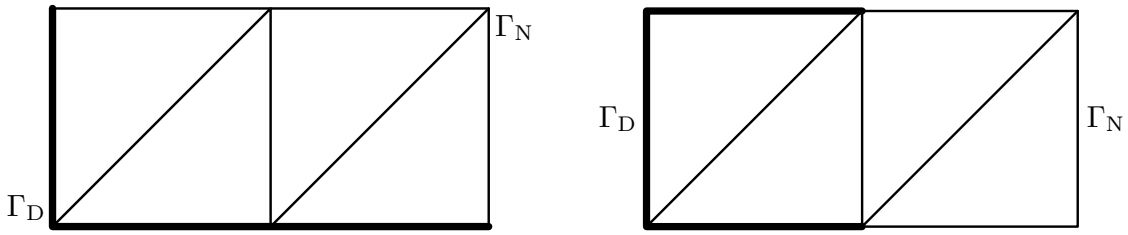

Figure 1: The domain, the initial mesh and the sets Γ_D and Γ_N used in Example 1 (left) and 2 (right).

substitution $\lambda_* = \lambda_h$ and $u_* = u_h$, the minimization of (11) over W_h is equivalent to the following problem: find $\boldsymbol{q}_h \in W_h$ such that

$$(\operatorname{div} \boldsymbol{q}_h, \operatorname{div} \boldsymbol{\psi}_h) + \frac{\lambda_h}{\varrho}(\mathcal{A}^{-1}\boldsymbol{q}_h, \boldsymbol{\psi}_h) = \frac{\lambda_h}{\varrho}(\nabla u_h, \boldsymbol{\psi}_h) - (\lambda_h u_h, \operatorname{div} \boldsymbol{\psi}_h) \quad \forall \boldsymbol{\psi}_h \in W_h. \tag{12}$$

This problem can be approached by the standard Raviart-Thomas finite elements.

Example 1: Let us consider the Poisson equation in rectangle $\Omega = (0,2) \times (0,1)$ with mixed homogeneous boundary conditions:

$$-\Delta u = f \quad \text{in } \Omega, \quad u = 0 \text{ on } \Gamma_D, \quad \boldsymbol{n}^\top \nabla u = 0 \text{ on } \Gamma_N. \tag{13}$$

We define $\Gamma_D = \partial\Omega \cap \{(x_1, x_2) : x_1 < 2 - 2x_2\}$ and $\Gamma_N = \partial\Omega \cap \{(x_1, x_2) : x_1 > 2 - 2x_2\}$, see Figure 1 (left). For testing purposes, let us choose the right-hand side $f(x_1, x_2) = 5\pi^2/16 \sin(\pi x_1/4) \sin(\pi x_2/2)$ such that the exact solution to (13) is $u(x_1, x_2) = \sin(\pi x_1/4) \sin(\pi x_2/2)$. Clearly, this u is the eigenfunction corresponding to the smallest eigenvalue $\lambda_1 = 5\pi^2/16$ of the eigenproblem

$$-\Delta u_i = \lambda_i u_i \quad \text{in } \Omega, \quad u_i = 0 \text{ on } \Gamma_D, \quad \boldsymbol{n}^\top \nabla u_i = 0 \text{ on } \Gamma_N. \tag{14}$$

We solve problem (13) by the standard finite element method with continuous and piecewise linear test functions with respect to a triangulation of Ω. The roughest triangulation has four elements and it is depicted in Figure 1 (left). Subsequently, we compute a sequence of finite element solutions u_h on a sequence of successively and uniformly refined meshes.

We use (5) to estimate the error $u - u_h$. To evaluate the right-hand side of (5) we need a value for C_F. In this example the exact value is known to be $C_F = 1/\sqrt{\lambda_1} = 4/(\sqrt{5}\pi)$. However, we do not use this value and we use its upper bound computed by minimization of (11). More precisely, we set $\varrho = 1$ and in order to test the approach, we solve (12) in two subspaces W_h^1 and W_h^2, where

$$W_h^p = \{\boldsymbol{\psi}_h \in W : \boldsymbol{\psi}_h \text{ is piecewise polynomial of degree at most } p\}, \quad p = 1, 2.$$

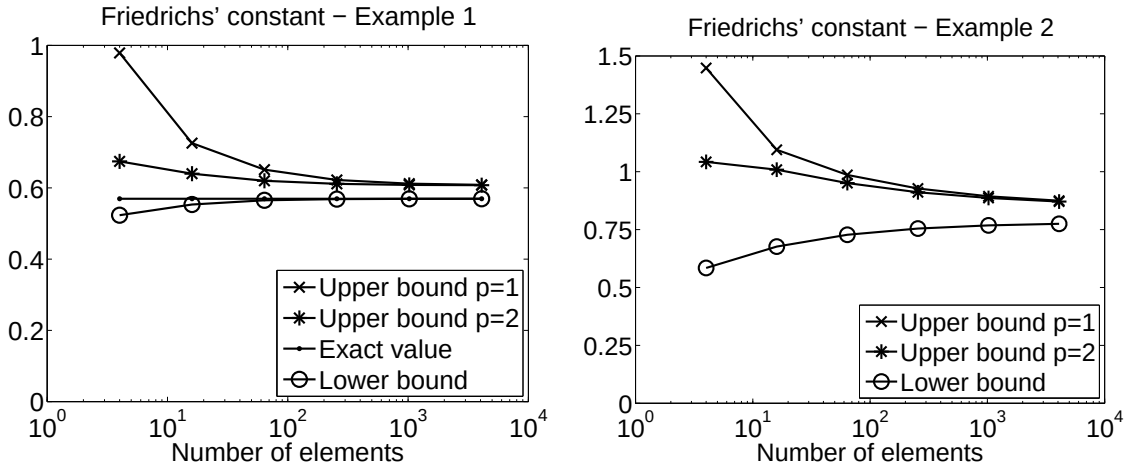

Figure 2: The lower and upper bounds for Friedrichs' constant.

The spaces W_h^1 and W_h^2 correspond to linear and quadratic Raviart-Thomas finite elements. The solutions $\boldsymbol{q}_h^1 \in W_h^1$ and $\boldsymbol{q}_h^2 \in W_h^2$ of (12) are used as described at the end of Section 4 to compute lower bounds on the smallest eigenvalue λ_1 and consequently the upper bounds on Friedrichs' constant C_{F}. We use the standard Rayleigh-Ritz method with $V_h = \{v_h \in V : v_h \text{ is piecewise linear}\}$ to compute the lower bound on C_{F}, see (7). The results are presented in Figure 2 (left).

The sharpest bounds were obtained on the finest mesh (the initial mesh refined five times, in total 4096 triangles). The lower bound $C_{\mathrm{F}}^{\mathrm{low}}$ rounded down, the exact value C_{F}, and the upper bound $C_{\mathrm{F}}^{\mathrm{up}}$ rounded up were

$$C_{\mathrm{F}}^{\mathrm{low}} = 0.5693, \quad C_{\mathrm{F}} = 0.5694, \quad C_{\mathrm{F}}^{\mathrm{up}} = 0.6075.$$

The upper bound $C_{\mathrm{F}}^{\mathrm{up}}$ can be improved by proper choice of the parameter ϱ. In this case, the upper bound decreases for great values of ϱ. Setting $\varrho = 10^6$, we calculate a sharper bound $C_{\mathrm{F}}^{\mathrm{up}} = 0.6004$.

We use this value in (5) to obtain guaranteed error bounds on $|||u - u_h|||$. The results are presented in Figure 3 (left). The right-hand side of (5) is minimized in the same way as the right-hand side of (10). We present results obtained by linear (denoted by $p = 1$ in Figure 3) and quadratic (denoted by $p = 2$) Raviart-Thomas finite elements. We also show a lower bound computed simply from a reference solution $u_h^{\mathrm{ref}} \in V_h^{\mathrm{ref}}$. If $V_h \subset V_h^{\mathrm{ref}} \subset V$, $u \in V$ is given by (2) and if $u_h \in V_h$ and $u_h^{\mathrm{ref}} \in V_h^{\mathrm{ref}}$ are the corresponding Galerkin approximations of u then it is easy to show that

$$|||u - u_h|||^2 = |||u - u_h^{\mathrm{ref}}|||^2 + |||u_h^{\mathrm{ref}} - u_h|||^2.$$

Consequently, the easily computable quantity $|||u_h^{\mathrm{ref}} - u_h||| = \left(|||u_h^{\mathrm{ref}}|||^2 - |||u_h|||^2\right)^{1/2}$ can be used as a lower bound on the error $|||u - u_h|||$. In this example, the space V_h^{ref}

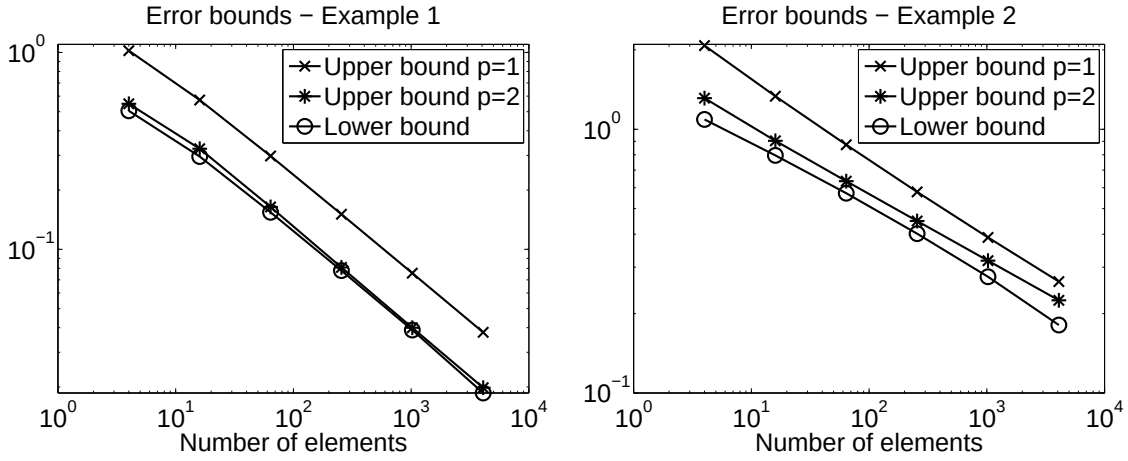

Figure 3: Lower and upper bounds on the energy norm of the error.

is based on a mesh that is constructed by seven uniform refinements of the initial mesh, i.e. it has 65 536 triangles. However, we note that more efficient methods for computing the lower bounds of the error exist and should be used, see e.g. [1].

In addition, since the exact solution is known, we can verify that the true error lies really within the computed bounds. It is indeed the case, but due to readability of Figure 3 (left), we do not plot it. Further, we note that taking the exact value C_{F} in (5) has no significant effects on the presented error bounds.

Example 2: Let us consider the same setting as in Example 1. The only difference is another choice of sets Γ_{D} and Γ_{N}. We set $\Gamma_{\mathrm{D}} = \partial\Omega \cap \{(x_1, x_2) : x_2 < 1\}$ and $\Gamma_{\mathrm{N}} = \partial\Omega \cap \{(x_1, x_2) : x_2 > 1\}$, see Figure 1 (right) for an illustration. For this choice the exact solution to problem (13) as well as to the eigenproblem (14) is unknown. Moreover, the exact solution has singularities at points $(1, 0)$ and $(1, 1)$.

Using the same methods as in Example 1, we compute the bounds for Friedrichs' constant C_{F}. The results are summarized in Figure 2 (right). The sharpest bounds were again obtained on the finest mesh. The properly rounded results were

$$C_{\mathrm{F}}^{\mathrm{low}} = 0.7750, \quad C_{\mathrm{F}}^{\mathrm{up}} = 0.8712.$$

The upper bound was computed for $\varrho = 1$, but as in Example 1, we can improve it by choosing $\varrho = 10^6$ and obtain $C_{\mathrm{F}}^{\mathrm{up}} = 0.8557$.

By the experience from Example 1, we expect the exact value of C_{F} to be close to $C_{\mathrm{F}}^{\mathrm{low}}$. However, to be on the safe side, we use $C_{\mathrm{F}}^{\mathrm{up}} = 0.8557$ to compute the guaranteed bounds of the error $u - u_h$ by (5). The same approach as in Example 1 yields the results shown in Figure 3 (right).

6. Conclusions

This contribution presents an approach for computing upper bounds on Friedrichs' constant and illustrates the usage of these upper bounds in a posteriori error estimation for guaranteed upper bound on the error. In contrast to [9] we do not need the C^2 regularity of the test and trial functions and we can use the standard finite element methods. The performed numerical experiments show that it suffices to compute the upper bound by the linear Raviart-Thomas finite elements. However, the usage of quadratic Raviart-Thomas elements yields relatively sharp results even on very rough meshes.

The combination of the presented approach with the classical Rayleigh-Ritz method enables to compute both lower and upper bound on Friedrichs' constant. This provides very good information about the accuracy of the obtained approximations. Similarly, two-sided error bounds can be computed for numerical solutions of linear elliptic differential equations. Interestingly, in both cases it is quite easy to compute the lower bounds, but is much more difficult to compute the upper bounds.

Furthermore, let us point out that the computed bounds are not truly guaranteed, because the calculations are polluted by the quadrature and round-off errors. Both the error bound (5) as well as the eigenvalue bounds (8)–(9) assume that all involved integrals and arithmetic operations are performed exactly.

Anyway, the presented concept is not limited to Friedrichs' constant only. It can be used to bound the constants in the Poincaré inequality, in various trace inequalities, etc. A list of useful corresponding differential eigenproblems is given in [17]. The computed bounds of these constants have applications for wide range of differential operators including fourth-order operators. The method easily incorporates additional terms in the differential operator (e.g. reaction term), it is suitable for variable coefficients and mixed boundary conditions.

Thus, the future research in this area can concentrate on various generalizations of the presented approach. It is desirable to analyze the method and prove its convergence. From practical point of view there is a potential for improving both accuracy and performance of numerical computations. This interesting and practical area of research promises new results and we plan to work it out in near future.

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